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Exponential Stability of Delay Difference Equations with Applications to Neural Networks

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Dedicated to Professor Zhicheng Wang on the occasion of his seventieth birthday

Abstract. Global exponential stability for a class of delay difference equations is obtained in this paper via constructing a discrete Liapunov functional. Applications to some discrete-time neural networks with delays show that our results improve some existing ones.

1 Introduction

In this paper, we shall study the exponential stability of the following system

$$x_i(n+1) = a_i(x_i(n)) + \sum_{j=1}^{m} w_{ij} g_j(x_j(n)) + \sum_{j=1}^{m} b_{ij} f_j(\sum_{p=1}^{r_{ij}} k_{ij}(p) x_j(n-p)), \quad (1.1)$$

where $n = 0, 1, \cdots$, r_{ij} are positive integers, $a_i(0) = 0$, $g_i(0) = 0$, and $f_i(0) = 0$ and $i = 1, 2, \cdots, m$. When it comes to neural network models, in (1.1), $m \geq 2$ denotes the number of neurons in the network; x_i describes the *activation* of the ith neuron; the $m \times m$ connection matrices $W = (w_{ij})$ and $B = (b_{ij})$ tell how the neurons are connected in the network at present states and the past states,

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respectively; the activation functions $g_j, f_j, j = 1, 2, \cdots, m$ show how the neurons react to one another. It is seen that the discrete kernels k_{ij} in (1.1) show the neurons states depend on some kind of average over periods of past time.

Clearly, if one assumes $B = 0$, $a_i(x) = -a_i x$, for $i = 1, \cdots, m$, then (1.1) reduces to

$$x_i(n+1) = -a_i x_i(n) + \sum_{j=1}^{m} w_{ij} g_j(x_j(n)), \tag{1.2}$$

which represents a discrete-time neural network model whose global asymptotic stability was investigated in [10] via Liapunov function method. The following special form of (1.1)

$$x_i(n+1) = (1 - a_i h) x_i(n) + \sum_{j=1}^{m} h w_{ij} g_j(x_j(n)), \tag{1.3}$$

is the discrete analog of the Hopfield neural network model

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{m} w_{ij} g_j(x_j), i = 1, \cdots, m. \tag{1.4}$$

In fact, (1.3) can be obtained via Euler method with the uniform step size h. Using a different way, Mohamad and Gopalsamy [13] obtained another discrete version for (1.4), which is described by

$$x_i(n+1) = e^{-a_i h} x_i(n) + \frac{1 - e^{-a_i h}}{a_i} \sum_{j=1}^{m} w_{ij} g_j(x_j(n)), i = 1, 2, \ldots, m \tag{1.5}$$

For the continuous time Hopfield neural network (1.4), there have been a large number of papers addressing its dynamics and applications in literature since it was proposed by Hopfield in 1984 [7]. For example, stability and instability were discussed in [6], [13], [23]; applications to solve optimization problems such as linear and quadratic programming problems, variational inequalities, can be found in [21] and [22]. There have been lots of evidences showing that time delays should be taken into consideration when modelling the neural networks (for details, see the recent book, [19]). With appearance of time delays, much more complicated dynamic behavior may occur in the delayed neural networks. For impact of delays on neural dynamics, we refer to [1], [14] and the references therein. For studies on stability and applications of delayed Hopfield neural networks, we refer to [2], [5], [12], [15], [16], [17], [18] and [20]. However, for discrete-time neural networks, the results on stability and its applications are much less than that of continuous time neural networks (see, [8], [9], [10] and [11]). Especially, for the models with delays, the results are scarce (for references, see, [13]). The aim of this paper is to investigate the global exponential stability (GES) of system (1.1) and apply our results to some discrete-time neural networks. As we shall see, by using Liapunov function method, we can obtain a GES result which improved the known results in the literature.

The rest of the paper is organized as follows. In Section 2 we introduce our basic notations and assumptions. Section 3 is devoted to the globally exponential stability results and the proofs for (1.1). Applications and numerical simulations are given in Section 4.

2 Preliminaries

In this paper, we use the following notations: $N(a) = \{a, a+1, \cdots, \}$; $Z^+ = N(0)$; $N(a,b) = \{a, a+1, \cdots, b-1, b\}$; $\rho(W)$: the spectrum radius of the matrix W; W^T: the transpose of the matrix W and $|W| = (|w_{ij}|)$. The initial conditions associated with (1.1) are of the form

$$x_i(s) = \phi_i(s),\ i = N(1,m),\ s \in N(-k,0) \tag{2.1}$$

where $k = \max\{r_{ij}, i, j \in N(1,m)\}$. We are now ready to give some assumptions.

(H_1) For each $i \in N(1,m)$, a_i is globally Lipschitz continuous with

$$\sup_{u,v\in\mathbb{R},u\neq v} \frac{|a_i(u) - a_i(v)|}{|u-v|} \leq \alpha_i.$$

(H_2) For each $i \in N(1,m)$, $g_i, f_i : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz continuous with

$$\sup_{u,v\in\mathbb{R},u\neq v} \frac{|g_i(u) - g_i(v)|}{|u-v|} = \beta_i, \quad \sup_{u,v\in\mathbb{R},u\neq v} \frac{|f_i(u) - f_i(v)|}{|u-v|} = l_i.$$

(H_3) The discrete kernels k_{ij} are nonnegative and bounded with

$$\sum_{p=1}^{r_{ij}} k_{ij}(p) = \kappa_{ij}.$$

3 Exponential stability for (1.1)

The following is the main result.

Theorem 3.1 *Suppose that $(H_1) - (H_3)$ hold. If there exist positive real numbers q_i, $i \in N(1,m)$ such that*

$$\alpha_i + \beta_i \sum_{j=1}^m q_j q_i^{-1} |w_{ji}| + l_i \sum_{j=1}^m q_j q_i^{-1} |b_{ji}| \kappa_{ji} < 1,\ i \in N(1,m), \tag{3.1}$$

then the trivial solution of (1.1) is globally exponentially stable in the sense that the following inequality holds:

$$\sum_{i=1}^m |x_i(n)| \leq C \left(\frac{1}{\gamma^*}\right)^n \max_{i\in N(1,m)} \left\{ \sup_{s\in N(-k,0)} |\phi_i(s)| \right\} \tag{3.2}$$

where $C > 0$ and $\gamma^* > 1$ will be specified below.

Proof Combining (H_1) with (H_2), we have

$$|x_i(n+1)| \leq \alpha_i |x_i(n)| + \sum_{j=1}^m |w_{ij}| \beta_j |x_j(n)| + \sum_{j=1}^m \left(|b_{ij}| l_j \sum_{p=1}^{r_{ij}} k_{ij}(p) |x_j(n-p)| \right). \tag{3.3}$$

Let

$$\mu_i(\gamma) := 1 - \alpha_i \gamma - \beta_i \sum_{j=1}^m q_j q_i^{-1} \gamma |w_{ji}| - l_i \sum_{j=1}^m q_j q_i^{-1} |b_{ji}| \sum_{p=1}^{r_{ji}} k_{ji}(p) \gamma^{p+1} \tag{3.4}$$

for $i \in N(1,m)$. By virtue of (3.1), we see that for each $i \in N(1,m)$, $\mu_i(1) > 0$ and as a function of γ, $\mu_i(\gamma)$ is decreasing, hence there is a $\gamma_i > 1$ such that $\mu_i(\gamma) > 0$ for $\gamma \in (1, \gamma_i]$. Let

$$\gamma^* = \min\{\gamma_i, i \in N(1,m)\}. \tag{3.5}$$

Then, for $i \in N(1,m)$, $\mu_i(\gamma^*) > 0$. Letting

$$|x_i(n)| = q_i^{-1}(\gamma^*)^{-n} y_i(n), \tag{3.6}$$

and using (3.3), we can obtain

$$\begin{aligned} y_i(n+1) \quad &\leq \quad \gamma^*|\alpha_i| y_i(n) + \sum_{j=1}^{m} \gamma^* \beta_j q_i q_j^{-1} |w_{ij}| y_j(n) \\ &\quad + \sum_{j=1}^{m} \left(l_j q_i q_j^{-1} |b_{ij}| \sum_{p=1}^{r_{ij}} (\gamma^*)^{p+1} k_{ij}(p) y_j(n-p) \right). \end{aligned}$$

Define $V(n) = V(y)(n)$ by

$$V(n) = \sum_{i=1}^{m} \left\{ y_i(n) + \sum_{j=1}^{m} |b_{ij}| l_j q_i q_j^{-1} \left[\sum_{p=1}^{r_{ij}} (\gamma^*)^{p+1} k_{ij}(p) \sum_{s=n-p}^{n-1} y_j(s) \right] \right\}. \tag{3.7}$$

We can estimate $\Delta V(n)$ along the solution of (1.1) as below:

$$\begin{aligned} \Delta V(n) \quad &= \quad V(n+1) - V(n) \\ &= \quad \sum_{i=1}^{m} \left\{ \Delta y_i(n) + \Delta \sum_{j=1}^{m} |b_{ij}| l_j q_i q_j^{-1} \left[\sum_{p=1}^{r_{ij}} (\gamma^*)^{p+1} k_{ij}(p) \sum_{s=n-p}^{n-1} y_j(s) \right] \right\} \\ &= \quad \sum_{i=1}^{m} \left\{ y_i(n+1) - y_i(n) + \sum_{j=1}^{m} |b_{ij}| l_j q_i q_j^{-1} \right. \\ &\quad \left. \sum_{p=1}^{r_{ij}} (\gamma^*)^{p+1} k_{ij}(p)(y_j(n) - y_j(n-p)) \right\} \\ &\leq \quad \sum_{i=1}^{m} \left[(\gamma^* \alpha_i - 1) y_i(n) + \sum_{j=1}^{m} \gamma^* \beta_j q_i q_j^{-1} |w_{ij}| y_j(n) \right. \\ &\quad \left. + \sum_{j=1}^{m} l_j q_i q_j^{-1} |b_{ij}| \sum_{p=1}^{r_{ij}} k_{ij}(p)(\gamma^*)^{p+1} y_j(n) \right] \\ &= \quad - \sum_{i=1}^{m} \mu_i(\gamma^*) y_i(n) \\ &\leq \quad - \min_{i \in N(1,m)} \{\mu_i(\gamma^*)\} \sum_{i=1}^{m} y_i(n) \\ &\leq \quad 0. \end{aligned}$$

Therefore, the nonnegative function $V(y)(n)$ is decreasing in n, then we have

$$
\begin{aligned}
& V(y)(n) \\
\leq \;& V(y)(0) \\
= \;& \sum_{i=1}^{m}\left(y_i(0)+\sum_{j=1}^{m}|b_{ij}|l_j q_i q_j^{-1}\sum_{p=1}^{r_{ij}}k_{ij}(p)(\gamma^*)^{p+1}\sum_{s=-p}^{-1}y_j(s)\right) \\
\leq \;& \sum_{i=1}^{m}\left(1+\sum_{j=1}^{m}|b_{ij}|l_j q_i q_j^{-1}\sum_{p=1}^{r_{ij}}k_{ij}(p)(\gamma^*)^{p+1}\right)\max_{i\in N(1,m)}\left\{\sup_{s\in N(-k,0)} y_i(s)\right\} \\
:= \;& C_0\max_{i\in N(1,m)}\left\{\sup_{s\in N(-k,0)} y_i(s)\right\}.
\end{aligned}
$$

It follows from (3.6) that

$$
\begin{aligned}
\max_{i\in N(1,m)}\left\{\sup_{s\in N(-k,0)} y_i(s)\right\} &= \max_{i\in N(1,m)}\left\{\sup_{s\in N(-k,0)} |q_i(\gamma^*)^s x_i(s)|\right\} \\
&\leq \max_{i\in N(1,m)}\left\{q_i\sup_{s\in N(-k,0)} |\phi_i(s)|\right\}.
\end{aligned}
$$

Thus, we have

$$
\sum_{i=1}^{m} y_i(n) \;\leq\; V(n)\leq C_0\max_{i\in N(1,m)}\left\{q_i\sup_{s\in N(-k,0)} |\phi_i(s)|\right\}.
$$

Put

$$
C:=C_0\frac{\max\{q_i, i\in N(1,m)\}}{\min\{q_i, i\in N(1,m)\}}. \tag{3.8}
$$

Then, using (3.6), we finally obtain

$$
\sum_{i=1}^{m}|x_i(n)|\leq C\left(\frac{1}{\gamma^*}\right)^n\max_{i\in N(1,m)}\left\{\sup_{s\in N(-k,0)} |\phi_i(s)|\right\}.
$$

This completes the proof. □

If we denote $A=diag(\alpha_1,\alpha_2,\cdots,\alpha_m)$, $W=(w_{ij})$, $\Lambda=diag(\beta_1,\beta_2,\cdots,\beta_m)$, $B=(b_{ij})$, $\hat{B}=(b_{ij}\kappa_{ij})$ and $L=diag(l_1,l_2,\cdots,l_m)$. Then we have

Corollary 3.2 *Assume that $(H_1)-(H_3)$ hold. If one of the following condition holds, then (1.1) is exponentially stable.*

(1) *the matrix* $\left(I-(A+|W|\Lambda+|\hat{B}|L)\right)^T$ *is diagonally dominant;*

(2) *the matrix* $\left(I-(A+|W|\Lambda+|\hat{B}|L)\right)$ *is diagonally dominant;*

(3)

$$
\rho\left(I-(A+|W|\Lambda+|\hat{B}|L)\right)<1; \tag{3.9}
$$

(4)

$$
Re\left(\lambda\left(I-(A+|W|\Lambda+|\hat{B}|L)\right)\right)>0; \tag{3.10}
$$

(5) *there exist some real positive numbers* $p_i, i \in N(1,m)$ *such that*

$$\alpha_i + \sum_{j=1}^{m} \beta_j |w_{ij}| p_j p_i^{-1} + \sum_{j=1}^{m} |b_{ij}| l_j p_j p_i^{-1} < 1, \quad i \in N(1,m); \tag{3.11}$$

(6)

$$\alpha_i + \sum_{j=1}^{m} \beta_i |w_{ji}| + \sum_{j=1}^{m} |b_{ji}| l_i < 1, \quad i \in N(1,m); \tag{3.12}$$

(7)

$$\alpha_i + \sum_{j=1}^{m} \beta_j |w_{ij}| + \sum_{j=1}^{m} |b_{ij}| l_j < 1, \quad i \in N(1,m). \tag{3.13}$$

Proof Note that (1) $\Longleftrightarrow$ (3.1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (5). Moreover, since the matrix $A + |W|\Lambda + |\hat{B}|L$ is a positive matrix, from [4], we know that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). Therefore, the conclusion is true if each of (1) $-$ (5) holds. Let the positive real numbers $q_i, p_i, i \in N(1,m)$ be 1 in (3.1) and (3.11), respectively, we immediately have (3.12) and (3.13). Hence the proof is completed. □

4 Applications to discrete-time neural networks

Incorporating time delays and adding a fixed input $J = (J_1, \cdots, J_m)^T$ from outside of the network into (1.2), we have the following discrete-time neural network model with delays

$$x_i(n+1) = -a_i x_i(n) + \sum_{j=1}^{m} b_{ij} g_j(x_j(n - r_{ij})) + J_i. \tag{4.1}$$

Here $g_j(0), j \in N(1,m)$ are not necessarily be zero. Instead, we assume that

(H_4) for each $i \in N(1,m)$, g_i is bounded with $|g_i| \le M_i$.

Now we first establish an existence result for the equilibrium of system (4.1).

Theorem 4.1 *Assume that* (H_2) *and* (H_4) *hold. If* $a_i \ne -1, i \in N(1,m)$, *then there exists an equilibrium for system (4.1).*

Proof We know that x^* is an equilibrium of (4.1) if and only if $x^* = (x_1^*, \cdots, x_m^*)^T$ is a solution of the following system

$$-a_i x_i + \sum_{j=1}^{n} b_{ij} g_j(x_j) + J_i = x_i, i \in N(1,m). \tag{4.2}$$

From (H_4), we have

$$|\sum_{j=1}^{m} b_{ij} g_j(x_j) + J_i| \le \sum_{j=1}^{m} |b_{ij}| M_j + |J_i| =: P_i.$$

Consider

$$x_i = h_i(x_1, x_2, \cdots, x_m) = \frac{1}{a_i + 1} \left(\sum_{j=1}^{m} b_{ij} g_j(x_j) + J_i \right)$$

for $i \in N(1,m)$. We have

$$|h_i(x_1, x_2, \cdots, x_n)| \le \max\{\frac{P_i}{1+a_i}, -\frac{P_i}{1+a_i}\} =: D_i, \text{ for } i \in N(1,m).$$

It follows that $(h_1, h_2, \cdots, h_m)^T$ maps a bounded set $D := [-D_1, D_1] \times [-D_2, D_2] \times \cdots \times [-D_m, D_m]$ into itself. Then the existence of the equilibrium follows from the Brouwer's fixed point theorem (Theorem 3.2, [3]). The proof is thus completed. $\square$

Let x^* be an equilibrium of (4.1) and substitute $x(n)$ with $u(n) + x^*$ into (4.1) resulting

$$u_i(n+1) = -a_i u_i(n) + \sum_{j=1}^{m} b_{ij} g_j^*(u_j(n - r_{ij})), i \in N(1, m) \tag{4.3}$$

where $g_j^*(u_j(n)) = g_j(u_j(n) + x_j^*) - g_j(x_j^*)$.

Note that (4.3) is a special case of (1.1) with $a_i(x) = -a_i x, W = 0, k_{ij}(p) = 0, p \in N(1, r_{ij} - 1), k_{ij}(r_{ij}) = 1$ and thus $\alpha_i = |a_i|, \quad \kappa_{ij} = 1$ for $i, j \in N(1, m)$. Applying Theorem 3.1 to system (4.3), we have

Theorem 4.2 *Assume that (H_2) and (H_4) hold. If there exist $q_i > 0, i \in N(1, m)$ such that*

$$|a_i| + \sum_{j=1}^{m} \beta_i |b_{ji}| \frac{q_j}{q_i} < 1, \text{ for } i \in N(1, m), \tag{4.4}$$

then, for any input J, the equilibrium x^* of (4.1) is globally exponentially stable in the sense that the following inequality holds:

$$\sum_{i=1}^{m} |x_i(n) - x_i^*| \le C \left(\frac{1}{\lambda^*}\right)^n \max_{i \in N(1,m)} \left\{ \sup_{s \in N(-k,0)} |\phi_i(s) - x_i^*| \right\}, \tag{4.5}$$

where $C > 0$ and $\lambda^* > 1$ can be similarly given as in Theorem 3.1.

Proof The proof can be easily completed by applying Theorem 3.1 to (4.3). $\square$

Consequently, applying Corollary 3.2 to (4.3), we have

Corollary 4.3 *Assume that (H_2) and (H_4) hold. If one of the following condition is satisfied*

(i) *the matrix $(I - (A + |B|\Lambda))^T$ or $(I - (A + |B|\Lambda))$ is an M-matrix;*

(ii)

$$\rho (A + |B|\Lambda) < 1; \tag{4.6}$$

(iii)

$$Re (\lambda (I - (A + |B|\Lambda))) > 0; \tag{4.7}$$

(iv) *there exist real numbers $p_i, i \in N(1, m)$ such that*

$$|a_i| + \sum_{j=1}^{m} \beta_j |b_{ij}| \frac{q_j}{q_i} < 1, \quad i \in N(1, m); \tag{4.8}$$

(v)

$$|a_i| + \sum_{j=1}^{m} |b_{ij}| \beta_j < 1, \quad i \in N(1, m); \tag{4.9}$$

or

$$|a_i| + \sum_{j=1}^{m} |b_{ji}| \beta_i < 1, \quad i \in N(1, m), \tag{4.10}$$

then every solution of (4.1) exponentially converges to its unique equilibrium.

Remark 4.4 Note that Jin and Gupta [9] obtained similar results for the global stability of the corresponding differentiable system without delays. Here we extend their results to not necessarily differentiable delayed difference system and indeed we can obtain global exponential stability.

As in [13], we next consider

$$x_i(n+1) = e^{-a_i h} x_i(n) + \psi_i(h) \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \psi_i(h) J_i, \tag{4.11}$$

and

$$x_i(n+1) = e^{-a_i h} x_i(n) + \psi_i(h) \sum_{j=1}^{m} b_{ij} f_j(x_j(n - r_{ij})) + \psi_i(h) J_i, \tag{4.12}$$

with $\psi_i(h) = \frac{1-e^{-a_i h}}{a_i}$, for $i \in N(1,m), n \in N(0)$. Let $x_i(n) = \psi_i(h) y_i(n)$, then (4.11) and (4.12) reduce to

$$y_i(n+1) = e^{-a_i h} y_i(n) + \sum_{j=1}^{m} b_{ij} f_j(\psi_j(h) y_j(n)) + J_i, \tag{4.13}$$

and

$$y_i(n+1) = e^{-a_i h} y_i(n) + \sum_{j=1}^{m} b_{ij} f_j(\psi_j(h) y_j(n - r_{ij})) + J_i. \tag{4.14}$$

Using a similar way as in the above theorem and employing Corollary 4.3, we have

Corollary 4.5 *Assume that $a_i > 0, h > 0, f_i$ is globally Lipschitz continuous and bounded satisfying (H_2) and (H_4) for each $i \in N(1,m)$. If one of the following conditions is satisfied, then every solution of (4.11) (or (4.12)) exponentially converges to its unique equilibrium x^*.*

(I) *there exist real numbers $p_i, i \in N(1,m)$ such that*

$$a_i q_i > \sum_{j=1}^{m} l_j |b_{ij}| q_j, \quad i \in N(1,m); \tag{4.15}$$

or

$$a_i q_i > \sum_{j=1}^{m} l_i |b_{ji}| q_j, \quad i \in N(1,m); \tag{4.16}$$

(II)

$$\rho\left(diag(a_1, a_2, \cdots, a_m) - |B|L\right) < 1; \tag{4.17}$$

(III)

$$a_i > \sum_{j=1}^{m} |b_{ij}| l_j, \quad i \in N(1,m); \tag{4.18}$$

or

$$a_i > \sum_{j=1}^{m} |b_{ji}| l_i, \quad i \in N(1,m). \tag{4.19}$$

Indeed, we have

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\psi_i(h)} \le C \left(\frac{1}{\lambda^*}\right)^n \max_{i \in N(1,m)} \left\{ \sup_{s \in N(-k,0)} \frac{|x_i(s) - x_i^*|}{\psi_i h} \right\}. \tag{4.20}$$

Proof The existence of equilibrium x^* of (4.11) ((4.12)) is equivalent to that of (4.13) ((4.14)), which can be similarly proved. Suppose that y^* is an equilibrium of (4.13)((4.14)). The transformation $u_i(n) = y_i(n) - y_i^*, i \in N(1,m)$ leads to

$$u_i(n+1) = e^{-a_i h} u_i(n) + \sum_{j=1}^{m} b_{ij} f_j^*(\psi_j(h) u_j(n)), \tag{4.21}$$

with $f_i^*(u_i(n)) = f_i(u_i(n) + y_i^*) - f_i(y_i^*), i \in N(1,m)$. The proof can be easily completed by employing Corollary 4.3. For example, (4.16) implies

$$((1 - e^{-a_i h})(a_i q_i - l_i \sum_{j=1}^{m} |b_{ji}| q_j) > 0,$$

which gives

$$e^{-a_i h} + \psi_i(h) l_i \sum_{j=1}^{m} |b_{ji}| q_j q_i^{-1} < 1.$$

Hence the conclusion immediately follows from Corollary 4.3 and thus the proof is complete. □

Remark 4.6 Note that condition (4.19) was derived in [13] for the global exponential stability and which is just a special case of (4.16).

Now we are ready to give some examples to demonstrate our results.

Example 4.7 Consider

$$\begin{cases} x_1(n+1) &= a_1 x_1(n) + b_{11} g_1(x_1(n - r_{11})) + b_{12} g_2(x_2(n - r_{12}) + J_1 \\ x_2(n+1) &= a_2 x_2(n) + b_{21} g_1(x_1(n - r_{21})) + a_{22} g_2(x_2(n - r_{22}) + J_2 \end{cases} \tag{4.22}$$

$n \in N(0)$, where g_1 and g_2 are bounded Lipschitz continuous with $\beta_1 = \beta_2 = 1$.

If we let $a_1 = -1/2, a_2 = 1/2$, and $b_{11} = 1/4, b_{12} = 1/8, b_{21} = 1/4, b_{22} = 1/16$. Then it is easy to check that

$$|a_1| + |b_{11}| + |b_{12}| = 7/8 < 1, \quad |a_2| + |b_{21}| + |b_{22}| = 13/16 < 1.$$

Therefore, v) in Corollary 4.3 holds and thus for any given fixed input $J = (J_1, J_2)^T$, (4.22) has a unique equilibrium which is exponentially stable independent of delays $r_{ij},\ i,j = 1,2$. A numeric simulation is given in **Fig. 1**.

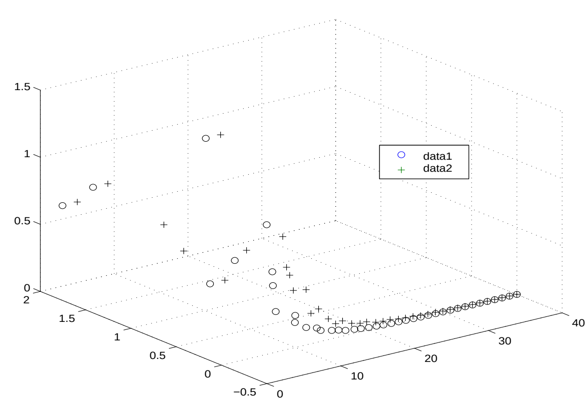

Figure 1 Numeric solutions of (4.22).

If we let $a_1 = a_2 = a, b_{11} = b_{22} = 0, b_{12} = b_{21} = b, r_{12} = r_{21}$ with $a + b < 1$, then Corollary 4.3 shows that the corresponding system has a unique equilibrium to which every solution converges. In this case if $a + b > 1$, the system may admit periodic solutions, for instance, if $a = 1/4$ and $b = 2$, $g_1(x) = g_2(x) = 1/2(|x+1| - |x-1|)$, there always exists a period two solution, say, $\{(1.6, -1.6)^T, (-1.6, 1.6)^T\}$.

Example 4.8 Consider

$$\left.\begin{array}{rcl} x_i(n+1) & = & e^{-a_i} x_i(n) + \psi_i(1) \sum_{j=1}^{3} b_{ij} \tanh(x_j(n - r_{ij}) \\ & & +\psi_i(1) J_i, \quad i = 1,2,3, \quad n \in N(0) \end{array}\right\} \tag{4.23}$$

with

$$\left.\begin{array}{lllll} a_1 = 1.0, & b_{11} = 0.2, & b_{12} = -0.4, & b_{13} = 1.4, & l_1 = 1 \\ a_2 = 2.5, & b_{21} = 0.5, & b_{22} = -0.5, & b_{23} = -1.5, & l_2 = 1 \\ a_3 = 3.0, & b_{31} = -0.3, & b_{32} = 0.6, & b_{33} = 0.1, & l_3 = 1 \end{array}\right\} \tag{4.24}$$

In this example, as we can see,

$$a_1 = |b_{11}| + |b_{21}| + |b_{31}| = 1, \ \ a_3 = |b_{13}| + |b_{23}| + |b_{33}| = 3.0,$$

which shows that (4.19) does not hold, that is, the main condition in [13] can not be satisfied and the result (Theorem 4.2, [13]) is not applicable. However, if we let $q_1 = 1, q_2 = 0.5$ and $q_3 = 0.8$, then

$$\left.\begin{array}{l} a_1 q_1 = 1.0 > \sum_{j=1}^{3} |b_{j1}| q_j = 0.69 \\ a_2 q_2 = 1.25 > \sum_{j=1}^{3} |b_{j2}| q_j = 1.13 \\ a_3 q_3 = 2.4 > \sum_{j=1}^{3} |b_{j3}| q_j = 2.23 \end{array}\right\} \tag{4.25}$$

which shows that (4.16) holds and thus (4.23) admits a unique equilibrium x^* for any fixed input J and every solution of it exponentially converges to x^* independent of choice of delays $r_{ij}, i, j \in N(1,3)$. The corresponding numeric simulations are given in **Fig. 2**.

Example 4.9 Consider a discrete-time neural network model with distributed delays

$$\left.\begin{array}{rl} x_i(n+1) = & a_i x_i(n) + \sum_{j=1}^{2} b_{ij} tanh\left(\sum_{p=1}^{r_{ij}} k_{ij}(p) x_j(n-p)\right) \\ & i = 1,2, \ n \in N(0) \end{array}\right\} \tag{4.26}$$

where

$$\left.\begin{array}{llll} a_1 = 1/2, & b_{11} = 1/4, & b_{12} = 1/8, & l_1 = 1 \\ a_2 = -1/4, & b_{21} = 1/4, & b_{22} = -1/4, & l_2 = 1 \end{array}\right\} \tag{4.27}$$

$$r_{11} = 2, \ \ r_{12} = 4, \ \ r_{21} = 5, \ \ r_{22} = 6 \tag{4.28}$$

$$\left.\begin{array}{llll} k_{11}(1) = 0.9, & k_{11}(2) = 0.1; & & \\ k_{12}(1) = 0.6, & k_{12}(2) = 0.3, & k_{12}(3) = 0.2, & k_{12}(4) = 0.1; \\ k_{21}(1) = 0.5, & k_{21}(2) = 0.4, & k_{21}(3) = 0.1, & k_{21}(4) = 0.05, \\ k_{21}(5) = 0.05; & & & \\ k_{22}(1) = 0.6, & k_{22}(2) = 0.3, & k_{22}(3) = 0.2, & k_{22}(4) = 0.2, \\ k_{22}(5) = 0.1, & k_{22}(6) = 0.1. & & \end{array}\right\} \tag{4.29}$$

It is easy to check that

$$\kappa_{11} = 1, \ \kappa_{12} = 1.2, \ \kappa_{21} = 1.1, \ \kappa_{22} = 1.5,$$

and

$$a_1 + |b_{11}| \kappa_{11} l_1 + |b_{12}| \kappa_{12} l_2 = 7/8 < 1,$$
$$a_2 |b_{21}| \kappa_{21} l_1 + |b_{22}| \kappa_{22} l_2 = 9/10 < 1.$$

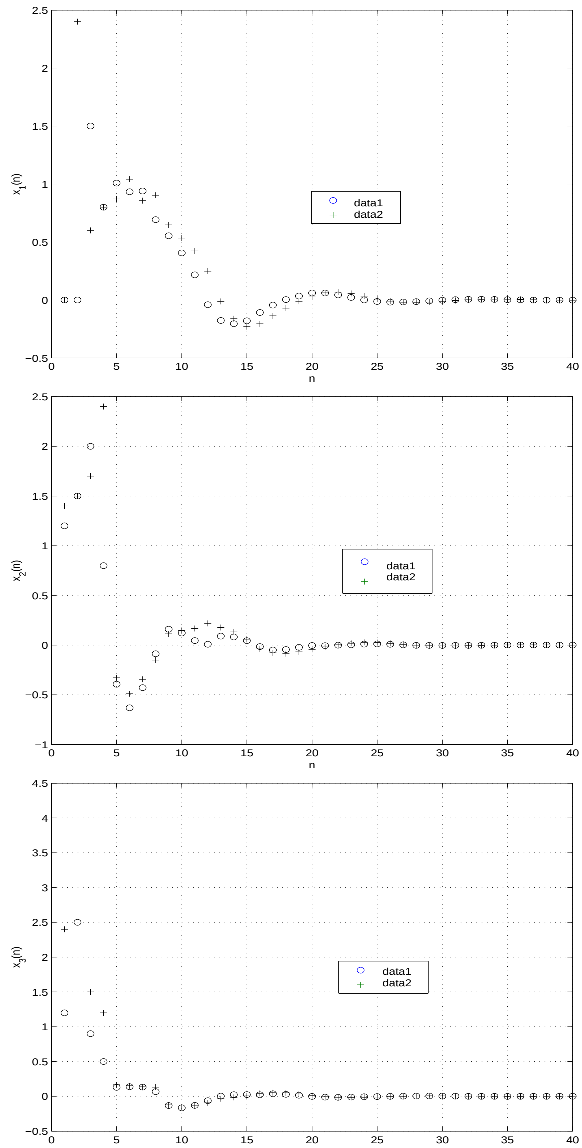

Figure 2 Exponential stability of discrete-time neural network (4.23)

Thus, (3.11) is satisfied and Corollary 3.2 shows that all solutions of (4.26) exponentially converge to zero. This conclusion is confirmed in our numerical simulation **Fig. 3**.

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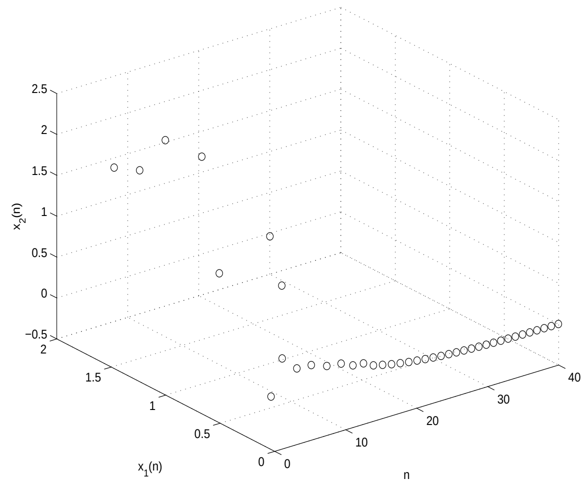

Figure 3 Numeric solutions of (4.26).

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