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Traveling Wavefronts in Lattice Differential Equations with Time Delay and Global Interaction

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Abstract. In this paper, we study the existence of traveling wave solutions in lattice differential equations with time delay and global interaction

$$\begin{aligned} u_n'(t) &= D\sum_{i\in Z^q\setminus\{0\}} J(i)[u_{n-i}(t)-u_n(t)] \\ &\quad +F\left(u_n(t),\sum_{i\in Z^q}K(i)\int_{-r}^{0}d\eta(\theta)g(u_{n-i}(t+\theta))\right). \end{aligned}$$

Following an idea in [10], we are able to relate the the existence of traveling wavefront to the existence of heteroclinic connecting orbits of the corresponding ordinary delay differential equations

$$u'(t)=F\left(u(t),\int_{-r}^{0}d\eta(\theta)g(u(t+\theta))\right).$$

1 Introduction

In a recent work [10], Faria et al. considered the existence of traveling wavefront for the following general class of delayed reaction-diffusion systems with non-local

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interaction:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\left(u(x,t), \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y)g(u(x+y,t+\theta))\right) \quad (1.1)$$

where $x \in R^m$ is the spatial variable, $t \geq 0$ is the time variable, $u(x,t) \in R^n$ is the unknown vector function, and $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ with $d_i > 0,\ i = 1, 2, \cdots, n$, $\Delta = \sum_{i=1}^{m} \partial^2/\partial x_i^2$ is the the Laplacian operator. They treated the wave profile equation for (1.1) as a perturbation of the following corresponding ordinary delay differential equation

$$u'(t) = F\left(u(t), \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(u(t+\theta))\right) \quad (1.2)$$

where $\mu_\Omega = \int_\Omega d\mu$. Then by choosing some appropriate Banach space and applying the perturbation theory to the associated Fredholm operator with some careful estimation of the nonlinear perturbation, the authors were able to relate the existence of traveling wave solution of (1.1) to the existence of heteroclinic connecting orbits of (1.2).

In this paper, we apply the novel approach used in [10] to tackle the existence of traveling wavefront for a very general class of lattice differential equations with time delay and global interaction:

$$\begin{aligned} u_n'(t) &= D\textstyle\sum_{i\in Z^q\setminus\{0\}} J(i)[u_{n-i}(t) - u_n(t)] \\ &\quad + F\left(u_n(t), \textstyle\sum_{i\in Z^q} K(i)\int_{-r}^{0} d\eta(\theta)g(u_{n-i}(t+\theta))\right), \end{aligned} \quad (1.3)$$

where $n \in Z^q$, q is a positive integer, $t \geq 0$, $u_n(t) \in R^N$, $D = \mathrm{diag}(d_1, d_2, \cdots, d_N)$ with $d_j \geq 0, j = 1, \cdots, N$, $r \geq 0$ and $\eta : [-r, 0] \to R^{N\times N}$ is of bounded variation, $F : R^N \times R^N \to R^N$ and $g : R^N \to R^N$ are C^k-smooth functions, $k \geq 2$. Here, the first term in (1.3) accounts for diffusion to point n in the the lattice from all other points, while the second term explains global nonlinear interactions. System (1.3) includes, as special cases, many model systems arising from various fields among which is population biology where the mobility of the immature individuals is responsible for the non-locality of the interaction term (see, e.g., [22] and the references therein). In such a context, the choice of spatially discrete domain correspond to a patch environment in which the species lives. Due to the biological background, traveling wave solutions to such equations are an important type of solutions since they explain spatial spread/invasion of the species within the lattice (patch) environment. In recent years, this topic has attracted the attention of the mathematical community and has resulted in many research papers; see, e.g., [11, 12, 13, 14, 15, 22, 24] and the reference therein.

We point out that as far as traveling waves are concerned, a system of lattice differential equations may demonstrate essentially different behavior from that of its continuous version (reaction-diffusion system). For example, pinning phenomenon may occur in a lattice differential system, while this phenomenon would be impossible in its spatially continuous version (a reaction diffusion equation); see, e.g., [11, 13]. Another example is that the direction of the waves play a role in the existence of traveling wavefront for a system on a lattice with a higher dimension, but in the case of continuous reaction diffusion equation with a higher spatial dimension, the direction has no such impact; see, e.g., [3, 16, 26]. Therefore, one can not expect that a method that works for (1.2) would automatically work for (1.1).

This motivates us to see if the ideas used in [10] for (1.1) could be applied to (1.3) for the existence of traveling wavefront. It turns out that after some non-trivial and careful explorations on properties of some operators resulted from the wave equation for (1.1) and the associated ordinary functional differential equation

$$u'(t) = F\left(u(t), \int_{-r}^{0} d\eta(\theta) g(u(t+\theta))\right), \tag{1.4}$$

we can also establish a similar result to that in [10], that is, relating the existence of traveling wavefront of (1.3) to the existence of heteroclinic connecting orbits of (1.4).

To proceed further, and also from the practical background of (1.3), we assume throughout the paper that the kernel functions J and K satisfy

$$\sum_{i\in Z^q\setminus\{0\}} J(i) = 1, \qquad \sum_{i\in Z^q\setminus\{0\}} |J(i)|\cdot|i| < +\infty,$$

and

$$\sum_{i\in Z^q} K(i) = 1, \qquad \sum_{i\in Z^q} |K(i)|\cdot|i| < +\infty,$$

where $|i| = \sum_{j=1}^{q} |i_j|$ for $i = (i_1, \cdots, i_q) \in Z^q$.

Let $F_u(u,v)$ and $F_v(u,v)$ denote the partial derivatives of F with respect to the variables $u \in R^N$ and $v \in R^N$, respectively, and let $g_u(u)$ be the derivative of g with respect to the variable $u \in R^N$. Suppose that Eq. (1.4) has two equilibria $E_j, j = 1,2$ (i.e., $F(E_i, E_i) = 0$, $i = 1,2$), and let

$$A_j = F_u\left(E_j, \int_{-r}^{0} d\eta(\theta) g(E_j)\right), \quad B_j = F_v\left(E_j, \int_{-r}^{0} d\eta(\theta) g(E_j)\right).$$

For a complex number λ, denote

$$\Lambda_j(\lambda) = \det\left[\lambda I - A_j - B_j \int_{-r}^{0} d\eta(\theta) g_u(E_j) e^{\lambda\theta}\right].$$

We assume that the following hypotheses hold:

(H1) E_1 is hyperbolic and the unstable manifold at the equilibrium E_1 is M-dimensional with $M \geq 1$. In other words, $\Lambda_1(i\beta) \neq 0$ for all $\beta \in R$ and $\Lambda_1(\lambda) = 0$ has exactly M roots with positive real parts, where the multiplicities are taken into account.

(H2) All eigenvalues corresponding to the equilibrium E_2 have negative real parts, that is, $\sup\{\Re\lambda : \Lambda_2(\lambda) = 0\} < 0$.

(H3) Eq. (1.2) has a heteroclinic solution $u_* : R \to R^N$ from E_1 to E_2. Namely, Eq. (1.2) has a solution $u_*(t)$ defined for all $t \in R$ such that

$$u_*(-\infty) := \lim_{t\to-\infty} u_*(t) = E_1, \quad u_*(+\infty) := \lim_{t\to+\infty} u_*(t) = E_2.$$

As usual, a traveling wave solution of (1.3) is a solution of the form $u_n(t) = U(\nu\cdot n + ct)$, where $U(\cdot)$ is called the profile of the wave and c is the wave speed. If U satisfy

$$U(-\infty) = E_1, \qquad U(\infty) = E_2, \tag{1.5}$$

then the traveling wave solutions is called a wavefront.

We can now formulate our main result as follows, which states that the existence of traveling wave solutions with large wave speeds for Eq. (1.1) is related to the existence of heteroclinic orbit of Eq. (1.2) connecting the two equilibria.

Theorem 1.1. *Assume that (H1), (H2) and (H3) hold. Then there exists $c^* > 0$ such that*

(i) for each fixed unit vector $\nu \in \mathrm{R}^q$ and $c > c^*$, Eq. (1.1) has a traveling wavefront $u_n(t) = U(\nu \cdot n + ct)$ connecting E_1 to E_2 (that is, (1.5) holds);

(ii) if restricted to a small neighborhood of the heteroclinic solution u_* in the space $BC(\mathrm{R}, \mathrm{R}^N)$ of bounded continuous functions equipped with the sup-norm, then for each $c > c^*$ and $\nu \in \mathrm{R}^q$, the set of all traveling wave solutions connecting E_1 to E_2 in the neighborhood forms a M-dimensional manifold $\mathsf{M}_\nu(c)$;

(iii) $\mathsf{M}_\nu(c)$ is a C^{k-1}-smooth manifold which is also C^{k-1}-smooth with respect to c. More precisely, there is a C^{k-1}-function $h : O \times (c^*, +\infty) \to C(\mathrm{R}, \mathrm{R}^N)$, where O is an open set in R^N, such that $\mathsf{M}_\nu(c)$ has the form

$$\mathsf{M}_\nu(c) = \{\psi : \ \psi = h(z, c), \quad z \in O\}.$$

Let $\nu \cdot n + ct = s \in R$ and $u_n(t) = U(\nu \cdot n + ct)$. Then, by straightforward substitution, one find that the profile function $U(s)$ satisfies the following associated wave equation

$$\begin{aligned} cU'(s) \quad &= \quad D\sum_{i\neq 0} J(i)[U(s-\nu\cdot i) - U(s)] \\ &\quad +F\left(U(s), \sum_i K(i)\int_{-r}^{0} d\eta(\theta) g(U(s-\nu\cdot i + c\theta))\right). \end{aligned} \tag{1.6}$$

Let $V(s) = U(cs)$ and $\epsilon = 1/c$, then (1.3) leads to

$$\begin{aligned} V'(s) \quad &= \quad D\sum_{i\neq 0} J(i)[V(s-\epsilon\nu\cdot i) - V(s)] \\ &\quad +F\left(V(s), \sum_i K(i)\int_{-r}^{0} d\eta(\theta) g(V(s-\epsilon\nu\cdot i + \theta))\right). \end{aligned} \tag{1.7}$$

Thus, existence of traveling wavefront solutions to (1.3) is equivalent to existence of solutions to (1.6) or (1.7) with the asymptotically boundary conditions (1.5). In Section 2, we will further transform (1.7) into some operational equation, and in Section 3, we will explore the properties of the operators in the equations obtained in Section 2. After the preparation in Sections 2-3, we give the proof of Theorem 1 in Section 4. Section 5 is devoted to applications of the main theorem to some cases where the heteroclinic orbits of the corresponding ODE equation (1.4) can be guaranteed by the connecting orbit theorem for monotone dynamical systems.

2 Operational equations for profile of traveling waves

We denote by $C = C(R, R^N)$ the space of continuous and bounded functions from R to R^N equipped with the standard sup norm: $\|\psi\|_C = \sup\{|\psi(t)| : \ t \in R\}$, where $|\cdot|$ is the Euclid norm in R^N.

Using the idea in [10], we relate (1.4) to an equivalent operational equation in a suitable Banach space. For this purpose, we further transform Eq. (1.4) by introducing the variable $w(s) = V(s) - u_*(s)$ for $s \in R$. Then we obtain the equation for w as follows

$$\begin{aligned}
w'(s) \quad &= V'(s) - u_*'(s) \\
&= D\sum_{i\neq 0} J(i)[V(s-\epsilon\nu\cdot i) - V(s)] \\
&\quad + F(V(s), \sum_i K(i)\int_{-r}^0 d\eta(\theta) g(V(s-\epsilon\nu\cdot i+\theta))) \\
&\quad - F(u_*(s), \int_{-r}^0 d\eta(\theta) g(u_*(s+\theta))) \\
&= D\sum_{i\neq 0} J(i)[[w+u_*](s-\epsilon\nu\cdot i) - [w+u_*](s)] \\
&\quad + F([w+u_*](s), \sum_i K(i)\int_{-r}^0 d\eta(\theta) g([w+u_*](s-\epsilon\nu\cdot i+\theta))) \\
&\quad - F(u_*(s), \int_{-r}^0 d\eta(\theta) g(u_*(s+\theta))) \\
&= L_0 w(s) + J_\epsilon w(s) + G(\epsilon, s, w) + J_\epsilon u_*(s),
\end{aligned} \tag{2.1}$$

where $[w+u_*](s) = w(s) + u_*(s)$ for $s\in R$, and the linear operators $L_0 : C \to C$ and $J_\epsilon : C\to C$ are defined by

$$L_0\psi(s) = A(s)\psi(s) + B(s)\int_{-r}^0 d\eta(\theta) g_u(u_*(s+\theta))\psi(s+\theta), \tag{2.2}$$

with

$$A(s) = F_u(u_*(s), \int_{-r}^0 d\eta(\theta) g(u_*(s+\theta))), \tag{2.3}$$

$$B(s) = F_v(u_*(s), \int_{-r}^0 d\eta(\theta) g(u_*(s+\theta))), \tag{2.4}$$

and

$$J_\epsilon\psi(s) = D\sum_{i\neq 0} J(i)[\psi(s-\epsilon\nu\cdot i) - \psi(s)], \tag{2.5}$$

respectively, and the nonlinear operator $G(\epsilon,\cdot,\cdot) : C\to C$ is defined by

$$\begin{aligned}
G(\epsilon, s, \psi) \quad = \quad & F(\psi(s) + u_*(s), \sum_i K(i)\int_{-r}^0 d\eta(\theta) g([\psi+u_*](s-\epsilon\nu\cdot i+\theta))) \\
& - F(u_*(s), \int_{-r}^0 d\eta(\theta) g(u_*(s+\theta))) - L_0\psi(s).
\end{aligned} \tag{2.6}$$

Next we further transform Eq. (2.1) into an integral equation as follows. We first rewrite (2.1) as

$$w'(s) + w(s) = w(s) + L_0 w(s) + J_\epsilon w(s) + G(\epsilon, s, w) + J_\epsilon u_*(s). \tag{2.7}$$

Clearly, $w : R\to R^N$ is a bounded solution of (2.7) if and only if it is a bounded solution of the integral equation

$$w(s) = \int_{-\infty}^s e^{-(s-t)}\{[Id + L_0]w(t) + J_\epsilon w(t) + G(\epsilon, t, w) + J_\epsilon u_*(t)\}dt.$$

Therefore, w is a bounded solution of (2.7) if and only if it solves the operational equation

$$\mathcal{L}w = \mathcal{G}(\cdot, w, \epsilon), \tag{2.8}$$

where the linear operator $\mathcal{L} : C\to C$ is defined by

$$\mathcal{L}w(s) = w(s) - \int_{-\infty}^s e^{-(s-t)}[Id + L_0]w(t)dt, \tag{2.9}$$

and the nonlinear operator $\mathcal{G}(\cdot,\cdot,\epsilon): C \to C$ is defined by

$$\mathcal{G}(s,w,\epsilon) = \int_{-\infty}^{s} e^{-(s-t)}[J_\epsilon w(t) + G(\epsilon,t,w) + J_\epsilon u_*(t)]dt. \tag{2.10}$$

3 Properties of the operators $\mathcal{L}$ and $\mathcal{G}$

Let $C^1(R,R^N) = \{\psi \in C: \ \psi' \in C\}$ be the Banach space equipped with the standard norm $\|\psi\|_{C^1} = \|\psi\|_C + \|\psi'\|_C$. Let $C_0 = \{\psi \in C: \ \lim_{t\to\pm\infty}\psi(t) = 0\}$ and $C_0^1 = \{\psi \in C_0: \ \psi' \in C_0\}$ equipped with the same norms as C and C^1, respectively.

If restricted to the subspace C_0, we then have $\mathcal{L}: C_0 \to C_0$. Let $\mathcal{N}(\mathcal{L})$ and $\mathcal{R}(\mathcal{L})$ denote the kernel and the range of the operator $\mathcal{L}$, then we have the following result.

Proposition 3.1. $\dim\mathcal{N}(\mathcal{L}) = M$ *and* $\mathcal{R}(\mathcal{L}) = C_0$.

For a proof of the Proposition 3.1, we refer the reader to the recent paper due to Faria et al. [10].

In order to complete the proof of Theorem 1.1, we need further information about the behavior of the nonlinear operator $\mathcal{G}(\cdot,w,\epsilon)$ when $\epsilon \geq 0$ is small and w is near the origin. To simplify the presentation, for any $\epsilon \geq 0$, we let $R(\epsilon,\cdot): C \to C$ be defined by

$$R(\epsilon,\psi)(s) = \sum_i K(i)\int_{-r}^{0} d\eta(\theta)g(\psi(s-\epsilon\nu\cdot i+\theta)), \tag{3.1}$$

and let the linear operator $L_\epsilon: C_0 \to C$ be defined by

$$L_\epsilon\psi(s) = A^\epsilon(s)\psi(s) + B^\epsilon(s)\sum_i K(i)\int_{-r}^{0} d\eta(\theta)g_u(u_*(s-\epsilon\nu\cdot i+\theta))\psi(s-\epsilon\nu\cdot i+\theta), \tag{3.2}$$

with

$$A^\epsilon(s) = F_u(u_*(s), R(\epsilon,u_*)(s)), \tag{3.3}$$

$$B^\epsilon(s) = F_v(u_*(s), R(\epsilon,u_*)(s)). \tag{3.4}$$

Then

$$\begin{aligned} G(\epsilon,s,\psi) &= F(\psi(s)+u_*(s), R(\epsilon,\psi+u_*)(s)) - F(u_*(s),R(0,u_*)(s)) - L_0\psi(s) \\ &= [L_\epsilon - L_0]\psi(s) + G^1(\epsilon,s,\psi) + G^2(\epsilon,s), \end{aligned} \tag{3.5}$$

where

$$G^1(\epsilon,s,\psi) = F(\psi(s)+u_*(s), R(\epsilon,\psi+u_*)(s)) - F(u_*(s),R(\epsilon,u_*)(s)) - L_\epsilon\psi(s), \tag{3.6}$$

and

$$G^2(\epsilon,s) = F(u_*(s),R(\epsilon,u_*)(s)) - F(u_*(s),R(0,u_*)(s)). \tag{3.7}$$

Therefore, we can express $\mathcal{G}$ as

$$\mathcal{G}(s,\psi,\epsilon) = \mathcal{J}_\epsilon\psi(s) + \mathcal{L}_\epsilon\psi(s) + \mathcal{G}^1(s,\psi,\epsilon) + \mathcal{G}^2(s,\epsilon) + \mathcal{J}_\epsilon u_*(s), \tag{3.8}$$

where

$$\mathcal{J}_\epsilon\psi(s) = \int_{-\infty}^{s} e^{-(s-t)}J_\epsilon\psi(t)dt,$$

$$\mathcal{L}_\epsilon\psi(s) = \int_{-\infty}^{s} e^{-(s-t)}[L_\epsilon - L_0]\psi(t)dt,$$

$$\mathcal{G}^1(s,\psi,\epsilon)=\int_{-\infty}^{s}e^{-(s-t)}G^1(\epsilon,t,\psi)dt,$$

and

$$\mathcal{G}^2(s,\epsilon)=\int_{-\infty}^{s}e^{-(s-t)}G^2(\epsilon,t)dt.$$

Lemma 3.2. *Let $\{f_j(x)\}$, $j\in \mathrm{Z}^q$, $x\in \mathrm{R}$, be a sequence of functions such that $\sum_j f_j(x)$ exits for any $x\in\mathrm{R}$ and $f_j(x)\to\bar{f}_j$ as $x\to x_0\in\{\mathrm{R},-\infty,+\infty\}$ for all $j\in\mathrm{Z}^q$. If there exists a summable sequence $\{g_j\}$ such that $|f_j(x)|\le g_j$ for all $j\in\mathrm{Z}^q$ and $x\in\mathrm{R}$, then*

$$\sum_j f_j(x)\to\sum_j\bar{f}_j,\qquad \text{as}\quad x\to x_0.$$

The proof of Lemma 3.2 is similar to that of the Lebesgue' dominated convergence theorem and is omitted.

Proposition 3.3. *For each $\epsilon\ge 0$ and $\psi\in C_0$, $\mathcal{G}(\cdot,\psi,\epsilon)\in C_0$. In other words, $\mathcal{G}(\cdot,C_0,\epsilon)\subseteq C_0$ for each $\epsilon\ge 0$.*

Proof At first, we note that for each $\epsilon\ge 0$ and each $\varphi\in C$, if $\lim_{s\to\pm\infty}\varphi(s)=\varphi(\pm\infty)$ exist, then it follows from Lemma 3.2 that

$$\lim_{s\to\pm\infty}J_\epsilon\varphi(s)=\lim_{s\to\pm\infty}D\sum_{i\neq 0}J(i)[\varphi(s-\epsilon\nu\cdot i)-\varphi(s)]=0.$$

Therefore, we have

$$\lim_{s\to\pm\infty}\mathcal{J}_\epsilon u_*(s)=\lim_{s\to\pm\infty}J_\epsilon u_*(s)=0,\tag{3.9}$$

and

$$\lim_{s\to\pm\infty}\mathcal{J}_\epsilon\psi(s)=\lim_{s\to\pm\infty}J_\epsilon\psi(s)=0,\tag{3.10}$$

for all $\epsilon\ge 0$ and $\psi\in C_0$.

In a similar way, let $\varphi\in C$ be such that $\lim_{s\to\pm\infty}\varphi(s)=\varphi(\pm\infty)$ exist, then it follows from Lemma 3.2 that

$$\lim_{s\to\pm\infty}R(\epsilon,\varphi)(s)=\lim_{s\to\pm\infty}\sum_i K(i)\int_{-r}^{0}d\eta(\theta)g(\varphi(s-\epsilon\nu\cdot i+\theta))=\int_{-r}^{0}d\eta(\theta)g(\varphi(\pm\infty)),\tag{3.11}$$

for all $\epsilon\ge 0$. Therefore, we have

$$\lim_{s\to-\infty}A^\epsilon(s)=F_u(E_1,\int_{-r}^{0}d\eta(\theta)g(E_1)),\quad \lim_{s\to+\infty}A^\epsilon(s)=F_u(E_2,\int_{-r}^{0}d\eta(\theta)g(E_2)),$$

and

$$\lim_{s\to-\infty}B^\epsilon(s)=F_v(E_1,\int_{-r}^{0}d\eta(\theta)g(E_1)),\quad \lim_{s\to+\infty}B^\epsilon(s)=F_v(E_2,\int_{-r}^{0}d\eta(\theta)g(E_2)).$$

Hence, it follows from Lemma 3.2 that for each $\epsilon\ge 0$ and $\psi\in C_0$,

$$\lim_{s\to\pm\infty}L_\epsilon\psi(s)=0,\tag{3.12}$$

and hence

$$\begin{aligned}&\lim_{s\to\pm\infty}G^1(\epsilon,s,\psi)\\&=\lim_{s\to\pm\infty}[F(\psi(s)+u_*(s),R(\epsilon,\psi+u_*)(s))-F(u_*(s),R(\epsilon,u_*)(s))]\\&\quad-\lim_{s\to\pm\infty}L_\epsilon\psi(s)\\&=0.\end{aligned}\tag{3.13}$$

Notice that for each $\epsilon \geq 0$,

$$\lim_{s\to\pm\infty} G^2(\epsilon, s) = \lim_{s\to\pm\infty}[F(u_*(s), R(\epsilon, u_*)(s)) - F(u_*(s), R(0, u_*)(s))] = 0, \quad (3.14)$$

it follows from (3.9)-(3.10) and (3.12)-(3.14) that for each $\epsilon \geq 0$ and $\psi \in C_0$,

$$\begin{aligned}
&\lim_{s\to\pm\infty} \mathcal{G}(s, \psi, \epsilon) \\
&= \lim_{s\to\pm\infty} \mathcal{J}_\epsilon\psi(s) + \lim_{s\to\pm\infty} \mathcal{L}_\epsilon\psi(s) \\
&\quad + \lim_{s\to\pm\infty} \mathcal{G}^1(s, \psi, \epsilon) + \lim_{s\to\pm\infty} \mathcal{G}^2(s, \epsilon) + \lim_{s\to\pm\infty} \mathcal{J}_\epsilon u_*(s) \\
&= \lim_{s\to\pm\infty} J_\epsilon\psi(s) + \lim_{s\to\pm\infty}[L_\epsilon - L_0]\psi(s) \\
&\quad + \lim_{s\to\pm\infty} G^1(\epsilon, s, \psi) + \lim_{s\to\pm\infty} G^2(\epsilon, s) + \lim_{s\to\pm\infty} J_\epsilon u_*(s) \\
&= 0.
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *For each $\epsilon \geq 0$, $\|\mathcal{J}_\epsilon\psi\|_{C_0} \leq 2\epsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|\psi\|_{C_0}$ for $\psi \in C_0$ and $\|\mathcal{J}_\epsilon u_*\|_{C_0} \leq 2\epsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|u_*\|_C$.*

Proof If $\psi \in C_0^1$, by exchanging the order of integration and integration by parts, we get

$$\begin{aligned}
\mathcal{J}_\epsilon\psi(s) &= \int_{-\infty}^s e^{-(s-t)} J_\epsilon\psi(t)dt \\
&= \int_{-\infty}^s e^{-(s-t)}\{D\sum_{i\neq 0} J(i)[\psi(t-\epsilon\nu\cdot i) - \psi(t)]\}dt \\
&= -\int_{-\infty}^s e^{-(s-t)}\{D\sum_{i\neq 0} J(i)\int_0^1 \psi'(t-\tau\epsilon\nu\cdot i)\epsilon\nu\cdot i d\tau\}dt \\
&= -\epsilon D\sum_{i\neq 0} J(i)\nu\cdot i\int_0^1\int_{-\infty}^s e^{-(s-t)}\psi'(t-\tau\epsilon\nu\cdot i)dtd\tau \\
&= -\epsilon D\sum_{i\neq 0} J(i)\nu\cdot i\int_0^1[\psi(s-\tau\epsilon\nu\cdot i) - \int_{-\infty}^s e^{-(s-t)}\psi(t-\tau\epsilon\nu\cdot i)dt]d\tau,
\end{aligned}$$

which yields

$$\|\mathcal{J}_\epsilon\psi\|_{C_0} \leq 2\epsilon\|D\|\sum_{i\neq 0}|J(i)|\cdot|i|\cdot\|\psi\|_{C_0}. \quad (3.15)$$

Since $\mathcal{J}_\epsilon : C_0 \to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the last inequality holds for all $\psi \in C_0$. This completes the proof. □

Proposition 3.5. *There exists $M_0 > 0$ such that for all $\epsilon \geq 0$ and $\psi \in C_0$,*

$$\|\mathcal{L}_\epsilon\psi\|_{C_0} \leq \epsilon M_0\|\psi\|_{C_0}.$$

Proof We first note that

$$\begin{aligned}
&[L_\epsilon - L_0]\psi(s) \\
&= [A^\epsilon(s) - A(s)]\psi(s) \\
&\quad + [B^\epsilon(s) - B(s)]\sum_i K(i)\int_{-r}^0 d\eta(\theta)g_u(u_*(s-\epsilon\nu\cdot i+\theta))\psi(s-\epsilon\nu\cdot i+\theta) \\
&\quad + B(s)\sum_i K(i)\int_{-r}^0 d\eta(\theta)[g_u(u_*(s-\epsilon\nu\cdot i+\theta)) - g_u(u_*(s+\theta))] \\
&\qquad \times \psi(s-\epsilon\nu\cdot i+\theta) \\
&\quad + B(s)\sum_i K(i)\int_{-r}^0 d\eta(\theta)g_u(u_*(s+\theta))[\psi(s-\epsilon\nu\cdot i+\theta) - \psi(s+\theta)],
\end{aligned}$$

where

$$\begin{aligned}
&A^\epsilon(s) - A(s) \\
&= F_u(u_*(s), R(\epsilon, u_*)(s)) - F_u(u_*(s), R(0, u_*)(s)) \\
&= \int_0^1 F_{uv}(u_*(s), R(0, u_*)(s) \\
&\quad + \tau[R(\epsilon, u_*)(s) - R(0, u_*)(s)])d\tau\cdot[R(\epsilon, u_*)(s) - R(0, u_*)(s)],
\end{aligned}$$

and

$$\begin{aligned}
&B^\epsilon(s) - B(s) \\
&= F_v(u_*(s), R(\epsilon, u_*)(s)) - F_v(u_*(s), R(0, u_*)(s)) \\
&= \int_0^1 F_{vv}(u_*(s), R(0, u_*)(s) \\
&\quad + \tau[R(\epsilon, u_*)(s) - R(0, u_*)(s)])d\tau \cdot [R(\epsilon, u_*)(s) - R(0, u_*)(s)].
\end{aligned}$$

Since

$$\begin{aligned}
&R(\epsilon, \psi)(s) - R(0, \psi)(s) \\
&= \sum_i K(i) \int_{-r}^0 d\eta(\theta)[g(\psi(s - \epsilon\nu \cdot i + \theta)) - g(\psi(s + \theta))] \\
&= -\sum_i K(i) \int_{-r}^0 d\eta(\theta) \int_0^1 g_u(\psi(s - \tau\epsilon\nu \cdot i + \theta))\psi'(s - \tau\epsilon\nu \cdot i + \theta)\epsilon\nu \cdot i d\tau,
\end{aligned}$$

we have

$$\|R(\epsilon, u_*) - R(0, u_*)\|_{C_0} \leq \epsilon \sum_i |K(i)| \cdot |i| \|\eta\| \|g_u\| \|u_*'\|_{C_0}, \tag{3.16}$$

where $\|\eta\| = \bigvee_{[-r,0]} \eta$ and $\|g_u\| = \max\{\|g_u(u_*(s))\| : \ s \in R\}$.

Since F is C^2-smooth, there exist $\epsilon_0 > 0$ and $M > 0$ such that for all $s \in R$ and $\epsilon \in [0, \epsilon_0]$,

$$\left\| \int_0^1 F_{uv}(u_*(s), R(0, u_*)(s) + \tau[R(\epsilon, u_*)(s) - R(0, u_*)(s)])d\tau \right\| \leq M, \tag{3.17}$$

and

$$\left\| \int_0^1 F_{vv}(u_*(s), R(0, u_*)(s) + \tau[R(\epsilon, u_*)(s) - R(0, u_*)(s)])d\tau \right\| \leq M. \tag{3.18}$$

Therefore, we have

$$\|A^\epsilon(s) - A(s)\| \leq \epsilon M \sum_i |K(i)| \cdot |i| \|\eta\| \|g_u\| \|u_*'\|_{C_0},$$

and

$$\|B^\epsilon(s) - B(s)\| \leq \epsilon M \sum_i |K(i)| \cdot |i| \|\eta\| \|g_u\| \|u_*'\|_{C_0}.$$

Therefore, it follows that for all $s \in R$ and $\psi \in C_0$,

$$\left\| \int_{-\infty}^s e^{-(s-t)}[A^\epsilon(t) - A(t)]\psi(t)dt \right\| \leq \epsilon M_1 \|\psi\|_{C_0}, \tag{3.19}$$

and

$$\begin{aligned}
&\left\| \int_{-\infty}^s e^{-(s-t)}[B^\epsilon(t) - B(t)] \sum_i K(i) \int_{-r}^0 d\eta(\theta) g_u(u_*(t - \epsilon\nu \cdot i + \theta)) \right. \\
&\quad \left. \times \psi(t - \epsilon\nu \cdot i + \theta)dt \right\| \leq \epsilon M_2 \|\psi\|_{C_0},
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
M_1 &= M\|\eta\| \|g_u\| \|u_*'\|_{C_0} \sum_i |K(i)| \cdot |i|, \\
M_2 &= M\|\eta\|^2 \|g_u\|^2 \|u_*'\|_{C_0} \sum_i |K(i)||i| \cdot \sum_i |K(i)|.
\end{aligned}$$

Since

$$g_u(u_*(s - \epsilon\nu \cdot i)) - g_u(u_*(s)) = -\epsilon\nu \cdot i \int_0^1 g_{uu}(u_*(s - \tau\epsilon\nu \cdot i))u_*'(s - \tau\epsilon\nu \cdot i)d\tau,$$

we have

$$\|g_u(u_*(s - \epsilon\nu \cdot i)) - g_u(s)\| \leq \epsilon |i| \|g_{uu}\| \|u_*'\|_{C_0},$$

where $\|g_{uu}\| = \max\{\|g_{uu}(u_*(s))\| :\ s \in R\}$. Therefore, for all $s \in R$ and $\psi \in C_0$, we have

$$\begin{aligned} &\left\| \int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i) \int_{-r}^{0} d\eta(\theta) [g_u(u_*(t - \epsilon\nu \cdot i + \theta)) \right.\\ &\qquad -g_u(u_*(t+\theta))]\psi(t - \epsilon\nu \cdot i + \theta) dt\| \\ &\leq \epsilon M_3 \|\psi\|_{C_0}, \end{aligned} \tag{3.21}$$

where

$$M_3 = \sup_{t \in R} \|B(t)\| \|\eta\| \|g_{uu}\| \|u'_*\|_{C_0} \sum_i |K(i)| \cdot |i|.$$

If $\psi \in C_0^1$, by exchanging the order of integration and integration by parts, we have

$$\begin{aligned} &\int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(t+\theta)) [\psi(t - \epsilon\nu \cdot i + \theta) - \psi(t+\theta)] dt \\ &= \int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(t+\theta)) \\ &\qquad \times \int_0^1 \psi'(t - \tau\epsilon\nu \cdot i + \theta)(-\epsilon\nu \cdot i) d\tau dt \\ &= -\epsilon \int_0^1 \int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i)(\nu \cdot i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(t+\theta)) \\ &\qquad \times \psi'(t - \tau\epsilon\nu \cdot i + \theta) dt d\tau \\ &= -\epsilon \int_0^1 \big\{ B(s) \sum_i K(i)(\nu \cdot i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(s+\theta)) \psi(s - \tau\epsilon\nu \cdot i + \theta) \\ &\quad - \int_{-\infty}^{s} e^{-(s-t)} (B(t) + B'(t) \sum_i K(i)(\nu \cdot i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(t+\theta)) \\ &\qquad \times \psi(t - \tau\epsilon\nu \cdot i + \theta) dt \\ &\quad - \int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i)(\nu \cdot i) \int_{-r}^{0} d\eta(\theta) g_{uu}(u_*(t+\theta)) u'_*(t+\theta) \\ &\qquad \times \psi(t - \tau\epsilon\nu \cdot i + \theta) \big\} d\tau, \end{aligned}$$

Therefore, we have that for all $s \in R$,

$$\begin{aligned} &\left\| \int_{-\infty}^{s} e^{-(s-t)} B(t) \sum_i K(i) \int_{-r}^{0} d\eta(\theta) g_u(u_*(t+\theta)) [\psi(t - \epsilon\nu \cdot i + \theta) - \psi(t+\theta)] dt \right\| \\ &\leq \epsilon M_4 \|\psi\|_{C_0}, \end{aligned} \tag{3.22}$$

where

$$M_4 = \sup_{t \in R} (\|B(t)\| + \|B'(t)\|)(2\|g_u\| + \|g_{uu}\| \|u'_*\|_{C_0}) \|\eta\| \sum_i |K(i)| \cdot |i|.$$

Thus, for $\epsilon \in [0, \epsilon_0]$ and $\psi \in C_0^1$,

$$\|\mathcal{L}_\epsilon \psi\|_{C_0} = \sup_{s \in R} \left\| \int_{-\infty}^{s} e^{-(s-t)} [L_\epsilon - L_0] \psi(t) dt \right\| \leq \epsilon M_0 \|\psi\|_{C_0}, \quad M_0 = \sum_{j=1}^{4} M_j. \tag{3.23}$$

Since $\mathcal{L}_\epsilon : C_0 \to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the inequality (3.23) holds for all $\psi \in C_0$. This completes the proof. $\square$

Proposition 3.6. *$\mathcal{G}^1(\cdot,0,\epsilon)=0$ and for each $\delta>0$, there is a $\sigma>0$ such that*

$$\|\mathcal{G}^1(\epsilon,\cdot,\phi)-\mathcal{G}^1(\epsilon,\cdot,\psi)\|_{C_0}\le\delta\|\phi-\psi\|_{C_0}$$

for all $\epsilon\in[0,1]$ and all $\phi,\psi\in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof From the definition of $G^1(\epsilon,\cdot,\psi)$, it is obvious that $G^1_\psi(\epsilon,\cdot,\psi)$ and $G^1_{\psi\psi}(\epsilon,\cdot,\psi)$ are continuous for $\epsilon\in[0,1]$ and for ψ in a neighborhood of the origin in C_0. Moreover, we have $G^1_\psi(\epsilon,\cdot,0)=0$ for all $\epsilon\in[0,1]$. It therefore follows that

$$\|G^1(\epsilon,\cdot,\psi)\|_{C_0}=O(\|\psi\|^2_{C_0}),\quad\text{as}\quad\|\psi\|_{C_0}\to 0 \tag{3.24}$$

uniformly for $\epsilon\in[0,1]$, and the Proposition 3.6 follows from (3.24) and the definition of $\mathcal{G}^1(\epsilon,\cdot,\cdot)$. □

Proposition 3.7. *$\|\mathcal{G}^2(\epsilon,\cdot)\|_{C_0}=O(\epsilon)$ as $\epsilon\to 0$.*

Proof Note that

$$\begin{aligned}&F(u_*(s),R(\epsilon,u_*)(s))-F(u_*(s),R(0,u_*)(s))\\&=\textstyle\int_0^1 F_v(u_*(t),R(0,u_*)(t)+\tau[R(\epsilon,u_*)(t)-R(0,u_*)(t)])d\tau\\&\qquad\times[R(\epsilon,u_*)(t)-R(0,u_*)(t)].\end{aligned}$$

Since

$$\|R(\epsilon,u_*)-R(0,u_*)\|_{C_0}\le\epsilon\sum_i|K(i)|\cdot|i|\|\eta\|\|g_u\|\|u_*'\|_{C_0},$$

and there exist $\epsilon_0>0$ and $M>0$ such that for all $s\in R$ and $\epsilon\in[0,\epsilon_0]$,

$$\left\|\int_0^1 F_v(u_*(s),R(0,u_*)(s)+\tau[R(\epsilon,u_*)(s)-R(0,u_*)(s)])d\tau\right\|\le M,$$

where $\|\eta\|=\bigvee_{[-r,0]}\eta$ and $\|g_u\|=\max\{\|g_u(u_*(s))\|:\ s\in R\}$. Therefore, we have

$$\|G^2(\epsilon,\cdot)\|_{C_0}=\|F(u_*(\cdot),R(\epsilon,u_*)(\cdot))-F(u_*(\cdot),R(0,u_*)(\cdot))\|_{C_0}=O(\epsilon),\quad\text{as}\quad\epsilon\to 0. \tag{3.25}$$

Thus, the proposition 3.7 follows from (3.25) and the definition of $\mathcal{G}^2(\epsilon,\cdot)$, and the proof is completed. □

By Proposition 3.1, there exist functions $\upsilon_1,\cdots,\upsilon_M\in C_0$ which give a basis of $\mathcal{N}(\mathcal{L})$. Hence, there exist linear functionals $h_1,\cdots,h_M:C_0\to R$ such that

$$h_j(\upsilon_j)=1,\quad h_j(\upsilon_i)=0,\quad i\ne j,\ i,j=1,\cdots,M.$$

Let $X=\{\phi\in C_0:\ h_j(\phi)=0,\ \ j=1,\cdots,M\}$. Clearly $X\subset C_0$ is a Banach space and

$$C_0=X\oplus\mathcal{N}(\mathcal{L}). \tag{3.26}$$

Moreover, if we let $\mathcal{S}=\mathcal{L}|_X$ be the restriction of $\mathcal{L}$ on X, then $\mathcal{S}:X\to C_0$ is one-to-one and onto, since $\mathcal{R}(\mathcal{L})=C_0$ by Proposition 3.1. Therefore, $\mathcal{S}$ has an inverse $\mathcal{S}^{-1}:C_0\to X$ which is a bounded linear operator.

4 Proof of the main theorem

We shall complete the proof of our main theorem 1.1 in this section. The proof is similar to that of the main result in [10] and for the reader's convenience, we present the details here.

Proof of Theorem 1.1. Firstly, by Proposition 3.1, there exist functions $\upsilon_1, \cdots, \upsilon_M \in C_0$ which give a basis of $\mathcal{N}(\mathcal{L})$. Hence, there exist linear functionals $h_1, \cdots, h_M : C_0 \to R$ such that

$$h_j(\upsilon_j) = 1, \quad h_j(\upsilon_i) = 0, \quad i \neq j, \ i, j = 1, \cdots, M.$$

Let $X = \{\phi \in C_0 : \ h_j(\phi) = 0, \ \ j = 1, \cdots, M\}$. Clearly $X \subset C_0$ is a Banach space and

$$C_0 = X \oplus \mathcal{N}(\mathcal{L}). \tag{4.1}$$

Moreover, if we let $\mathcal{S} = \mathcal{L}|_X$ be the restriction of $\mathcal{L}$ on X, then $\mathcal{S} : X \to C_0$ is one-to-one and onto, since $\mathcal{R}(\mathcal{L}) = C_0$ by Proposition 3.1. Therefore, $\mathcal{S}$ has an inverse $\mathcal{S}^{-1} : C_0 \to X$ which is a bounded linear operator.

For each $\psi \in C_0$, there exist unique $\xi \in \mathcal{N}(\mathcal{L})$ and $\phi \in X$ such that $\psi = \phi + \xi$. Hence, ψ is a solution of (2.8) if and only if

$$\mathcal{L}\phi = \mathcal{G}(\cdot, \phi + \xi, \epsilon), \tag{4.2}$$

or equivalently, if and only if ϕ is a solution of the equation

$$\phi = \mathcal{S}^{-1}\mathcal{G}(\cdot, \phi + \xi, \epsilon). \tag{4.3}$$

It follows from Propositions 3.3-3.7 that there exist $\sigma > 0$, $\epsilon^* > 0$ and $\rho \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon^*]$ and $\psi, \varphi \in \overline{B(\sigma)} \subset C_0$,

$$\|\mathcal{G}(\cdot, \psi, \epsilon)\|_{C_0} \leq \frac{1}{3\|\mathcal{S}^{-1}\|}(\|\psi\|_{C_0} + \sigma), \tag{4.4}$$

and

$$\|\mathcal{G}(\cdot, \psi, \epsilon) - \mathcal{G}(\cdot, \varphi, \epsilon)\|_{C_0} \leq \frac{\rho}{\|\mathcal{S}^{-1}\|}\|\psi - \varphi\|_{C_0}. \tag{4.5}$$

For each fixed $\xi \in \mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}$, (4.4) implies that

$$\|\mathcal{S}^{-1}\mathcal{G}(\cdot, \phi+\xi, \epsilon)\|_{C_0} \leq \frac{1}{3}(\|\phi+\xi\|_{C_0}+\sigma) \leq \sigma, \quad \text{for } \epsilon \in (0, \epsilon^*], \quad \phi \in X \cap \overline{B(\sigma)}. \tag{4.5}$$

Therefore, from (4.5) and (4.6), we conclude that the mapping

$$\mathcal{F} : (X \cap \overline{B(\sigma)}) \times (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*) \to X \times \overline{B(\sigma)}$$

given by

$$\mathcal{F}(\phi, \xi, \epsilon) = \mathcal{S}^{-1}\mathcal{G}(\cdot, \phi + \xi, \epsilon)$$

is a uniform contraction mapping of $\phi \in X \cap \overline{B(\sigma)}$. Hence, for each $(\xi, \epsilon) \in (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*)$, there is a unique fixed point $\phi(\xi, \epsilon) \in X \cap \overline{B(\sigma)}$ of the mapping $\mathcal{F}(\cdot, \xi, \epsilon)$. In other words, $\phi(\xi, \epsilon)$ is the unique solution in $X \cap \overline{B(\sigma)}$ of Eq. (4.3). Thus, for $\epsilon \in (0, \epsilon^*)$ fixed, $\psi(\xi, \epsilon) = \phi(\xi, \epsilon) + \xi$ is a solution of (2.8). Notice that $\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}$ is M-dimensional. It follows that for each $\epsilon \in (0, \epsilon^*)$ and for each unit vector $\nu \in R^q$, the set

$$\Gamma_\nu(\epsilon) = \{\psi(\xi, \epsilon) : \ \xi \in \mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}\}$$

is an M-dimensional manifold. This proves that claims (i) and (ii) in the statement of the theorem.

To prove (iii), we first note that if F, g are C^k $(k \geq 2)$, then $\mathcal{G}(\cdot, \psi, \epsilon)$ is continuous on (ψ, ϵ) and C^{k-1}-smooth with respect to ψ. Hence, $\mathcal{F}(\phi, \xi, \epsilon)$ is continuous on (ϕ, ψ, ϵ) and C^{k-1}-smooth with respect to ϕ and ξ. The uniform contraction mapping principle (see Chow and Hale [7]) implies that the fixed point $\phi(\xi, \epsilon)$ is a continuous mapping on (ξ, ϵ) and C^{k-1}-smooth with respect to ξ. Therefore, in addition we conclude that for each $\epsilon \in (0, \epsilon^*)$ and for each unit vector $\nu \in R^q$, $\Gamma_\nu(\epsilon)$ is a C^{k-1}-manifold. It is locally given as the graph of a C^{k-1}-mapping that is also continuous with respect to ϵ.

Let $c = 1/\epsilon$ with $\epsilon \in (0, \epsilon^*)$ and

$$\mathsf{M}_\nu(c) = \{U : \; U(s) = \psi_\xi(s/c) + u_*(s/c), \quad s \in R, \; \psi_\xi \in \Gamma_\nu(1/c)\}.$$

Then $\mathsf{M}_\nu(c)$ is an M-dimensional manifold in a neighborhood of u_* consisting of traveling wave solutions of Eq. (1.1) with wave speed c and direction ν. Moreover, for each $c > c^* := 1/\epsilon^*$ and for each unit vector $\nu \in R^q$, $\mathsf{M}_\nu(c)$ is a C^{k-1}-manifold that is given by the graph of a C^{k-1}-mapping that is continuous on c.

Recall that F and g are assumed to be C^k-smooth. It remains to prove that the above fixed point $\phi(\xi, \epsilon)$ is also C^{k-1}-smooth with respect to ϵ. We will achieve this in several steps.

Assume the functions F and g are $C^k (k \geq 2)$. For $p \in N$, define $C_0^p := \{\phi \in C_0 : \; \phi \text{ is } C^p - \text{smooth}\}$.

Claim 1. From the definition of L_0 in (2.2), it is clear that $L_0 : C_0 \to C_0$ is linear bounded and that $L_0(C_0^p) \subset C_0^p$, for $0 \leq p \leq k-1$.

Claim 2. From the definition of $\mathcal{L}$ in (2.9), $\mathcal{L} : C_0 \to C_0$ is linear bounded and $\mathcal{L}(C_0^p) \subset C_0^p$, for $0 \leq p \leq k$.

Claim 3. From the definition of $\mathcal{G}$ in (2.10), we have $\mathcal{G}(\cdot, C_0^{p-1}, \epsilon) \subset C_0^p$ for $\epsilon > 0$ and $p = 1, 2, \cdots, k$, where $C_0^0 = C_0$.

Claim 4. $\mathcal{N}(\mathcal{L}) \subset C^k$.

In fact, by definition, $\phi \in C_0$ and $\mathcal{L}\phi = 0$ if and only if

$$\phi(s) = \int_{-\infty}^{s} e^{-(s-t)}[\phi(t) + L_0\phi(t)]dt, \quad s \in R.$$

Hence, ϕ is continuously differentiable. By differentiating the last equation, we find that $\mathcal{L}\phi = 0$ if and only if $\phi'(s) = L_0\phi(s), \quad s \in R$. Therefore, $\mathcal{N}(\mathcal{L}) = \{\phi \in C^1 : \phi'(t) = L_0\phi(t), \; t \in R\}$. From Claim 1, by induction we conclude that $\mathcal{N}(\mathcal{L}) \subset C_0^k$.

Claim 5. For each $(\xi, \epsilon) \in (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*)$, the fixed point $\phi^* := \phi(\xi, \epsilon) \in C_0^1$.

To prove this claim, we fix $(\xi, \epsilon) \in (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*)$, and define $\psi^* = \phi^* + \xi$. From $\phi^* = \mathcal{F}(\phi^*, \xi, \epsilon)$, we obtain $\mathcal{L}\psi^* = \mathcal{G}(\cdot, \psi^*, \epsilon)$, or equivalently,

$$\psi^*(s) = \mathcal{G}(s, \psi^*, \epsilon) + \int_{-\infty}^{s} e^{-(s-t)}[\psi^*(t) + L_0\psi^*(t)]dt, \quad s \in R.$$

Hence, $\psi^* \in C_0^1$. From Claim 4, we conclude that $\phi^* \in C_0^1$.

Claim 6. The fixed point $\phi^* = \phi(\xi, \epsilon)$ is C^1-smooth with respect to ϵ.

Consider $\mathcal{F}$ restricted to $\phi \in X \cap \overline{B(\sigma)} \cap C_0^1$, more precisely, using Claim 2 and Claim 3, we consider

$$\mathcal{F}^1 : (X \cap \overline{B(\sigma)} \cap C_0^1) \times (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*) \to X \cap \overline{B(\sigma)} \cap C_0^1,$$

$$\mathcal{F}^1(\phi, \xi, \epsilon) = \mathcal{F}(\phi, \xi, \epsilon).$$

Notice that $\mathcal{F}^1$ is a uniform contraction of $\phi \in X \cap \overline{B(\sigma)} \cap C_0^1$ for the norm $\|\cdot\|_{C_0}$, and that $\mathcal{F}^1$ is a C^1-mapping on (ϕ, ξ, ϵ). In fact, for $\psi(s) = \phi(s) + \xi(s)$ C^1-smooth on s, from the definition of $\mathcal{G}$ in (2.10), we conclude that $\frac{\partial \mathcal{G}}{\partial \epsilon}(s, \psi, \epsilon)$ exists and is continuous. In Claim 5, we have proved that there exits a fixed point $\phi^* = \phi(\xi, \epsilon)$ of $\mathcal{F}^1$. By repeating the arguments used to prove the differentiability of the fixed point in the uniform contraction principle (see Chow and Hale [7]), we conclude that $\phi(\xi, \epsilon)$ is a C^1-smooth mapping on (ξ, ϵ).

Claim 7. The fixed point $\phi^* = \phi(\xi, \epsilon)$ is C^{k-1}-smooth with respect to ϵ.

As in Claim 5, by induction, we prove that $\phi(\xi, \epsilon) \in C_0^p, p = 1, 2, \cdots, k$. By using Claim 2 and Claim 3, we consider

$$\mathcal{F}^p : (X \cap \overline{B(\sigma)} \cap C_0^p) \times (\mathcal{N}(\mathcal{L}) \cap \overline{B(\sigma)}) \times (0, \epsilon^*) \to X \cap \overline{B(\sigma)} \cap C_0^p,$$

$$\mathcal{F}^p(\phi, \xi, \epsilon) = \mathcal{F}(\phi, \xi, \epsilon), \quad p = 2, \cdots, k-1.$$

As in the proof of the uniform contraction principle, by an inductive argument we conclude that $\phi^* = \phi(\xi, \epsilon)$ is C^{k-1}-smooth with respect to ϵ.

5. Applications

Among the conditions for Theorem 1.1, (H1) and (H2) are verified by analyzing the characteristic equations of (1.2) at E_1 and E_2. To verify (H3), the connecting orbit theorem in monotone dynamical system theory is useful, which is stated below. See, e.g., Wu et al. [23], Dance and Hess [9] and Smith [17, 18] and the reference therein.

Let X be an ordered Banach space with a closed cone K. For $u, v \in X$, we write $u \geq v$ if $u - v \in K$, and $u > v$ if $u \geq v$ but $u \neq v$.

Lemma 5.1. *Let U be a subset of X and $\Phi : [0, +\infty) \times U \to U$ be a semiflow such that*

(i) Φ is strictly order-preserving, i.e., $\Phi(t, u) > \Phi(t, v)$ for $t \geq 0$ and for all $u, v \in U$ with $u > v$;

(ii) for some $t_0 > 0$, $\Phi(t_0, \cdot) : U \to U$ is set-condensing with respect to a measure of non-compactness.

Suppose $u_2 > u_1$ are two equilibria of Φ and assume $[u_1, u_2] := \{u : u_2 \geq u \geq u_1\}$ contains no other equilibria. Then there exists a full orbit connecting u_1 and u_2. Namely, there is a continuous function $\phi : \mathrm{R} \to U$ such that $\Phi(t, \phi(s)) = \phi(t + s)$ for all $t \geq 0$ and all $s \in \mathrm{R}$, either (a): $\phi(t) \to u_1$ as $t \to +\infty$ and $\phi(t) \to u_2$ as $t \to -\infty$; or (b): $\phi(t) \to u_1$ as $t \to -\infty$ and $\phi(t) \to u_2$ as $t \to +\infty$.

Returning to the system (1.2), we use the standard phase space for (1.2). In this section, C will denote the Banach space $C([-r, 0]; R^N)$ of continuous R^N-valued functions on $[-r, 0]$ with the usual supremum norm. Under the smoothness condition on F and g, the system (1.2) generates a (local) semiflow on C given by

$$\Phi(t, \phi) = u(\phi)(t + \cdot), \quad t \geq 0, \ \phi \in C,$$

for all those t for which a unique solution $u(\phi)$ of (1.2) with $u(\phi)(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$ is defined. Let B be an $N \times N$ quasi-positive matrix, that is, $B + \lambda I \geq 0$ for all sufficiently large λ. Here and in what follows, we write $A \geq B$ for $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ if and only if $a_{ij} \geq b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. Define

$$K_B = \left\{ \phi \in C : \ \phi \geq 0, \phi(t) \geq e^{B(t-s)} \phi(s), -r \leq s \leq t \leq 0 \right\}.$$

Then K_B is a closed cone in C and this induces a partial order on C, denoted by $\geq_B$. Namely, $\phi \geq_B \psi$ if and only if $\phi - \psi \in K_B$.

We shall need the following conditions:

(E_B) $\hat{E}_2 \geq_B \hat{E}_1$, here $\hat{E}_j$ is the constant mapping on $[-r, 0]$ with value $E_j, j = 1, 2$.

(M$_B$) Whenever $\phi, \psi \in C$ with $\phi \geq_B \psi$, then

$$F(\phi(0), \int_{-r}^{0} d\eta(\theta) g(\phi(\theta))) - F(\psi(0), \int_{-r}^{0} d\eta(\theta) g(\psi(\theta))) \geq B[\phi(0) - \psi(0)].$$

Under the above assumptions, Smith and Thieme [20] proved the following

Lemma 5.2. *Assume that there exists an $N \times N$ quasi-positive matrix B such that (E_B) and (M_B) are satisfied. Then*

(i) $[E_1, E_2]_B := \{\phi \in C : \hat{E}_2 \geq_B \phi \geq_B \hat{E}_1\}$ is positive invariant for the semiflow Φ;

(ii) the semiflow $\Phi : [0, +\infty) \times [E_1, E_2]_B \to [E_1, E_2]_B$ is strictly monotone with respect to $\geq_B$ in the sense that if $\phi, \psi \in [E_1, E_2]_B$ with $\phi >_B \psi$, then $\Phi(t, \phi) >_B \Phi(t, \psi)$ for all $t \geq 0$.

In Smith and Thieme [20], it was also shown that (M$_B$) holds if for all $u, v \in R^N$ with $\hat{u}, \hat{v} \in [E_1, E_2]_B$, the following is satisfied:

$$\begin{cases} F_u(u, \int_{-r}^{0} d\eta(\theta) g(v)) \geq B, \\ [F_u(u, \int_{-r}^{0} d\eta(\theta) g(v)) - B] e^{Br} + F_v(u, \int_{-r}^{0} d\eta(\theta) g(v)) g'(v) \geq 0. \end{cases}$$

In the case $N = 1$, it was shown that in Smith and Thieme [19] that (M$_B$) holds for some $B < 0$ if

(S$_B$) $\quad L_2 < 0, L_1 + L_2 < 0, r|L_2| < 1$ and $rL_1 - \ln(rL_2|) > 1$,

where

$$L_1 = \inf_{E_1 \leq u, v \leq E_2} F_u(u, \int_{-r}^{0} d\eta(\theta) g(v)), \quad L_2 = \inf_{E_1 \leq u, v \leq E_2} F_v(u, \int_{-r}^{0} d\eta(\theta) g(v)) g'(v).$$

Note that $[E_1, E_2]_B$ is a bounded set in C and that $\Phi(t, \cdot) : C \to C$ is compact for $t > r$. Therefore, for $t_0 > r$, the mapping $\Phi(t_0, \cdot) : [E_1, E_2]_B \to [E_1, E_2]_B$ is compact, and hence is set-condensing. This observation allows us to derive from Lemma 5.1, 5.2 and Theorem 1.1 the following result.

Theorem 5.1. *Assume that*

(i) (H1) and (H2) are satisfied;

(ii) there exists an $N \times N$ quasi-positive matrix B such that (E_B) and (M_B) are satisfied;

(iii) there exist no other equilibria in $[E_1, E_2]_B$.

Then the conclusions of Theorem 1.1 hold.

As a first application of our main result, we consider the following lattice differential equation

$$u_n'(t) = D \sum_{i \in Z \setminus \{0\}} J(i)[u_{n-i}(t) - u_n(t)] - du_n(t) + \sum_{i \in Z} K(i) b(u_{n-i}(t - r)), \quad (5.1)$$

where $n \in Z$, $t \geq 0$, $u_n(t) \in R$, $D > 0$, $r \geq 0$, $d > 0$ and $b(\cdot)$ is of class C^2. We assume that $b(0) = dK - b(K) = 0$ for some $K > 0$ and

$$\sum_{i \in Z \setminus \{0\}} J(i) = 1, \qquad \sum_{i \in Z \setminus \{0\}} |J(i)| \cdot |i| < +\infty, \tag{5.2}$$

$$\sum_{i \in Z} K(i) = 1, \quad \sum_{i \in Z} |K(i)| \cdot |i| < +\infty. \tag{5.3}$$

When $\sum_{|i| \geq 2} J(i) = 0$, Eq. (5.1) has been derived by Weng et al. [22] as a discrete non-local model parallel to the continuous nonlocal model in So et al. [21]. Clearly, the auxiliary ordinary delay differential equation reads as

$$u'(t) = -du(t) + b(u(t-r)), \tag{5.4}$$

and it is easily seen that the corresponding characteristic equations at the equilibria $E_1 = 0$ and $E_2 = K$ of Eq. (5.4) are

$$\Lambda_1(\lambda) := \lambda + d - b'(0)e^{-\lambda r}, \tag{5.5}$$

and

$$\Lambda_2(\lambda) := \lambda + d - b'(K)e^{-\lambda r}, \tag{5.6}$$

respectively.

Theorem 5.2. *Assume that $b'(0) > d > b'(K)$ and $b(u) > du$ for all $u \in (0, K)$. Let $I_0 = [0, r^1) \cap [0, r^2) \cap [0, r_1)$ and $I_j = [0, r^1) \cap [0, r^2) \cap (r_j, r_{j+1}), j \in \mathrm{N}$, where*

$$r^1 := \sup\left\{r \geq 0 : \ re^{dr} \min\{b'(u) : \ u \in [0, K]\} \geq -e^{-1}\right\},$$

$$r^2 := \begin{cases} +\infty, & \text{if} \quad -d \leq b'(K) < d, \\ \frac{\arccos(\frac{d}{b'(K)})}{\sqrt{(b'(K))^2 - d^2}}, & \text{if} \quad b'(K) < -d, \end{cases}$$

and

$$r_j := \frac{2j\pi - \arccos(\frac{d}{b'(0)})}{\sqrt{[b'(0)]^2 - d^2}}, \quad j \in \mathrm{N}.$$

Then for any $j \in \mathrm{N}$ with $I_{j-1} \neq \emptyset$ and for any $r \in I_{j-1}$, there exists $c^* > 0$ such that for every $c > c^*$, the set of all traveling wave solutions $u_n(t) = U(n + ct)$ with $U(-\infty) = 0$ and $U(+\infty) = K$ of Eq. (5.1) forms a $(2j-1)$-dimensional C^1-manifold, which is also C^1-smooth with respect to c.

Theorem 5.2 is a direct consequence of Theorem 5.1 and the following three lemmas.

Lemma 5.3. *If $r \in I_{j-1}, j \in \mathrm{N}$, then the equilibrium $E_1 = 0$ of Eq. (5.4) is hypobolic and its unstable manifold is exactly $2j-1$-dimensional.*

Proof Clearly, if $r \in [0, r_1)$, then (5.5) has a simple eigenvalue $\lambda > 0$. A straightforward calculation shows that $E_1 = 0$ is hyperbolic if $r \neq r_j, j \in N$, and (5.5) has a pair of simple eigenvalues $\lambda = \pm i\beta_j$ with $\beta_j > 0$ at $r = r_j, j \in N$. For any $r \geq 0$, suppose that $\lambda = \lambda(r) = \alpha(r) + i\beta(r)$ with $\beta(r) \geq 0$ is a eigenvalue of (5.5). It suffices to show that $\alpha'(r) > 0$ whenever $|\alpha(r)|$ is small enough.

Substituting $\lambda = \lambda(r) = \alpha(r) + i\beta(r)$ into (5.5), we get

$$\begin{cases} (\alpha + d)e^{\alpha r} &= b'(0)\cos\beta r, \\ \beta e^{\alpha r} &= -b'(0)\sin\beta r. \end{cases} \tag{5.7}$$

Therefore, we have

$$(\alpha+d)^2+\beta^2=[b'(0)]^2e^{-2\alpha r},$$

and hence

$$\beta\beta'=-\{\alpha+d+r[b'(0)]^2e^{-2\alpha r}\}\alpha'. \tag{5.8}$$

On the other hand, differentiating (5.5) with respect to r to get

$$\begin{cases} \alpha' e^{\alpha r}+(\alpha+d)[\alpha' r+\alpha]e^{\alpha r} &= -b'(0)[\beta' r+\beta]\sin\beta r, \\ \beta' e^{\alpha r}+\beta[\alpha' r+\alpha]e^{\alpha r} &= -b'(0)[\beta' r+\beta]\cos\beta r, \end{cases}$$

which yields

$$\alpha'\beta-\beta'(\alpha+d)=b'(0)e^{-\alpha r}[\beta' r+\beta]Q,$$

where $Q=(\alpha+d)\cos\beta r-\beta\sin\beta r=\frac{1}{b'(0)}[(\alpha+d)^2+\beta^2]e^{\alpha r}>0$. Multiplying the above equality by β, then (5.8) implies that

$$\begin{aligned} \alpha'\beta^2 &= \beta\beta'[\alpha+d+b'(0)e^{-\alpha r}rQ]+b'(0)e^{-\alpha r}\beta^2Q \\ &= -[\alpha+d+r(b'(0))^2e^{-2\alpha r}][\alpha+d+b'(0)e^{-\alpha r}rQ]\alpha'+b'(0)e^{-\alpha r}\beta^2Q. \end{aligned}$$

Therefore, we have

$$\alpha'=\alpha'(r)=\frac{b'(0)e^{-\alpha r}\beta^2Q}{\beta^2+[\alpha+d+r(b'(0))^2e^{-2\alpha r}][\alpha+d+b'(0)e^{-\alpha r}rQ]}>0.$$

This completes the proof. □

Lemma 5.4. *There exists $B<0$ such that (E_B) and (M_B) are satisfied.*

Proof In the case where $b'_{\min}:=\min\{b'(u):\ u\in[0,K]\}\ge 0$. Choose $B=-d$, then for any $u,v\in[0,K]$, we have $F_u(u,g(v))=-d\ge B$ and $[F_u(u,g(v))-B]e^{Br}+F_v(u,g(v))g'(v)=b'(v)\ge 0$. Therefore ($\mathrm{M}_B$) holds for $B=-d<0$.

In the case where $b'_{\min}<0$, we have

$$L_1:=\inf_{E_1\le u,v\le E_2}F_u(u,g(v))=-d,$$

and

$$L_2:=\inf_{E_1\le u,v\le E_2}F_v(u,g(v))g'(v)=b'_{\min}.$$

Therefore, $L_2<0, L_1+L_2<0$. Thus there is some $B<0$ so that (S_B) (and hence (M_B)) holds if

$$0<-rb'_{\min}<1,$$

and

$$\ln(-rb'_{\min})<-dr-1,$$

from which we conclude that (M_B) holds if $re^{dr}b'_{\min}>-e^{-1}$. Thus (M_B) holds for all $r\in[0,r^1)$ and some $B<0$.

Since $B<0$, we see that (E_B) also holds. This completes the proof. □

Lemma 5.5. *If $r\in[0,r^2)$, then the equilibrium $E_2=K$ of Eq. (5.4) is asymptotic stable.*

Proof We claim that if $r\in[0,r^2)$, then all zeros of $\Lambda_2(\lambda)=0$ have negative real parts. Since $b'(K)<d$, we first note that if $\Lambda_2(\lambda)=0$, then $\lambda\ne 0$. Suppose otherwise that there exists $\lambda=\alpha+i\beta$ with $\alpha\ge 0$ and $\beta\ge 0$ such that $\Lambda_2(\lambda)=\Lambda_2(\alpha+i\beta)=0$. Then we have

$$\begin{cases} \alpha &= -d+b'(K)e^{-\alpha r}\cos\beta r, \\ \beta &= -b'(K)e^{-\alpha r}\sin\beta r. \end{cases}$$

If $-d < b'(K) < d$, then $d > b'(K)\cos\beta r = (\alpha + d)e^{-\alpha r} \geq d$, which leads to a contradiction. If $b'(K) = -d$, then $\alpha > 0$. Suppose otherwise that $\alpha = 0$, we then have $\beta > 0$ and $\cos\beta r = -1$, and hence $\sin\beta r = 0$, which yields $\beta = -b'(k)e^{-\alpha r}\sin\beta r = 0$, a contradiction. Therefore, $d \geq b'(K)\cos\beta r = (\alpha + d)e^{\alpha r} > d$, which is also a contradiction. Thus the assertion is valid for $-d \leq b'(K) < d$.

In the case where $b'(K) < -d$, let $\lambda = i\beta$ with $\beta > 0$ be such that $\Lambda_2(\lambda) = 0$. Then we have $d = b'(K)\cos\beta r$ and $\beta = -b'(K)\sin\beta r$, from which we find that $\frac{\arccos(\frac{d}{b'(K)})}{\sqrt{[b'(K)]^2 - d^2}}$ is the minimal value of r so that $\Lambda_2(i\beta) = 0$ has a solution $\beta > 0$. This completes the proof of the lemma. $\square$

As anther application of our main result, we consider the following lattice differential equation

$$u_n' = \sum_{i \in Z \setminus \{0\}} J(i)u_{n-i} - u_n - f(u_n), \quad n \in Z, \tag{5.9}$$

where f is in C^2 and $f(-1) = f(1) = 0$, and the kernel $J(i)$ satisfies (5.2). Eq. (5.9) was derived in [2] as an l_2-gradient flow for a Helmholtz free energy functional with general long range interaction (see [1] for its continuum form). In [2], the authors constructed traveling waves and stationary solutions, and obtained the uniqueness of traveling wavefronts with non-zero speed and the multiplicity of stationary solutions in the case where f is bistable. In a very recent paper, Carr and Chmaj [4] established the uniqueness of traveling wavefronts in the case where f is monostable. As a direct consequence of Theorem 5.1, we have the following

Theorem 5.3. *Assume that $f'(-1) < 0$, $f'(1) > 0$ and $f(u) < 0$ for $u \in (-1, 1)$. Then there exists $c^* > 0$ such that for any $c > c^*$, Eq. (5.9) has a traveling wave solution $u_n(t) = U(n + ct)$ satisfying $U(-\infty) = -1$ and $U(+\infty) = 1$.*

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