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PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Periodic neutral functional differential equations are considered. Sufficient conditions for existence, uniqueness and global attractivity of periodic solutions are established by combining the theory of monotone semiflows generated by neutral functional differential equations and Krasnosel'skii's fixed-point theorem. These results are applied to a concrete neutral functional differential equation that can model single-species growth, the spread of epidemics, and the dynamics of capital stocks in a periodic environment.

1. *Introduction*

Existence, uniqueness and global attractivity of periodic solutions of functional differential equations are of great interest in mathematics and its applications to the modeling of various practical problems. There is an extensive literature related to this topic for autonomous models (see [**1**, **2**] and references cited therein). Since physical environments vary, there are sufficient reasons to consider nonautonomous functional differential equations, and in particular, periodic cases (for example seasonal effects of weather, food supplies, and mating habits). To our knowledge, most existing results on the existence of periodic solutions of functional differential equations are for the retarded type, and these existence results are usually obtained by the technique of bifurcation [**6**, **7**], by fixed point theorems [**3**, **27**, **28**, **30**], or by degree theory [**18**, **19**]. In general, it is more difficult to study the uniqueness or global attractivity of the periodic solutions. The Liapunov direct method is attractive for the general case, but it often needs more mathematical restrictions, such as diagonal dominance [**17**, **29**].

It should be pointed out that the monotone semiflow theory developed in recent years plays an important role in investigation of the behavior of solutions of dynamic systems [**10, 15, 16, 20–26, 31–33**]. After the fundamental work of Hirsch [**12**, **13**] on general monotone dynamical systems, Smith *et al.* [**20–26**] applied the theory of general monotone semiflows to retarded functional differential equations and partial functional differential equations. For neutral functional differential equations, Wu and his collaborators [**15**, **16**, **31**, **32**] successfully constructed a partially ordered space $(C_r, \leqslant_D)$ in which the monotone property of semiflows generated by neutral functional differential equations could also be established, and thus the general theory of monotone dynamical systems was also made applicable to neutral functional differential equations.

In addition to generic convergence results, the monotone semiflow theory also

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provides a useful tool in the investigation of periodic and asymptotically periodic solutions of dynamic systems. Tang and Kuang [**27**] developed a method of finding periodic solutions of a general Lotka–Volterra type n-dimensional periodic retarded functional differential equation

$$\dot{x}_i(t) = x_i(t)F_i(t, x_1(t), \dots, x_n(t), x_1(t-\tau_1(t)), \dots, x_n(t-\tau_n(t))), \qquad i = 1, 2, \dots, n, \quad (1.1)$$

by combining the theory of monotone semiflows generated by retarded functional differential equations with Horn's fixed point theorem [**27**]. In [**30**], Wang, Chen adn Lu considered periodic solutions of the general periodic retarded functional differential equations

$$\dot{x}(t) = F(t, x_t), \quad (1.2)$$

and obtained results on existence, uniqueness and global attractivity of a periodic solution of (1.2) by combining the basic theory of monotone semiflows for retarded functional differential equations established in [**24**] with the Krasnosel'skii's fixed point theorem.

Motivated by the work of [**27**] and [**30**], we are concerned in this paper with the general periodic neutral functional differential equations

$$\frac{d}{dt}D(x_t) = F(t, x_t). \quad (1.3)$$

Assuming that the generalized difference operator $D : C_r \longrightarrow R^n$ is quasimonotone (see below), and making use of the theory of monotone semiflow for neutral functional differential equations established in [**15**, **16**, **31**, **32**], we develop a technique of a monotone and concave operator that is similar to the Poincaré mapping of neutral functional differential equations (1.3). Then, by virtue of such an operator, we show that under some conditions on F, system (1.3) admits a unique positive periodic solution that attracts all solutions in $C^+_{r,D}$.

This paper is organized as follows. In the next section, we present some notation and preliminaries adopted from [**32**]. The main results for periodic solutions of (1.3) are given in Section 3. The final section provides an application of our main results to a concrete neutral functional differential equation that can model the growth of a single species in population dynamics, the spread of epidemics, and the dynamics of capital stocks in a periodic environment [**4**, **5**, **16**].

2. *Notation and preliminaries*

Let R^n_+ be the subset of non-negative vectors in R^n. The partially ordered space $(R^n, \leqslant)$ is induced by R^n_+. Given $r = (r_1, \dots, r_n) \in R^n_+$, we define $|r| = \max\{r_i : 1 \leqslant i \leqslant n\}$, $C_r = \Pi_{i=1}^n C([-r_i, 0], R)$, $C_r^+ = \Pi_{i=1}^n C([-r_i, 0], R^+)$. Equip C_r with the uniform convergence topology defined by the norm

$$\|\varphi\|_{C_r} = \max_{1 \leqslant i \leqslant n} \sup_{-r_i \leqslant \theta \leqslant 0} |\varphi_i(\theta)|, \qquad \varphi \in C_r.$$

It is obvious that C_r is a Banach space with this topology.

Suppose that $D : C_r \longrightarrow R^n$ is a given bounded and linear operatory that is represented as follows:

$$D_i(\varphi) = \sum_{j=1}^n \int_{-r_j}^0 d\mu_{ij}(\theta)\varphi_j(\theta), \qquad \varphi \in C_r,\ 1 \leqslant i \leqslant n, \quad (2.1)$$

where function $\mu_{ij} : [-r_j, 0] \longrightarrow R$, $1 \leqslant i, j \leqslant n$, is of bounded variation on $[-r_j, 0]$. Let

$$\bar{\mu}_{ij}(\theta) = \begin{cases} \mu_{ij}(\theta) & \theta \in [-r_j, 0) \\ \mu_{ij}(0^-) & \theta = 0, \end{cases} \tag{2.2}$$

$$L_i(\varphi) = \sum_{j=1}^{n} \int_{-r_j}^{0} d\bar{\mu}_{ij}(\theta)\varphi_j(\theta), \qquad \phi \in C_r,\ 1 \leqslant i \leqslant n. \tag{2.3}$$

Then $D(\varphi) = A\varphi(0) + L(\varphi)$, where $A = (\mu_{ij}(0) - \mu_{ij}(0^-))$. Recall (see [**11**, **32**]) that D is said to be atomic at zero if A is nonsingular and there exists a nonnegative scalar continuous function β on $[0, \min_{j\in J} r_j]$ with $\beta(0) = 0$ and

$$\left| \sum_{j\in J} \int_{-s}^{0} d\bar{\mu}_{ij}(\theta)\varphi_j(\theta) \right| \leqslant \beta(s)\|\varphi\|_{C_r},$$

where $J = \{i \in N;\, r_i \neq 0\}$ and $N = \{1, \ldots, n\}$.

DEFINITION 2.1 ([**32**]). A bounded and linear operator $D : C_r \longrightarrow R^n$ defined by (2.1) is said to be quasimonotone if the following hold:

(i) It is atomic at zero.

(ii) $b_{ii} > 0$ and $b_{ij} \geqslant 0$ for $i, j \in N = \{1, 2, \ldots, n\}$, where $(b_{ij}) = A^{-1}$.

(iii) $\mu_{ij} : [-r_j, 0) \longrightarrow R$, $i, j \in N$ is nonincreasing and continuous from the left.

Note that the D-operator associated with usual equations has the form

$$D\varphi = \varphi(0) + L(\varphi),$$

and therefore it is quasimonotone if (iii) is satisfied, where $L(\varphi)$ is defined by (2.2) and (2.3). Throughout the paper, we assume that the operator D is quasimonotone.

Now we define an ordering, denoted by $\leqslant_D$, as follows:

$$\varphi \leqslant_D \psi \quad \text{iff} \quad \begin{cases} \varphi_i(\theta) \leqslant \psi_i(\theta) & \theta \in [-r_i, 0],\ 1 \leqslant i \leqslant n \\ D(\varphi) \leqslant D(\psi) & \varphi, \psi \in C_r. \end{cases}$$

We also write $\psi \geqslant_D \varphi$ if $\varphi \leqslant_D \psi$. Let $C_{r,D}^+ = \{\varphi \in C_r : \varphi \geqslant_D 0\}$. Then $\mathrm{Int}C_{r,D}^+$ is not empty, provided that D is quasimonotone. We write $\varphi <_D \psi$ if $\varphi \leqslant_D \psi$ and $\varphi \neq \psi$, and $\varphi \ll_D \psi$ if $\psi - \varphi \in \mathrm{Int}C_{r,D}^+$. Let $\wedge$ denote the inclusion $R^n \longrightarrow C_r$ by $x \longrightarrow \hat{x}$, $\hat{x}_i(\theta) \equiv x_i, \theta \in [-r_i, 0]$ and $i \in N$. Thus $(C_r, \leqslant_D)$ is a strongly ordered space, meaning that for every open set $U \subset C_r$ and for any $\varphi \in U$, there exist $\varphi_1, \varphi_2 \in U$ such that $\varphi_1 \ll_D \varphi \ll_D \varphi_2$. We refer to Wu and Freedman [**32**] for detailed discussion of C_r and its ordering induced by $C_{r,D}^+$.

Consider the following neutral functional differential equations:

$$\frac{d}{dt} D(x_t) = F(t, x_t), \tag{2.4}$$

where $D : C_r \longrightarrow R^n$ is quasimonotone, and $F : \Omega \longrightarrow R^n$, where Ω is an open subset in $R^+ \times C_r$, is assumed to satisfy the following basic conditions:

(C1) F is continuous and Lipschitz in the second variable on any compact subset of Ω.

(C2) $F(t, \hat{0}) = 0$ for all $t \in R^+$.

Under these conditions, we know that (see [**11**, **32**]), for any $(\sigma, \varphi) \in \Omega$, there exists a unique solution of (3.1) through (σ, φ). As usual, the unique solution through (σ, φ) is denoted by $x_t(\sigma, \varphi)$ or $x(t, \sigma, \varphi)$, and $x_t = (x_t^1, \ldots, x_t^n)$ is an element of C_r with $x_t^i(\theta) = x^i(t+\theta)$, $-r_i \leqslant \theta \leqslant 0$, $i \in N$. Throughout this paper, we always assume that, for any $\varphi \in C_r$, the solution $x(t, \sigma, \varphi)$ can be extended to $[\sigma, \infty)$. Note that (C2) implies that $x(t, \sigma, \hat{0}) = 0$ for all $t \geqslant 0$.

Now we can start with the following monotonicity principle.

Lemma 2.2 [**32**, Lemma 3.1, Remark 3.2]. *Assume the following*:

$$\text{If } (t,\varphi),(t,\psi) \in \Omega \text{ with } \varphi \leqslant_D \psi \text{ and } D_i(\varphi) = D_i(\psi) \text{ for some } i \in N, \\ \text{then } F_i(t,\varphi) \leqslant F_i(t,\psi). \tag{M}$$

Then the following hold:

(i) *For any* $(\sigma, \varphi), (\sigma, \psi) \in \Omega$ *with* $\varphi \leqslant_D \psi$, *we have* $x_t(\sigma, \varphi) \leqslant_D x_t(\sigma, \psi)$ *for all* $t \geqslant \sigma$.

(ii) $C_{r,D}^+$ *is positively invariant for* (2.4), *that is, for any* $(\sigma, \varphi) \in R^+ \times C_{r,D}^+$, *we have* $x_t(\sigma, \varphi) \geqslant_D 0$ *for all* $t \geqslant \sigma$.

To proceed further, we need to assume further that F is continuously differentiable. By [**11**], we know that $x(t, \sigma, \varphi)$ or $x_t(\sigma, \varphi)$ is continuously Fréchet differentiable. Let $D_\varphi F(t, \varphi)$ be the Fréchet derivative of $F(t, \varphi)$ with respect to φ in C_r. For any given $\varphi \in C_r$, by the Riesz representation theorem, there exists $\eta_{ij}(\varphi, t, \cdot) : [-r_j, 0] \longrightarrow R$, $i, j \in N$, of bounded variation on $[-r_j, 0]$, such that

$$[D_\varphi F(t,\varphi)\psi]_i = \sum_{j=1}^{n} \int_{-r_j}^{0} d_\theta \eta_{ij}(\varphi, t, \theta)\psi_j(\theta), \qquad i \in N, \ \psi \in C_r. \tag{2.5}$$

Let

$$B(\varphi, t) = (\eta_{ij}(\varphi, t, 0) - \eta_{ij}(\varphi, t, 0^-)) \tag{2.6}$$

and define $K(\varphi, t) : C_r \longrightarrow R^n$ by

$$(K(\varphi,t)\psi)_i = \sum_{j=1}^{n} \int_{-r_j}^{0} d_\theta \bar{\eta}_{ij}(\varphi, t, \theta)\psi_j(\theta), \qquad i \in N, \ \psi \in C_r, \tag{2.7}$$

where

$$\bar{\eta}_{ij}(\varphi, t, \theta) = \begin{cases} \eta_{ij}(\varphi, t, \theta) & \theta \in [-r_j, 0), \\ \eta_{ij}(\varphi, t, 0^-) & \theta = 0, \end{cases} \tag{2.8}$$

then $D_\varphi F(t, \varphi)\psi = B(\varphi, t)\psi(0) + K(\varphi, t)\psi$.

Definition 2.3. The neutral functional differential equation (2.4) is said to be cooperative if, for any $\varphi \in C_r$, all off-diagonal elements of $B(\varphi, t)A^{-1}$ are non-negative and $(K(\varphi, t) - B(\varphi, t)A^{-1}L)C_{r,D}^+ \subset R_+^n$ for all $t \in R$.

We need the following assumptions (see [**32**]):

(I) If $(t, \varphi), (t, \psi) \in \Omega$ with $\varphi \leqslant_D \psi$ and $\varphi_i(-r_i) < \psi_i(-r_i)$ for some $i \in N$, then there exists $j \in N$ such that either (i) $D_j(\varphi) < D_j(\psi)$ or (ii) $D_j(\varphi) = D_j(\psi)$ and $F_j(t, \varphi) < F_j(t, \psi)$.

(T) For any proper subset $K \subset N$ and any φ, $\psi \in C_r$ with $\varphi \leqslant_D \psi$, $\varphi_j(\theta) < \psi_j(\theta)$

and $D_j(\varphi) < D_j(\psi)$ for $j \in K$ and $\theta \in [-r_j, 0]$, there exists $i \in N\backslash K$ such that either (i) $D_i(\varphi) < D_i(\psi)$ or (ii) $D_i(\varphi) = D_i(\psi)$ and $F_i(t,\varphi) < F_i(t,\psi)$.

(P) There exists a continuous functional $l_i : R \times C_r \times C_r \longrightarrow R$ such that, for any $(t,\varphi), (t,\psi) \in \Omega$ with $\varphi \leqslant_D \psi$, we have

$$F_i(t,\psi) - F_i(t,\varphi) \geqslant l_i(t,\varphi,\psi)(D_i(\psi) - D_i(\varphi)), \qquad i \in N.$$

Note that (P) implies (M), but sometimes (P) is easier to verify. Note also that if (2.4) is cooperative, then (P) holds for (2.4) (see [**32**, proof of Corollary 3.2]). From Lemma 2.2, we know that under (M), one only knows that the solution semiflow is monotone. The additional assumptions (I) and (P) give *stronger* but *more convenient* conditions than the so-called *irreducibility*, which together with (M) or (P) would guarantee that the solution semiflow was eventually *strongly* monotone.

LEMMA 2.4 [**32**, Theorem 3.1]. *Let* (I), (P) *and* (T) *hold. If* $\psi, \varphi \in C_r$ *with* $\varphi <_D \psi$, *then* $x(t,\sigma,\varphi) \ll x(t,\sigma,\psi)$ *and* $D(x_t(\sigma,\varphi)) \ll D(x_t(\sigma,\psi))$ *for all* $t \geqslant \sigma + n|r|$.

LEMMA 2.5. *For any* $\varphi \in C^+_{r,D}$, $\beta \in \operatorname{Int} C^+_{r,D}$, *let* $y(t,\beta) = D_\varphi x(t,0,\varphi)\beta$. *Then*

(i) $y(t,\beta)$ *satisfies the linear variational equation*

$$\begin{cases} \dfrac{d}{dt}D(y_t) = D_{x_t}F(t,x_t(0,\varphi))y_t \quad t \geqslant 0 \\ y_0 = \beta; \end{cases} \tag{2.9}$$

(ii) *if neutral functional differential equation* (2.4) *is cooperative, then* $y_t(\beta) \in \operatorname{Int} C^+_{r,D}$, $t \geqslant 0$.

Proof. By [**11**], $y(t,\beta)$ satisfies the linear variational equation

$$\begin{cases} \dfrac{d}{dt}D_{x_t}(D(x_t(0,\varphi)))y_t = D_{x_t}F(t,x_t(0,\varphi))y_t, \quad t \geqslant 0 \\ y_0 = \beta. \end{cases}$$

Since the operator D is bounded and linear, $D_{x_t}(D(x_t(0,\varphi)))y_t = D(y_t)$. Thus the first assertion of Lemma 2.5 is true. To prove (ii), let $C(\varphi,t) = B(\varphi,t)A^{-1} = (C_{ij}(\varphi,t))$. For any $h \in C_r$, $D(h) = Ah(0) + L(h)$ such that

$$\begin{aligned} h(0) &= A^{-1}[D(h) - L(h)], \\ D_{x_t}F(t,x_t(\varphi))h &= B(\varphi,t)h(0) + K(\varphi,t)h \\ &= C(\varphi,t)Dh + (K(\varphi,t) - C(\varphi,t)L)h. \end{aligned}$$

For any $h, k \in C_r$ with $h \leqslant_D k$, we have

$$\begin{aligned} &[D_{x_t}F(t,x_t(\varphi))k]_i - [D_{x_t}F(t,x_t(\varphi))h]_i \\ &\quad = \sum_{j=1}^n C_{ij}(\varphi,t)(D_j(k) - D_j(h)) + [(K(\varphi,t) - C(\varphi,t)L)(k-h)]_i \\ &\quad \geqslant C_{ii}(\varphi,t)(D_i(k) - D_i(h)), \end{aligned}$$

where the inequality follows from cooperativity hypothesis, and the subscript i denotes the ith component. Therefore condition (P) holds for equation (2.9). Then, by [**32**, Lemma 3.3], we have $y_t(\beta) \in \operatorname{Int} C^+_{r,D}$, since $\beta \in \operatorname{Int} C^+_{r,D}$, and equation (2.9) is linear. The proof of this lemma is complete. □

Definition 2.6. Let $(E, \leqslant)$ be a partially ordered Banach space induced by its positive cone P, which has nonempty interior $\text{Int}\, P$ in E. The nonlinear operator $U : P \longrightarrow P$ is said to be

(i) monotone if for any $\varphi, \psi \in P$ with $\varphi \leqslant \psi$ one has $U(\varphi) \leqslant U(\psi)$;

(ii) strongly positive if for any $\varphi \in P$ with $\varphi \neq 0$ one has $U(\varphi) \in IntP$;

(iii) strongly concave if for any $\varphi \in IntP$ and $\tau \in (0,1)$, there exists a number $\delta > 0$ such that $U(\tau\varphi) \geqslant (1+\delta)\tau U(\varphi)$.

Lemma 2.7 (Krasnosel'skii [**14**]). *A monotone, strongly positive and strongly concave operator U defined on a positive cone P can have no more than one nonzero fixed point in P.*

3. *Periodic solutions of periodic neutral functional differential equations*

In this section, we establish some results for periodic solutions of the following *periodic* neutral functional differential equation:

$$\frac{d}{dt} D(x_t) = F(t, x_t). \tag{3.1}$$

Here, in addition to those basic assumptions on D and F in Section 2, F is further assumed to be *periodic* in the first variable, that is, $F(t+\omega, \varphi) = F(t, \varphi)$ for all $(t, \varphi) \in R^+ \times C_r$, where $\omega > 0$ is a given constant.

We first have the following lemma.

Lemma 3.1. *Let* (M) *hold. If there exists* $h \in R^n$ *such that* $f(t, \hat{h}) \geqslant 0$ ($f(t, \hat{h}) \leqslant 0$) *for any* $t \in R$, *then* $\{x_{m\omega}(0, \hat{h})\}_{m=0}^{\infty}$ *is nondecreasing (nonincreasing) in* $(C_r, \leqslant_D)$ *with respect to m.*

Proof. Suppose that $f(t, \hat{h}) \geqslant 0$. Then the proof in the case of $f(t, \hat{h}) \leqslant 0$ is similar. If $\varphi \geqslant_D \hat{h}$, $D_i(\varphi) = D_i(\hat{h})$ for some $i \in N$, then, by (M), we have $F_i(t, \varphi) \geqslant F_i(t, \hat{h}) \geqslant 0$. Therefore, by a standard comparison method (see, for example, [**32**, Lemma 3.1, Remark 3.2], we know that the set $[\hat{h}, \infty) = \{\varphi \in C_r : \varphi \geqslant_D \hat{h}\}$ is positively invariant for (3.1). In particular, $x_\omega(0, \hat{h}) \geqslant_D \hat{h}$. Now Lemma 2.2(i) implies that

$$x_\omega(0, x_\omega(0, \hat{h})) \geqslant_D x_\omega(0, \hat{h}) \geqslant_D \hat{h}.$$

By the uniqueness of solutions of (3.1) and the periodicity of f, one has

$$x_\omega(0, x_\omega(0, \hat{h})) = x_{2\omega}(0, \hat{h}),$$

and hence

$$x_{2\omega}(0, \hat{h}) \geqslant_D x_\omega(0, \hat{h}) \geqslant_D \hat{h}.$$

Continuing in this manner, we see that $\{x_{m\omega}(0, \hat{h})\}_{m=0}^{\infty}$ is nondecreasing in $(C_r, \leqslant_D)$ with respect to m, and the proof is complete. □

Recall that a bounded and linear operator $D : C_r \longrightarrow R^n$ is stable if the zero solution of the generalized difference equation

$$\begin{cases} D(y_t) = 0 \\ y_0 = \varphi \end{cases}$$

is uniformly asymptotically stable. From [**11**], D is stable if and only if there are constants $a > 0$ and $b > 0$ such that, for any $\varphi \in C_r$ and $h \in C([0,\infty), R^n)$, the solution $y(t)$ of the nonhomogeneous equation

$$\begin{cases} D(y_t) = h(t) & t \geqslant t_0 \\ y_{t_0} = \varphi \end{cases}$$

satisfies $\|y_t\| \leqslant b\mathrm{e}^{-a(t-t_0)}\|y_{t_0}\| + b\sup_{t_0 \leqslant s \leqslant t}|h(s)|$, $t \geqslant t_0$.

The next theorem gives sufficient conditions for the existence of positive ω-periodic solutions to (3.1).

THEOREM 3.2. *Let* (M) *hold. Assume that the following hold*:

(H1) *There exist* $0 < a < b$ *such that*

$$F(t, \hat{A}) \geqslant 0 \quad \text{and} \quad F(t, \hat{B}) \leqslant 0, \qquad t \in R^+,$$

where $A = (a, a, \ldots, a) \in R^n$, $B = (b, b, \ldots, b) \in R^n$.

(H2) *The operator* D *is stable and* F *maps bounded sets of* $R \times C_r$ *into bounded sets of* R^n.

Then, each of $x(t, 0, \hat{A})$ *and* $x(t, 0, \hat{B})$ *converges to a positive* ω*-periodic solution of* (3.1) *as* $t \to +\infty$.

Proof. Let $y(t) \equiv B$, $t \in R^+$. Then

$$\frac{d}{dt}D(y_t) = 0 \geqslant F(t, \hat{B}) = F(t, y_t).$$

Thus, by Lemma 2.2,

$$0 \leqslant x(t, 0, \hat{B}) \leqslant y(t) = B, \qquad t \geqslant 0.$$

Similarly, $0 \leqslant A \leqslant x(t, 0, \hat{A})$, $t \geqslant 0$. Since $A \ll B$, again by Lemma 2.2, we have $A \leqslant x(t, 0, \hat{A}) \leqslant x(t, 0, \hat{B}) \leqslant B$, $t \geqslant 0$, that is, $\{x_t(0, \hat{A}) : t \geqslant 0\}$ and $\{x_t(0, \hat{B}) : t \geqslant 0\}$ are bounded sets of C_r since $C^+_{r,D}$ is a normal cone of C_r.

Next we show that $\{x_t(0, \hat{A}) : t \geqslant 0\}$ and $\{x_t(0, \hat{B}) : t \geqslant 0\}$ are precompact in C_r. Let $\gamma^+(\hat{A}) = \{x_t(0, \hat{A}) : t \geqslant 0\}$. For any $s \geqslant 0$, there is a nonnegative integer m such that $s = m\omega + s_0$, where $s_0 \in [0, \omega)$. Then $F(s, \varphi) = F(s_0, \varphi)$ for any $\varphi \in C_r$. Hence there exists a constant number $M > 0$ such that $|F(s, \gamma^+(\hat{A}))| \leqslant M$ for any $s \geqslant 0$ since $\gamma^+(\hat{A})$ is bounded. Now we have

$$D(x_{t+\tau}(0, \hat{A}) - x_t(0, \hat{A})) = \int_t^{t+\tau} F(s, x_s(0, \hat{A}))\, ds \qquad \text{for any } \tau \geqslant 0,\ t \geqslant 0.$$

From (H2), we know that

$$\|x_{t+\tau}(0, \hat{A}) - x_t(0, \hat{A})\| \leqslant b\|x_\tau(0, \hat{A}) - \hat{A}\| + bM\tau.$$

Since $x_\tau(0, \hat{A})$ is continuous at $\tau = 0$, $x(t, 0, \hat{A})$ is uniformly continuous in $[-r, +\infty]$. Hence $\gamma^+(\hat{A})$ is precompact in C_r since $x(t, 0, \hat{A})$ is bounded in $[-r, +\infty)$. A similar argument can prove that $\{x_t(0, \hat{B}) : t \geqslant 0\}$ is also precompact in C_r.

Now we define operator $P : C^+_{r.D} \longrightarrow C^+_{r.D}$ by

$$P(\varphi) = x_\omega(0, \varphi), \tag{3.2}$$

and denote by $P^m(\varphi)$ the mth iterate of φ under P. Then, by Lemma 3.1, the sequence $\{P^m(\hat{A})\}_{m=0}^{\infty}$ ($\{P^m(\hat{B})\}_{m=0}^{\infty}$) is nondecreasing and bounded above by $\hat{B}$ (nonincreasing and bounded below by $\hat{A}$). Therefore, each sequence converges as $m \to \infty$ from the precompactness of each sequence.

Let $\varphi^* = \lim_{m\to\infty} P^m(\hat{A})$ and $\psi^* = \lim_{m\to\infty} P^m(\hat{B})$. Then, since P is continuous by [**11**], we know that $P(\varphi^*) = \varphi^*$ and $P(\psi^*) = \psi^*$, that is, $x(t, 0, \varphi^*)$ and $x(t, 0, \psi^*)$ are positive ω-periodic solutions of (3.1). It is easy to see that $x(t, 0, \hat{A})$ $(x(t, 0, \hat{B}))$ tends to $x(t, 0, \varphi^*)$ $(x(t, 0, \psi^*))$ as $t \to +\infty$. This completes the proof of Theorem 3.2. □

REMARK 3.3. In Theorem 3.2, if (H1) is replaced by the following (H1)′, which is stronger than (H1):

$$\text{there exist } 0 < a < b \text{ such that } F(t, \hat{A}s) \geqslant 0, \text{ for } t \in R^+,\ 0 \leqslant s \leqslant 1, \quad \text{and } F(t, \hat{B}\xi) \leqslant 0 \text{ for } t \in R^+,\ 1 \leqslant \xi, \tag{H$'$}$$

and if (3.1) admits a unique positive ω-periodic solution, then Lemma 2.4 and Theorem 3.2 imply that this periodic solution attracts each solution $x(t, 0, \varphi)$ of (3.1) with $\varphi \in C^+_{r,D}$ and $\varphi \neq 0$.

Note that Lemma 2.4 implies that there exists a positive integer k_0 such that $x_t(0, \varphi) \gg_D 0$ for $t \geqslant k_0\omega - |r| > 0$, $\varphi \in C^+_{r,D}$ and $\varphi \neq 0$. For this fixed k_0, we define an operator $U : C^+_{r,D} \longrightarrow C^+_{r,D}$ by

$$U\varphi = x_{k_0\omega}(0, \varphi). \tag{3.3}$$

Then we have the following conclusions.

PROPOSITION 3.4. *Assume that* (I), (P) *and* (T) *hold. Then the operator U defined by* (3.3) *is strongly monotone and strongly positive.*

Proof. This is an immediate consequence of Lemma 2.2 and Lemma 2.4. □

To derive sufficient conditions under which the operator U is strongly concave, we define an auxiliary operator W by

$$W\varphi = U\varphi - D_\varphi U(\varphi)\varphi, \qquad \varphi \in C^+_{r,D}, \tag{3.4}$$

where $D_\varphi U(\varphi)$ is the Fréchet derivative of U with respect to φ.

PROPOSITION 3.5. *Under the assumptions of Proposition* 3.4, *if* $W(\operatorname{Int} C^+_{r,D}) \subset \operatorname{Int} C^+_{r,D}$, *then* $U(\tau\varphi) \geqslant_D \tau U(\varphi)$ *for all* $\varphi \in C^+_{r,D}$ *and* $\tau \in [0, 1]$.

Proof. Since the operator U is continuous by [**11**], it is sufficient to prove that

$$U(\tau\varphi) \geqslant_D \tau U(\varphi), \qquad \varphi \in \operatorname{Int} C^+_{r,D},\ \tau \in (0, 1). \tag{3.5}$$

Suppose that (3.5) is not true. Then there exist $\varphi_0 \in \operatorname{Int} C^+_{r,D}$ and $\tau_0 \in (0, 1)$ such that $\psi_0 = U(\tau_0\varphi_0) - \tau_0 U(\varphi_0)$ is not a point in the cone $C^+_{r,D}$ of C_r. Then a well known result from convex analysis (see, for example, [**34**, Corollary 2.4.16]) implies that there exists a continuous linear functional $g : C_r \longrightarrow R$ and a real number b such that

$$g(\varphi) < b < g(\psi_0) \qquad \text{for all } \varphi \in C^+_{r,D}. \tag{3.6}$$

Let us define an auxiliary function

$$\alpha(\tau) = g\left(\frac{1}{\tau}U(\tau\varphi_0) - U\varphi_0\right) - \frac{1}{\tau}b, \qquad 0 < \tau \leqslant 1.$$

Then $\alpha(\tau)$ is obviously differentiable and

$$\alpha'(\tau) = \frac{1}{\tau^2}\{b - g(W(\tau\varphi_0))\}.$$

Since $W(\operatorname{Int} C^+_{r,D}) \subset \operatorname{Int} C^+_{r,D}$, (3.6) implies that $\alpha'(\tau) > 0$, $0 < \tau \leqslant 1$. It follows that $\alpha(\tau_0) < \alpha(1) = g(0) - b < 0$. However, again by (3.6), $\alpha(\tau_0) = (1/\tau_0)(g(\psi_0) - b) > 0$. This contradiction implies that (3.5) is true, and the proof of Proposition 3.5 is complete. □

PROPOSITION 3.6. *Under the assumptions of Proposition* 3.5, *the operator U defined by* (3.3) *is strongly concave on* $C^+_{r,D}$.

Proof. For the sake of contradiction, assume that U is not strongly concave on $C^+_{r,D}$. Then there exist $\varphi_0 \in \operatorname{Int} C^+_{r,D}$ and $\tau_0 \in (0, 1)$ such that, for any $\delta > 0$,

$$U(\tau_0\varphi_0) - (1 + \delta)\tau_0 U\varphi_0 \notin C^+_{r,D}. \tag{3.7}$$

Since $U(\tau_0\varphi_0) - \tau_0 U\varphi_0 \in C^+_{r,D}$ by Proposition 3.5, it follows that $\psi = U(\tau_0\varphi_0) - \tau_0 U\varphi_0$ is a boundary point of the cone $C^+_{r,D}$. (Otherwise, $\psi \in \operatorname{Int} C^+_{r,D}$. Since $\tau_0 U\varphi_0 \in C^+_{r,D}$, there exists a sufficiently small $\delta > 0$ such that $\psi - \delta_0\tau_0 U\varphi_0 \in C^+_{r,D}$, that is, $U(\tau_0\varphi_0) - (1 + \delta_0)\tau_0 U\varphi_0 \in C^+_{r,D}$. This contradicts (3.7).) Define $\psi^\tau = (1/\tau)U(\tau\varphi_0) - U\varphi_0$, and let ψ^τ_i be the ith component of ψ^τ. Then we have two cases to consider: either $\psi_i(\theta_0) = 0$ for some $i \in N$ and some $\theta_0 \in [-r_i, 0]$, or $D_i(\psi) = 0$ for some $i \in N$.

Case 1: There exists an $i \in N$ such that $\psi_i(\theta_0) = 0$ for some $\theta_0 \in [-r_i, 0]$.

Set $\alpha(\tau, \theta) = \psi^\tau_i(\theta) = (1/\tau)U_i(\tau\varphi_0)(\theta) - U_i(\varphi_0)(\theta)$, $0 < \tau \leqslant 1$, $\theta \in [-r_i, 0]$.

It is easy to check that $\alpha'_\tau(\tau, \theta) = -(1/\tau^2)W_i(\tau\varphi_0)(\theta)$, $0 < \tau \leqslant 1$, $\theta \in [-r_i, 0]$. By $W(\operatorname{Int} C^+_{r,D}) \subset \operatorname{Int} C^+_{r,D}$, we know that $\alpha'_\tau(\tau, \theta) < 0$, $0 < \tau \leqslant 1, \theta \in [-r_i, 0]$. Thus $\alpha(\tau_0, \theta_0) > \alpha(1, \theta_0) = 0$, contradicting $\alpha(\tau_0, \theta_0) = (1/\tau_0)\psi_i(\theta_0) = 0$.

Case 2: There exists an $i \in N$ such that $D_i(\psi) = 0$.
Let $\beta(\tau) = D_i(\psi^\tau)$, $0 < \tau \leqslant 1$. It is easy to check that

$$\beta'(\tau) = -D_i(W(\tau\varphi_0))/\tau^2.$$

Again by $W(\operatorname{Int} C^+_{r,D}) \subset \operatorname{Int} C^+_{r,D}$, we know that $\beta'(\tau) < 0$, $0 < \tau \leqslant 1$. Thus $\beta(\tau_0) > \beta(1) = D_i(\hat{0}) = 0$, a contradiction to $\beta(\tau_0) = D_i(\psi) = 0$.

Summarizing case 1 and cass 2 completes the proof of the proposition. □

In what follows, we write $x(t, \varphi)$ $(x_t(\varphi))$ for $x(t, 0, \varphi)$ in $R^n(x_t(0, \varphi)$ in $C_r)$, where $x(t, 0, \varphi)$ is the solution of (3.1) through $(0, \varphi)$. We are now in a position to establish sufficient conditions under which the operator W defined by (3.4) maps interior points of the cone $C^+_{r,D}$ into its interior points. To this end, we need the following.

LEMMA 3.7. *For any* $\varphi \in C_r$, *let* $q(t, \varphi) = x(t, \varphi) - D_\varphi x(t, \varphi)\varphi$ (*that is,* $q_t(\varphi) =$

$x_t(\varphi) - D_\varphi x(t,\varphi)\varphi)$. Then $q_t(\varphi)$ satisfies the following equation:

$$\begin{cases} \dfrac{d}{dt}D(q_t(\varphi)) = D_{x_t}F(t,x_t(\varphi))q_t(\varphi) + f(t,x_t(\varphi)) \\ q_0(\varphi) = 0, \end{cases} \tag{3.8}$$

where

$$f(t,x_t(\varphi)) = F(t,x_t(\varphi)) - D_{x_t}F(t,x_t(\varphi))x_t(\varphi). \tag{3.9}$$

Proof. By (3.1), we have

$$D(x_t(\varphi)) = D(\varphi) + \int_0^t F(s,x_s(\varphi))\,ds.$$

Differentiating both sides of the above equation with respect to φ and using the linearity of D, we have

$$D(D_\varphi x_t(\varphi)\psi) = D(\psi) + \int_0^t D_{x_s}F(s,x_s(\varphi))D_\varphi x_s(\varphi)\psi\,ds.$$

In particular, let $\varphi = \psi$. We have

$$D(D_\varphi x_t(\varphi)\varphi) = D(\varphi) + \int_0^t D_{x_s}F(s,x_s(\varphi))D_\varphi x_s(\varphi)\varphi\,ds.$$

By hypothesis, $D_\varphi x_t(\varphi)\varphi = x_t(\varphi) - q_t(\varphi)$, so we have the following formulating procedure:

$$\begin{aligned} D(q_t(\varphi)) &= D(x_t(\varphi)) - D(D_\varphi x_t(\varphi)\varphi) \\ &= \int_0^t F(s,x_s(\varphi))\,ds - \int_0^t D_{x_s}F(s,x_s(\varphi))D_\varphi x_s(\varphi)\varphi\,ds \\ &= \int_0^t F(s,x_s(\varphi)\,ds - \int_0^t D_{x_s}F(s,x_s(\varphi))[x_s(\varphi) - q_s(\varphi)]\,ds \\ &= \int_0^t D_{x_s}F(s,x_s(\varphi))q_s(\varphi)\,ds + \int_0^t f(s,x_s(\varphi))\,ds, \end{aligned}$$

where $f(s,x_s(\varphi))$ is defined by (3.9). Note that $q_0(\varphi) = x_0(\varphi) - D_\varphi(x_0(\varphi))\varphi = 0$, and thus (3.8) is true. □

PROPOSITION 3.8. *Let neutral functional differential equation* (3.1) *be cooperative, and* (I), (T) *hold. If $f(t,\varphi) \gg 0$ for all $\varphi \in \operatorname{Int} C^+_{r,D}$ and $t \in R$, then $q_t(\varphi) \in \operatorname{Int} C^+_{r,D}$ for $\varphi \in \operatorname{Int} C^+_{r,D}$ and $t \geqslant k_0\omega$.*

Proof. Let $\varphi \in \operatorname{Int} C^+_{r,D}$ be arbitrarily fixed. Then we have

$$x(t,\varphi) = \int_0^1 D_\varphi x(t,s\varphi)\varphi\,ds \tag{3.10}$$

since $x(t,\hat{0}) = 0$. If $\xi \in C^+_{r,D}$ and $\beta \in \operatorname{Int} C^+_{r,D}$, let $y(t,\beta) = D_\varphi x(t,\xi)\beta$. Then, by Lemma 2.5, we have

$$\begin{cases} \dfrac{d}{dt}D(y_t) = D_{x_t}F(t,x_t(\xi))y_t \\ y_0 = \beta, \end{cases}$$

and $y_t(\beta) \in \operatorname{Int} C^+_{r,D}$, $t \geqslant 0$. Therefore, by (3.10) and the definition of $y(t,\beta)$, we know that $x_t(\varphi) \gg_D \hat{0}$. Hence, by the hypotheses of the proposition, $f(t, x_t(\varphi)) \gg 0$, $t \geqslant 0$. Since neutral functional differential equation (3.1) is cooperative, we know that

$$\frac{d}{dt} D(q_t(\varphi))|_{t=0} \gg 0$$

from (3.8). As a consequence, there exists $\varepsilon > 0$ such that $D(q_t(\varphi)) \gg 0$, $0 \leqslant t < \varepsilon$. By [**32**, Lemma 2.3], $q(t, \varphi) \gg_D 0$, $0 \leqslant t < \varepsilon$. Now we claim that $q_t(\varphi) \gg_D 0$ for all $t \geqslant 0$. If not, there is a $t_1 > 0$ such that $q_t(\varphi) \gg_D 0$, $0 \leqslant t < t_1$, and $D_i(q_{t_1}(\varphi)) = 0$ for some $i \in N$. It is evident that

$$\frac{d}{dt} D_i(q_t(\varphi))|_{t=t_1} \leqslant 0.$$

However, by continuity and the cooperative property of (3.1), F satisfies (M) and $D_\varphi F(t, \varphi)$ satisfies (P) (see the proof of Lemma 2.5). Therefore, we know that

$$\begin{aligned} \frac{d}{dt} D_i(q_t(\varphi))\bigg|_{t=t_1} &= [D_{x_{t_1}} F(t_1, x_{t_1}(\varphi)) q_{t_1}(\varphi)]_i + f_i(t_1, x_{t_1}(\varphi)) \\ &> [D_{x_{t_1}} F(t_1, x_{t_1}(\varphi)) q_{t_1}(\varphi)]_i \\ &= [D_{x_{t_1}} F(t_1, x_{t_1}(\varphi)) q_{t_1}(\varphi)]_i \\ &\quad - t[D_{x_{t_1}} F(t_1, x_{t_1}(\varphi))\hat{0}]_i \\ &\geqslant l_i(t_1, q_{t_1}(\varphi), \hat{0})[D_i(q_{t_1}(\varphi)) - D_i(\hat{0})] \\ &= l_i(t_1, q_{t_1}(\varphi), \hat{0})(0 - 0) = 0. \end{aligned}$$

This contradiction implies that such a t_1 cannot exist, and it thus establishes the above assertion. The proof is complete. □

Now, we are able to give the main result of the paper.

THEOREM 3.9. *Suppose that the conditions of Proposition* 3.8 *are satisfied. Further, assume that* (H1)$'$ *and* (H2) *are also satisfied and that* (3.1) *has no positive constant solution. Then* (3.1) *has a unique nonconstant positive ω-periodic solution that which attracts each solution $x(t, 0, \varphi)$ of* (3.1) *with $\varphi \in C^+_{r,D}$ and $\varphi \neq 0$.*

Proof. Note that Proposition 3.8 implies that $W(\operatorname{Int} C^+_{r,D}) \subset \operatorname{Int} C^+_{r,D}$. It follows from Proposition 3.4, Proposition 3.6 and Lemma 2.7 that U has no more than one nonzero fixed point on $C^+_{r,D}$. Thus (3.1) has no more than one positive periodic solution with period $k_0\omega$. Combining this with Theorem 3.2, we know that (3.1) has a unique positive ω-periodic solution. Now, the assumption that (3.1) has no positive *constant* solution guarantees that this unique periodic solution is *nonconstant*. Finally, by Remark 3.3, this nonconstant ω-periodic solution attracts each solution $x(t, 0, \varphi)$ of (3.1) with $\varphi \in C^+_{r,D}$ and $\varphi \neq 0$. The proof of Theorem 3.9 is complete. □

4. *An example*

In this section, we give an example to demonstrate the results obtained in Section 3. Consider the following scalar periodic neutral functional differential equation:

$$\frac{d}{dt}[x(t) - gx(t-r)] = a(t)x(t) + b(t)x(t-r) - c(t)[x(t) - gx(t-r)]^\alpha, \tag{4.1}$$

where $0 < g < 1$, $r \geqslant 0$, $\alpha > 1$, and functions $a(t)$, $b(t)$ and $c(t)$ are continuous, positive and ω-periodic.

Note that when $\alpha = 2$, (4.1) reduces to following equation:

$$\frac{d}{dt}[x(t) - gx(t-r)] = x(t)\left\{a(t) - c(t)\left[x(t) - \frac{2g}{c(t)}x(t-r)\right]\right\} \\ + x(t-r)[b(t) - g^2c(t)x(t-r)]. \quad (4.2)$$

In the case where $g = 0$, equation (4.2) models one-species growth in a ω-periodic environment with delayed recruitment, the spread of epidemics, and the dynamics of capital stocks [**4**, **5**, **16**]. In (4.2) with $0 < g < 1$, the additional term $gx(t-r)$ on the left-hand side can be viewed as a certain feedback control mechanism that adjusts the change in the system according to its past growth rate, or it can be regarded as the relapse of the infectious disease considered in the Cooke–Kaplan model of epidemics. The second term $x(t-r)[b(t) - g^2c(t)x(t-r)]$ on the right-hand side of (4.2) can be justified as nonlinear delayed recruitment or nonlinear delayed feedback control [**16**].

Let $C_r = C([-r,0],R)$. For any $\varphi \in C_r$, operator $D : C_r \longrightarrow R$ is defined by

$$D\varphi = \varphi(0) - g\varphi(-r). \quad (4.3)$$

Then $D : C_r \longrightarrow R$ is quasimonotone and stable since $g \in (0,1)$. $F(t,\cdot) : C_r \longrightarrow C_r$ is defined by

$$F(t,\varphi) = a(t)\varphi(0) + b(t)\varphi(-r) - c(t)(D\varphi)^\alpha \quad \text{for any } \varphi \in C_r,\ t \in R^+. \quad (4.4)$$

Then F maps bounded sets of $R^+ \times C_r$ into bounded sets of R, and hence (H2) holds. Obviously (T) is naturally satisfied since equation (4.1) is scalar.

In the remainder of this section, by verifying those conditions in Theorem 3.9, we show that (4.1) admits a unique nonconstant positive ω-periodic solution that attracts each solution $x(t,0,\varphi)$ of (4.1) with $\varphi \in \operatorname{Int} C^+_{r,D}$ and $\varphi \neq 0$, that is, this periodic solution is globally attractive in $C^+_{r,D}\backslash\{0\}$.

LEMMA 4.1. *Conditions* (I) *and* (H1)$'$ *hold for equation* (4.1).

Proof. Let $\varphi \leqslant_D \psi$ and $\varphi(-r) < \psi(-r)$ for any $\varphi, \psi \in C_r$. Then either $D\varphi < D\psi$ or $D\varphi = D\psi$ holds. If $D\varphi = D\psi$, then $\psi(0) - \psi(0) = g[\psi(-r) - \varphi(-r)]$. Hence we have

$$\begin{aligned} F(t,\psi) - F(t,\varphi) &= a(t)[\psi(0) - \varphi(0)] + b(t)[\psi(-r) - \varphi(-r)] - c(t)[(D\psi)^\alpha - (D\varphi)^\alpha] \\ &= [b(t) + ga(t)][\psi(-r) - \varphi(-r)] > 0 \quad \text{for all } t \in R^+, \end{aligned}$$

that is, (I) holds.

For any $\xi \in R$, we have

$$F(t,\hat{\xi}) = [b(t) + a(t)]\xi - c(t)(1-g)^\alpha\xi^\alpha \quad \text{for all } t \in R^+. \quad (4.5)$$

We know from (4.5) that $F(t,\hat{\xi}) \geqslant 0$ for sufficiently small $\xi > 0$ and $F(t,\hat{\xi}) \leqslant 0$ for sufficiently large $\xi > 0$, that is, (H1)$'$ holds. □

LEMMA 4.2. *Equation* (4.1) *is cooperative.*

Proof. For any $\varphi, h \in C_r$, $t \in R^+$, we have

$$\begin{aligned} D_\varphi F(t,\varphi)h &= a(t)h(0) + b(t)h(-r) - \alpha c(t)(D\varphi)^{\alpha-1}D_\varphi(D\varphi)h \\ &= a(t)h(0) + b(t)h(-r) - \alpha c(t)(D\varphi)^{\alpha-1}Dh \\ &= [a(t) - \alpha c(t)(D\varphi)^{\alpha-1}]h(0) + [b(t) + \alpha g c(t)(D\varphi)^{\alpha-1}]h(-r). \end{aligned}$$

Thus we have

$$B(\varphi, t) = a(t) - \alpha c(t)(D\varphi)^{\alpha-1}$$
$$K(\varphi, t) = b(t) + \alpha g c(t)(D\varphi)^{\alpha-1}.$$

Then, for any $h \in C_r^+$, we have

$$\begin{aligned}(K(\varphi, t) - B(\varphi, t)A^{-1}L)h \\ &= [b(t) + \alpha g c(t)(D\varphi)^{\alpha-1}]h(-r) - [-a(t) + \alpha c(t)(D\varphi)^{\alpha-1}]gh(-r) \\ &= [b(t) + ga(t)]h(-r) \geqslant 0, \qquad t \in R,\end{aligned}$$

that is, $[K(\varphi, t) - B(\varphi, t)A^{-1}L]C_r^+ \subset R^+$. □

LEMMA 4.3. $f(t, \varphi) = F(t, \varphi) - D_\varphi F(t, \varphi)\varphi > 0$ *for any* $\varphi \in \operatorname{Int} C_{r,D}^+$ *and* $t \in R^+$.

Proof. For any $\varphi \in \operatorname{Int} C_{r,D}^+$ and $t \in R^+$, we have

$$\begin{aligned}f(t, \varphi) &= F(t, \varphi) - D_\varphi F(t, \varphi)\varphi \\ &= a(t)\varphi(0) + b(t)\varphi(-r) - c(t)(D\varphi)^\alpha \\ &\quad -[a(t)\varphi(0) + b(t)\varphi(-r) - \alpha c(t)(D\varphi)^{\alpha-1}D\varphi] \\ &= (\alpha - 1)c(t)(D\varphi)^\alpha > 0,\end{aligned}$$

where the inequality follows from $D\varphi > 0$ since $\varphi \in \operatorname{Int} C_{r,D}^+$. □

Now, by Lemmas 4.1–4.3 and Theorem 3.9, we obtain the following.

PROPOSITION 4.4. *Equation* (4.1) *admits a unique positive ω-periodic solution that is globally attractive in* $C_{r,D}^+\backslash\{0\}$. *This periodic solution is a positive constant solution if $a(t)+b(t)$ is proportional to $c(t)$, and it is nonconstant if $a(t)+b(t)$ is not proportional to $c(t)$.*

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