Fields Institute Communications
Volume **00**, 0000

3/2 Type criteria for global attractivity of Lotka-Volterra discrete system with delays

X. H. Tang
Department of Applied Mathematics,Central South University,Changsha, Hunan 410083, P.R.China
xhtang@public.cs.hn.cn

Lin Wang and Xingfu Zou
Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NF, A1C5S7, Canada
lin@math.mun.ca, xzou@math.mun.ca

Abstract. This paper deals with a two-species competition system of discrete Lotka-Volterra type with delays. Motivated by an existing 3/2 global attractivity result for the *scaler* discrete logistic model with delay, we establish a new 3/2 type criterion for global attractivity of the positive equilibrium of the *system*.

1 Introduction

In [11], May first proposed the following discrete two-species competition model of Lotka-Volterra type

$$\begin{aligned} x_{n+1} &= x_n \exp[r_1 - a_{11}x_n - a_{12}y_n] \\ y_{n+1} &= y_n \exp[r_2 - a_{21}x_n - a_{22}y_n]. \end{aligned} \tag{1.1}$$

By rescaling, (1.1) can be rewritten as

$$\begin{aligned} x_{n+1} &= x_n \exp[r_1(1 - x_n - \mu_1 y_n)] \\ y_{n+1} &= y_n \exp[r_2(1 - \mu_2 x_n - y_n)]. \end{aligned} \tag{1.2}$$

Since May [11], system (1.1) or (1.2) has attracted great attention and interest of many authors, and it has been found that the system can demonstrate quite rich and complicated dynamics. For example, (1.1) can admit limit cycle, various bifurcations and even chaotic oscillations. For details of the various dynamics for (1.1), we refer to [1-4, 7-8, 10-14] and the references therein. However, a mathematical model is expected to serve at least two purposes: (i) explaining what is observed in

2000 *Mathematics Subject Classification.* 39A11 92D25.
This work is supported by NNSF of China and NSERC of Canada .

field or from experiments; (ii) predicting what would happen under certain circumstances. Keeping in mind these basic purposes, complicated and chaotic dynamics of the model is not what one would like to see. Instead, one would like to see some nice properties, among which are permanence and stability. Permanence is the basic requirement for a reasonably proposed population growth model, while existence of a globally asymptotically stable positive equilibrium accounts for co-existence of the species involved. In a recent paper, Lu and Wang [9] addressed these two properties for a more general system that includes (1.1) as well as a cooperative version of (1.1). More precisely, they proved the following results (Lu and Wang [9, Theorem 2-3]).

Theorem 1.1 *Assume*

$$\mu_1 < 1 \quad \text{and} \quad \mu_2 < 1. \tag{1.3}$$

Then
(i) system (1.2) is permanent;
(ii) the positive equilibrium (x^*, y^*) *is the global attractor of (1.2), provided that* $r = \max\{r_1, r_2\}$ *is sufficiently small, where*

$$x^* = \frac{1-\mu_1}{1-\mu_1\mu_2}, \qquad y^* = \frac{1-\mu_2}{1-\mu_1\mu_2}. \tag{1.4}$$

Note that for 1.1-(ii), no measurement or estimate for 'smallness' of r is given in [9]. But Smith [16, Proposition 6.1-(c)] established, among many other things, an estimate for r which is $r \le 1$. Applying the main result in Wang and Lu [18] to (1.2) also generates this estimate.

Related to 1.1-(i), Hafbauer et al [5], and Hutson and Moran [6] discussed permanence of more general systems of difference equations.

There have been many arguments and evidences that time delay always exists in real situations, and should be taken into consideration in modeling. For example, individuals of a species need time to grow mature, resources once consumed take time to recover, and the conversion of biomass of the captured prey into the biomass of the predator is also not instantaneous. For models of differential equations, there has been much work in the literature. For discrete models related to (1.2), Morris et al [12] incorporated delays into the system and observed chaotic behavior of the model. Recently, Saito et al [15] also introduced delays into (1.2) and considered the following system

$$\begin{aligned} x_{n+1} &= x_n \exp[r_1(1 - x_{n-k_1} - \mu_1 y_{n-l_2})] \\ y_{n+1} &= y_n \exp[r_2(1 - \mu_2 x_{n-l_1} - y_{n-k_2})] \end{aligned} \tag{1.5}$$

with initial conditions

$$\begin{array}{lll} x_{-i} \ge 0, & i = 0, 1, \cdots, \max\{k_1, l_1\}; & x_0 > 0, \\ y_{-i} \ge 0, & i = 0, 1, \cdots, \max\{k_2, l_2\}; & y_0 > 0, \end{array} \tag{1.6}$$

where $r_i > 0, \mu_i > 0, i, j = 1, 2$, and k_1, k_2, l_1, l_2 are non-negative integers. The main concern of Saito et al [15] is the permanence of (1.5). Here, the system (1.5) is said to be permanent if there exists a compact set D in the interior of R_+^2 such that any solution of (1.5)-(1.6) will ultimately stay in D. The main result of [15] is the the following necessary and sufficient condition.

Theorem 1.2 *The system (1.5) is permanent for all non-negative integers* k_1, k_2, l_1, l_2 *if and only if (1.3) holds.*

From the above two theorems, we see that as far as permanence is concerned, delays play no role for system (1.5). But, in respect to the global attractivity of the positive equilibrium (x^*, y^*) given by (1.4), it remains an open problem. This paper will address this problem. Note that (1.5) is a result of coupling of two scalar equations of the form

$$x_{n+1} = x_n \exp[r(1 - x_{n-k})]. \tag{1.7}$$

For (1.7), So and Yu [17, Corollary 3.2] showed that the if $r(k+1) \le 3/2$, then the positive equilibrium $x = 1$ of (1.7) is globally attractive. Thus, one naturally expects some criteria for the global attractivity of the positive equilibrium of (1.5) which are related to 3/2 and which reduce to the criterion in So and Yu [17, Corollary 3.2] when the coupling disappears (i.e., $\mu_1 = \mu_2 = 0$). This is the goal of this paper, and we will achieve this goal by proving the following theorem.

Theorem 1.3 *Assume that (1.3) holds and that*

$$r_i(k_i + 1) \le \frac{3(1-\mu)}{2(1+\mu)}, \qquad i = 1, 2, \tag{1.8}$$

where $\mu = \max\{\mu_1, \mu_2\}$. *Then the positive equilibrium point* (x^*, y^*) *given by (1.4) is a global attractor of (1.5)-(1.6).*

Remark 1.4 1.1 is included by 1.3. Moreover, 1.3 provides estimates for both r_i and k_i for $i = 1, 2$. It also shows that under the diagonally dominance condition (1.3), the off-diagonal delays $l_i,\ i = 1, 2$, have no impact on the global attractivity of the positive equilibrium.

Remark 1.5 When $\mu = 0$, 1.3 reproduces Corollary 3.2 in [17].

2 Preliminaries

Lemma 2.1 *Let* $0 < a, b \le 1, 0 < \mu < 1$. *The system of inequalities*

$$\begin{cases} y \le (a + \mu x) \exp\left[(1-\mu)x - \frac{(1-\mu)^2}{6(1+\mu)}x^2\right] - a \\ \\ x \le b - (b - \mu y) \exp\left[-(1-\mu)y - \frac{(1-\mu)^2}{6(1+\mu)}y^2\right] \end{cases} \tag{2.1}$$

has a unique solution: $(x, y) = (0, 0)$ *in the region* $D = \{(x, y) : 0 \le x < 1,\ 0 \le y < b/\mu\}$.

Proof Let

$$\varphi(x) = (1-\mu)x - \frac{(1-\mu)^2}{6(1+\mu)}x^2, \quad \psi(y) = (1-\mu)y + \frac{(1-\mu)^2}{6(1+\mu)}y^2.$$

Then (2.1) can be written as

$$\begin{cases} y \le (a + \mu x)e^{\varphi(x)} - a, \\ \\ x \le b - (b - \mu y)e^{-\psi(y)}. \end{cases} \tag{2.2}$$

Assume that (2.2) has another solution in the region D besides $(0, 0)$, say (x_0, y_0). Then $0 < x_0 < 1$ and $0 < y_0 < b/\mu$. Define two curves Γ_1 and Γ_2 as follows:

$$\Gamma_1 : \ y = (a + \mu x)e^{\varphi(x)} - a, \quad \Gamma_2 : \ x = b - (b - \mu y)e^{-\psi(y)}. \tag{2.3}$$

By direct calculation, we have for curve Γ_1:

$$\left.\frac{dy}{dx}\right|_{(0,0)} = a + (1-a)\mu < 1$$

and for curve Γ_2:

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1}{b + (1-b)\mu} > 1.$$

Hence Γ_2 lies above Γ_1 near (0,0). The existence of (x_0, y_0) implies that the curves Γ_1 and Γ_2 must intersect at some point(s) in the region D besides $(0,0)$. Let (x_1, y_1) be the first such point, i.e. x_1 is smallest. Then the slope of Γ_1 at (x_1, y_1) is not less than the slope of Γ_2 at (x_1, y_1), i.e.

$$[\mu + (a + \mu x_1)\varphi'(x_1)]e^{\varphi(x_1)} \geq \frac{1}{\mu + (b - \mu y_1)\psi'(y_1)} e^{\psi(y_1)}$$

or

$$[\mu + (a + \mu x_1)\varphi'(x_1)][\mu + (b - \mu y_1)\psi'(y_1)] \geq e^{\psi(y_1) - \varphi(x_1)}. \tag{2.4}$$

Now we claim that

$$x_1 < y_1. \tag{2.5}$$

It follows from (2.3) that $x_1 < b$ so that (2.5) is true if $y_1 \geq b$. Next we may assume that $y_1 < b$ and then from (2.3), we have

$$\begin{aligned}
-\ln\left(1 - \frac{x_1}{b}\right) &= -\ln\left(1 - \frac{\mu y_1}{b}\right) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)} y_1^2 \\
&< \left(\frac{\mu}{b} y_1 + \frac{\mu^2}{2b^2} y_1^2 + \frac{\mu^3}{3b^3} y_1^3 + \cdots\right) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)} y_1^2 \\
&\leq \frac{1}{b} y_1 + \frac{1}{2b^2} y_1^2 + \frac{1}{3b^3} y_1^3 + \cdots \\
&= -\ln\left(1 - \frac{y_1}{b}\right).
\end{aligned}$$

This implies that (2.5) holds and thus so does the claim. Using (2.5), we derive that

$$\begin{aligned}
& [\mu+(a+\mu x_1)\varphi'(x_1)][\mu+(b-\mu y_1)\psi'(y_1)] \\
\leq\; & [\mu+(1+\mu x_1)\varphi'(x_1)][\mu+(1-\mu y_1)\psi'(y_1)] \\
=\; & 1+\left[\frac{(1-\mu)^2}{3(1+\mu)}-\mu(1-\mu)\right](y_1-x_1)-\left[\frac{(1-\mu)^2}{3(1+\mu)}-\mu(1-\mu)\right]^2 x_1y_1 \\
& -\frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2+y_1^2)+\frac{\mu(1-\mu)^3}{3(1+\mu)}\left[\frac{1-\mu}{3(1+\mu)}-\mu\right]x_1y_1(y_1-x_1) \\
& +\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2y_1^2 \\
<\; & 1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)-\frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2+y_1^2) \\
& +\frac{\mu(1-\mu)^4}{9(1+\mu)^2}x_1y_1(y_1-x_1)+\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2y_1^2 \\
:=\; & 1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)+h(\mu,x_1,y_1) \\
\leq\; & 1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)
\end{aligned}$$

since that

$$\begin{aligned}
h(\mu,x_1,y_1) \;=\; & -\frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2+y_1^2)+\frac{\mu(1-\mu)^4}{9(1+\mu)^2}x_1y_1(y_1-x_1) \\
& +\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2y_1^2 \\
=\; & -\frac{\mu(1-\mu)^2}{9(1+\mu)^2}y_1^2\left[3(1+\mu)-(1-\mu)^2x_1\right] \\
& +\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2\left[y_1^2-\frac{1}{\mu}y_1-\frac{3(1+\mu)}{\mu(1-\mu)^2}\right] \\
\leq\; & -\frac{\mu(1-\mu)^2}{9(1+\mu)^2}y_1^2\left[3(1+\mu)-(1-\mu)^2\right] \\
& +\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2\max_{y_1\in[0,\frac{1}{\mu})}\left\{y_1^2-\frac{1}{\mu}y_1-\frac{3(1+\mu)}{\mu(1-\mu)^2}\right\} \\
\leq\; & -\frac{\mu(1-\mu)^2}{9(1+\mu)^2}(3\mu+2)y_1^2-\frac{\mu(1-\mu)^2}{3(1+\mu)}x_1^2 \\
\leq\; & 0 \text{ (in the proof we used the fact that} \\
& 0\leq x_1<1,\ 0\leq y_1<\frac{b}{\mu}\leq\frac{1}{\mu}\text{)}
\end{aligned}$$

and

$$\begin{aligned}
e^{\psi(y_1)-\varphi(x_1)} \;=\; & \exp\left[(1-\mu)(y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2)\right] \\
>\; & 1+(1-\mu)(y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
& e^{\psi(y_1)-\varphi(x_1)} - [\mu + (a+\mu x_1)\varphi'(x_1)][\mu + (b-\mu y_1)\psi'(y_1)] \\
> & \left[1 + (1-\mu)(y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2)\right] \\
& - \left[1 + (1-\mu)\left(\frac{1-\mu}{3(1+\mu)} - \mu\right)(y_1 - x_1)\right] \\
= & (1-\mu)\left[1 + \mu - \frac{1-\mu}{3(1+\mu)}\right](y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2) \\
> & 0,
\end{aligned}$$

which contradicts (2.4). The proof is complete. □

Lemma 2.2 *Assume that*

$$\mu_1 < 1, \qquad \mu_2 < 1. \tag{2.6}$$

Let $(\{x_n\}, \{y_n\})$ *be the solution of (1.5) and (1.6). Then*

$$0 < \liminf_{n\to\infty} x_n(t) \le \limsup_{n\to\infty} x_n(t) < \infty, \tag{2.7}$$

and

$$0 < \liminf_{n\to\infty} y_n(t) \le \limsup_{n\to\infty} y_n(t) < \infty. \tag{2.8}$$

Proof This is a direct result of 1.2. □

3 The Proof of Theorem

By the transformation

$$\bar{x}_n = x_n - x^*, \qquad \bar{y}_n = y_n - y^*,$$

system (1.5) becomes

$$\begin{aligned}
x_{n+1} + x^* &= (x_n + x^*)\exp[-r_1(x_{n-k_1} + \mu_1 y_{n-l_2})] \\
y_{n+1} + y^* &= (y_n + y^*)\exp[-r_2(\mu_2 x_{n-l_1} + y_{n-k_2})]
\end{aligned} \tag{3.1}$$

here we used x_n, y_n instead of $\bar{x}_n, \bar{y}_n$. Clearly, the global attractivity of (x^*, y^*) of system (1.5) is equivalent to that for (3.1),

$$\lim_{n\to\infty}(x_n, y_n) = (0, 0) \tag{3.2}$$

whenever $(x_0, y_0) > -(x^*, y^*)$. We will prove (3.2) in the following two cases:

Case 1. Both $x_{n-k_1} + \mu_1 y_{n-l_2}$ and $\mu_2 x_{n-l_1} + y_{n-k_2}$ are non-oscillatory. In this case, by the boundedness of $\{x_n\}, \{y_n\}$ and the fact $x_n + x^* > 0, y_n + y^* > 0$ (Lemma 2), $x_{n+1} - x_n$ and $y_{n+1} - y_n$ are sign-definite eventually which imply that $\{x_n\}$ and $\{y_n\}$ are monotonous eventually. It follows immediately that $x_n(t) \to c, y_n(t) \to d$ as $n \to \infty$, and

$$c + \mu_1 d = 0, \qquad \mu_2 c + d = 0,$$

which imply that $c = d = 0$, i.e., (3.2) holds.

Case 2. At least $x_{n-k_1} + \mu_1 y_{n-l_2}$ or $\mu_2 x_{n-l_1} + y_{n-k_2}$ is oscillatory, say, the former. Then there exists an infinite sequence $\{n_j\}$ of integers such that

$$x_{n_j-k_1} + \mu_1 y_{n_j-l_2} \le 0, \quad \text{and} \quad x_{n_j+1-k_1} + \mu_1 y_{n_j+1-l_2} \ge 0, \quad j = 1, 2, \cdots. \tag{3.3}$$

Set

$$V_1 = \liminf_{n\to\infty} x_n \quad U_1 = \limsup_{n\to\infty} x_n,$$

and

$$V_2 = \liminf_{n\to\infty} y_n \quad U_2 = \limsup_{n\to\infty} y_n.$$

In view of Lemma 2,

$$-x^* < V_1 \le U_1 < \infty, \quad \text{and} \quad -y^* < V_2 \le U_2 < \infty. \tag{3.4}$$

Let

$$-V = \min\{V_1, V_2\} \quad \text{and} \quad U = \max\{U_1, U_2\}.$$

Then from (3.3) and (3.4), we have

$$0 \le V < \max\{x^*, y^*\} < 1, \quad 0 \le U < \infty. \tag{3.5}$$

In what follows, we show that V and U satisfy the inequalities

$$a + U \le (a + \mu V)\exp\left[(1-\mu)V - \frac{(1-\mu)^2}{6(1+\mu)}V^2\right] \tag{3.6}$$

and

$$b - V \ge (b - \mu U)\exp\left[-(1-\mu)U - \frac{(1-\mu)^2}{6(1+\mu)}U^2\right]. \tag{3.7}$$

where $a, b = x^*$ or y^*. Without loss of generality, we may assume that $U = U_1$ and $V = -V_2$. Then $V < y^*$. Let $\epsilon > 0$ be sufficiently small such that $v_1 = V + \epsilon < \max\{x^*, y^*\}$. Choose an integer $N > 0$ such that

$$-v_1 < x_n, y_n < U + \epsilon \equiv u_1, \quad n \ge N - \max\{k_1, k_2, l_1, l_2\}. \tag{3.8}$$

Define two functions $x(t)$ and $y(t)$ as follows:

$$x(t) = (x_n + x^*)\left(\frac{x_{n+1} + x^*}{x_n + x^*}\right)^{t-n} - x^*, \qquad n \le t < n+1, \quad n = 0, 1, \cdots,$$

and

$$y(t) = (y_n + y^*)\left(\frac{y_{n+1} + y^*}{y_n + y^*}\right)^{t-n} - y^*, \qquad n \le t < n+1, \quad n = 0, 1, \cdots.$$

Then $x(t)$ and $y(t)$ are continuous on $[0, \infty)$ and differentiable except $t = 0, 1, 2, \ldots$, and satisfy

$$x(n) = x_n, \quad \min\{x_n, x_{n+1}\} \le x(t) \le \max\{x_n, x_{n+1}\}, \quad n \le t \le n+1,$$
$$y(n) = y_n, \quad \min\{y_n, y_{n+1}\} \le y(t) \le \max\{y_n, y_{n+1}\}, \quad n \le t \le n+1,$$

and

$$\begin{aligned} \dot{x}(t) &= -r_1(x(t) + x^*)(x([t-k_1]) + \mu_1 y([t-l_2])), \\ \dot{y}(t) &= -r_2(y(t) + y^*)(\mu_2 x([t-l_1]) + y([t-k_2])) \end{aligned} \tag{3.9}$$

where and in the sequel, $[x]$ denotes the greatest integer which is less or equal x, and $\dot{x}(t)$ and $\dot{y}(t)$ denote the left derivatives of $x(t)$ and $y(t)$, respectively. Set $v_2 = (1+\mu)v_1$ and $u_2 = (1+\mu)u_1$. Then from (3.9), we have

$$\frac{\dot{x}(t)}{x^* + x(t)} \le r_1(-x([t-k_1]) + \mu v_1] \le r_1 v_2, \quad n \ge N \tag{3.10}$$

and

$$\frac{\dot{y}(t)}{y^* + y(t)} \ge r_2(-\mu u_1 - y([t-k_2])) \ge -r_2 u_2, \quad n \ge N. \tag{3.11}$$

First, we prove that (3.6) holds. If $U \le \mu V$, then (3.6) obviously holds. Therefore, we will prove (3.6) only for the case when $U > \mu V$, which implies $U > \mu v_1$ (by choosing $\epsilon > 0$ sufficiently small). Thus, we cannot have $x(t) \le \mu v_1$ eventually. On the other hand, if $x(t) \ge \mu v_1$ eventually, then it follows from the first inequality in (3.10) that $x(t)$ is non-increasing and $U = \lim_{t\to\infty} x(t) = \mu v_1$. This is also impossible. Therefore, it follows that $x(t)$ oscillates about μv_1.

Let $\{p_j\}$ be an increasing sequence of integers such that $p_j \ge N + k_1, \dot{x}(p_j) \ge 0, x(p_j) \ge \mu v_1, \lim_{j\to\infty} p_j = \infty$ and $\lim_{j\to\infty} x(p_j) = U$. By (3.10), $x(p_j - k_1) \le \mu v_1$. Thus, there exists $\xi_j \in [p_j - k_1 - 1, p_j]$ such that $x(\xi_j) = \mu v_1$. For $t \in [\xi_j, p_j]$, integrating (3.10) from $[t - k_1]$ to ξ_j we get

$$-\ln \frac{x^* + x([t-k_1])}{x^* + x(\xi_j)} \le r_1 v_2(\xi_j + k_1 + 1 - t),$$

or

$$x([t-k_1]) \ge -x^* + (x^* + \mu v_1)\exp[-r_1 v_2(\xi_j + k_1 + 1 - t)], \quad \xi_j \le t \le p_j.$$

Substituting this into the first inequality in (3.10), we obtain

$$\frac{\dot{x}(t)}{x^* + x(t)} \le r_1(1 + \mu v_1)\{1 - \exp[-r_1 v_2(\xi_j + k_1 + 1 - t)]\}, \quad \xi_j \le t \le p_j.$$

Combining this with (3.10), we have

$$\frac{\dot{x}(t)}{x^* + x(t)} \le \min\{r_1 v_2, r_1(1+\mu v_1)\{1 - \exp[-r_1 v_2(\xi_j + k_1 + 1 - t)]\}\}, \quad \xi_j \le t \le p_j. \tag{3.12}$$

To prove (3.6), we consider the following two possible subcases.

Case 2.1. $r_1(p_j - \xi_j) \le -\frac{1}{v_2}\ln[1 - (1-\mu)v_1]$. Then by (1.8) and (3.12)

$$\begin{aligned}
& \ln \frac{x^* + x(p_j)}{x^* + \mu v_1} \\
\le\ & r_1(1+\mu v_1)(p_j - \xi_j) - r_1(1+\mu v_1)\int_{\xi_j}^{p_j} \exp[-r_1 v_2(\xi_j + k_1 + 1 - t)]dt \\
=\ & (1+\mu v_1) \\
& \left\{ r_1(p_j - \xi_j) - \frac{1}{v_2}\exp[-r_1 v_2(\xi_j + k_1 + 1 - p_j)][1 - \exp(-r_1 v_2(p_j - \xi_j))] \right\} \\
\le\ & (1+\mu v_1)\left\{ r_1(p_j - \xi_j) - \frac{1 - \exp(-r_1 v_2(p_j - \xi_j))}{v_2} \right. \\
& \left. \times \exp\left[-v_2\left(\frac{3(1-\mu)}{2(1+\mu)} - r_1(p_j - \xi_j) \right) \right] \right\}.
\end{aligned}$$

If

$$r_1(p_j - \xi_j) \le -\frac{1}{v_2}\ln[1 - (1-\mu)v_1] \le 3(1-\mu)/2(1+\mu),$$

then

$$
\begin{aligned}
& \ln \frac{x^* + x(p_j)}{x^* + \mu v_1} \\
\leq\ & (1+\mu v_1)\left\{ -\frac{1}{v_2} \ln[1-(1-\mu)v_1] \right. \\
& \left. -\frac{1-\mu}{1+\mu} \exp\left[-v_2 \left(\frac{3(1-\mu)}{2(1+\mu)} + \frac{\ln[1-(1-\mu)v_1]}{v_2} \right) \right] \right\} \\
\leq\ & (1+\mu v_1)\left\{ -\frac{1}{v_2} \ln[1-(1-\mu)v_1] \right. \\
& \qquad \left. -\frac{1-\mu}{1+\mu} \left[1 - v_2 \left(\frac{3(1-\mu)}{2(1+\mu)} + \frac{\ln[1-(1-\mu)v_1]}{v_2} \right) \right] \right\} \\
=\ & \frac{1+\mu v_1}{1+\mu} \\
& \left\{ -\frac{1}{v_1} \ln[1-(1-\mu)v_1] - (1-\mu)\left[1 - \frac{3(1-\mu)}{2} v_1 - \ln[1-(1-\mu)v_1] \right] \right\} \\
=\ & \frac{1+\mu v_1}{1+\mu} \left\{ -\frac{1}{v_1} [1-(1-\mu)v_1] \ln[1-(1-\mu)v_1] - (1-\mu) + \frac{3(1-\mu)^2}{2} v_1 \right\} \\
\leq\ & \frac{1+\mu v_1}{1+\mu} \left[(1-\mu)^2 v_1 - \frac{(1-\mu)^3}{6} v_1^2 \right] \\
<\ & (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)} v_1^2 .
\end{aligned}
$$

In the above third inequality, we have used the following inequality

$$
[1-(1-\mu)v_1] \ln[1-(1-\mu)v_1] \geq -(1-\mu)v_1 + \frac{(1-\mu)^2}{2} v_1^2 + \frac{(1-\mu)^3}{6} v_1^3 . \quad (3.13)
$$

If $r_1(p_j - \xi_j) \leq 3(1-\mu)/2(1+\mu) \leq -\frac{1}{v_2} \ln[1-(1-\mu)v_1]$, then

$$
\frac{3}{2}(1-\mu) \leq -\frac{1}{v_1} \ln[1-(1-\mu)v_1] \leq \frac{1-\mu}{1-(1-\mu)v_1} \left[1 - \frac{1-\mu}{2} v_1 - \frac{(1-\mu)^2}{6} v_1^2 \right],
$$

which implies that $(1-\mu)v_1 > 1/2$. Hence,

$$
\begin{aligned}
& \ln \frac{x^* + x(p_j)}{x^* + \mu v_1} \\
\leq\ & (1+\mu v_1) \left\{ \frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{v_2} \left[1 - \exp(-\frac{3}{2}(1-\mu)v_1 \right] \right\} \\
=\ & \frac{1+\mu v_1}{1+\mu} \left[\frac{3}{2}(1-\mu) - \frac{1}{v_1} \left(1 - e^{-3(1-\mu)v_1/2} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1+\mu v_1}{1+\mu}\Big(\frac{3}{2}(1-\mu)-[\frac{3}{2}(1-\mu)-\frac{9}{8}(1-\mu)^2v_1+\frac{9}{16}(1-\mu)^3v_1^2\\
&\quad -\frac{27}{128}(1-\mu)^4v_1^3]\Big)\\
&= \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[\frac{9}{8}(1-\mu)v_1-\frac{9}{16}(1-\mu)^2v_1^2+\frac{27}{128}(1-\mu)^3v_1^3\right]\\
&\le \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[(1-\mu)v_1-\frac{1}{6}(1-\mu)^2v_1^2\right]\\
&< (1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}
$$

Case 2.2. $-\frac{1}{v_2}\ln[1-(1-\mu)v_1]<r_1(p_j-\xi_j)\le 3(1-\mu)/2(1+\mu)$. Choose $l_j\in(\xi_j,p_j)$ such that $r_1(p_j-l_j)=-\frac{1}{v_2}\ln[1-(1-\mu)v_1]$. Then by (1.8) and (3.12),

$$
\begin{aligned}
&\ln\frac{x^*+x(p_j)}{x^*+\mu v_1}\\
\le\ & r_1v_2(l_j-\xi_j)+(1+\mu v_1)(r_1(p_j-l_j)\\
&-r_1\int_{l_j}^{p_j}\exp[-r_1v_2(\xi_j+k_1+1-t)]dt)\\
=\ & r_1v_2(l_j-\xi_j)+(1+\mu v_1)\\
&\times\left\{r_1(p_j-l_j)-\frac{1}{v_2}\exp[-r_1v_2(\xi_j+k_1+1-p_j)][1-\exp(-r_1v_2(p_j-l_j))]\right\}\\
=\ & r_1v_2(l_j-\xi_j)\\
&+(1+\mu v_1)\left\{r_1(p_j-l_j)-\frac{1-\mu}{1+\mu}\exp[-r_1v_2(\xi_j+k_1+1-p_j)]\right\}\\
\le\ & r_1v_2(l_j-\xi_j)\\
&+(1+\mu v_1)\left\{r_1(p_j-l_j)-\frac{1-\mu}{1+\mu}+\frac{1-\mu}{1+\mu}r_1v_2(\xi_j+k_1+1-p_j)\right\}\\
\le\ & r_1v_2(k_1+1)+(1-v_1)r_1(p_j-l_j)-\frac{1-\mu}{1+\mu}\\
=\ & r_1(k_1+1)v_2-\frac{1}{v_2}(1-v_1)\ln[1-(1-\mu)v_1]-\frac{1-\mu}{1+\mu}\\
\le\ & \frac{3}{2}(1-\mu)v_1-\frac{1}{1+\mu}\left[-(1-\mu)+\frac{(1-\mu)(1+\mu)}{2}v_1+\frac{(1-\mu)^2(1+2\mu)}{6}v_1^2\right]\\
&-\frac{1-\mu}{1+\mu}\\
=\ & (1-\mu)v_1-\frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}v_1^2\\
<\ & (1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}
$$

In the above fourth inequality, we have used the following inequality

$$
(1-v_1)\ln[1-(1-\mu)v_1]\ge -(1-\mu)v_1+\frac{(1-\mu)(1+\mu)}{2}v_1^2+\frac{(1-\mu)^2(1+2\mu)}{6}v_1^3.
$$

Combining Case 2.1 with Case 2.2, we have proved that

$$\ln\frac{x^*+x(p_j)}{x^*+\mu v_1}\le(1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2,\quad j=1,\ 2,\ \ldots.$$

Letting $j\to\infty$ and $\epsilon\to 0$, we have

$$\ln\frac{x^*+U}{x^*+\mu V}\le(1-\mu)V-\frac{(1-\mu)^2}{6(1+\mu)}V^2.$$

This shows that (3.6) holds. Next, we will prove that (3.7) holds as well. In the case where $V=0$, then it follows from (3.6) that $U=0$, the proof is complete. Hence, in what follows, we consider the case when $V>0$. From (3.6), we have

$$U<(a+\mu)e^{1-\mu}-a<2,\quad \mu U\le\mu[(a+\mu V)e^{(1-\mu)V}-a]<V<y^*. \tag{3.14}$$

Thus we may assume, without loss of generality, that $V>\mu u_1$. In view of this and (3.11), we can show that neither $y(t)\ge-\mu u_1$ eventually nor $y(t)\le-\mu u_1$ eventually. Therefore, $y(t)$ oscillates about $-\mu u_1$.

Let $\{q_j\}$ be an increasing sequence of integers such that $q_j\ge N+k_2, \dot y(q_j)\le 0, x(q_j)\le-\mu u_1, \lim_{j\to\infty}q_j=\infty$ and $\lim_{j\to\infty}y(q_j)=-V$. By (3.11), $y(q_j-k_2)\ge -\mu u_1$. Thus, there exists $\eta_j\in[q_j-k_2-1,q_j]$ such that $y(\eta_j)=-\mu u_1$. For $t\in[\eta_j,q_j]$, integrating (3.11) from $[t-k_2]$ to η_j, we have

$$y([t-k_2])\le(y^*-\mu u_1)\exp[r_2u_2(\eta_j+k_2+1-t)]-y^*,\quad \eta_j\le t\le q_j.$$

Substituting this into the first inequality in (3.11), we obtain

$$-\frac{\dot y(t)}{y^*+y(t)}\le r_2(1-\mu u_1)\{\exp[r_2u_2(\eta_j+k_2+1-t)]-1\},\quad \eta_j\le t\le q_j.$$

Combining this and (3.11), we have

$$-\frac{\dot y(t)}{y^*+y(t)}\le\min\{r_2u_2,r_2(1-\mu u_1)\{\exp[r_2u_2(\eta_j+k_2+1-t)]-1\}\},\quad \eta_j\le t\le q_j. \tag{3.15}$$

There are also two possibilities:

Case 2.3. $r_2(q_j-\eta_j)\le\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Integrating (3.15) from η_j to q_j and using the inequality

$$\ln[1+(1-\mu)u_1]\ge\frac{1}{2}(1-\mu)u_1-\frac{(1-\mu)^2}{6(1+\mu)}u_1^2,$$

we have

$$\begin{aligned}-\ln\frac{y^*+y(q_n)}{y^*-\mu u_1}&\le r_2u_2(q_j-\eta_j)\\ &\le u_2\left\{\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]\right\}\\ &= \frac{3}{2}(1-\mu)u_1-\ln[1+(1-\mu)u_1]\\ &\le (1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2.\end{aligned}$$

Case 2.4. $r_2(q_j-\eta_j)>\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Choose $h_j\in(\eta_j,q_j)$ such that

$$r_2(h_j-\eta_j)=\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1].$$

Then by (1.8) and (3.15) we have

$$
\begin{aligned}
& -\ln\frac{y^*+y(q_n)}{y^*-\mu u_1} \\
\leq\; & r_2u_2(h_j-\eta_j) \\
& +(1-\mu u_1)\left\{r_2\int_{h_j}^{q_j}\exp[r_2u_2(\eta_j+k_2+1-t)]dt-r_2(q_j-h_j)\right\} \\
=\; & r_2u_2(h_j-\eta_j)+(1-\mu u_1)\times\{\frac{1}{u_2}[\exp(r_2u_2(\eta_n+k_2+1-h_j)) \\
& -\exp(r_2u_2(\eta_j+k_2+1-q_j))]-r_2(q_j-h_j)\} \\
=\; & r_2u_2(h_j-\eta_j)-r_2(1-\mu u_1)(q_j-h_j)+\frac{1-\mu u_1}{u_2}e^{r_2(k_2+1)u_2} \\
& \left\{[1+(1-\mu)u_1]\exp\left(-\frac{3(1-\mu)}{2(1+\mu)}u_2\right)-e^{-r_2u_2(q_j-\eta_j)}\right\} \\
\leq\; & r_2u_2(h_j-\eta_j)-r_2(1-\mu u_1)(q_n-h_n) \\
& +\frac{1-\mu u_1}{u_2}\left\{1+(1-\mu)u_1-\exp\left[u_2\left(\frac{3(1-\mu)}{2(1+\mu)}-r_2(q_j-\eta_j)\right)\right]\right\} \\
\leq\; & r_2u_2(h_j-\eta_j)-r_2(1-\mu u_1)(q_j-h_j) \\
& +\frac{1-\mu u_1}{u_2}\left\{(1-\mu)u_1-(1+\mu)u_1\left[\frac{3(1-\mu)}{2(1+\mu)}-r_2(q_j-\eta_j)\right]\right\} \\
=\; & (1+u_1)r_2(h_j-\eta_j)-\frac{1-\mu}{2(1+\mu)}(1-\mu u_1) \\
=\; & \frac{3(1-\mu)}{2(1+\mu)}(1+u_1)-\frac{(1+u_1)\ln[1+(1-\mu)u_1]}{(1+\mu)u_1}-\frac{1-\mu}{2(1+\mu)}(1-\mu u_1) \\
\leq\; & \frac{1-\mu}{1+\mu}u_1+\frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}u_1^2 \\
\leq\; & (1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2.
\end{aligned}
$$

In the above fourth inequality, we have used the following inequality

$$(1+u_1)\ln[1+(1-\mu)u_1]\geq(1-\mu)u_1+\frac{(1-\mu)(1+\mu)}{2}u_1^2-\frac{(1-\mu)^2(1+2\mu)}{6}u_1^3.$$

Combining Case 2.3 with Case 2.4, we have shown that

$$-\ln\frac{y^*+y(q_j)}{y^*-\mu u_1}\leq(1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2,\quad j=1,\ 2,\ \ldots.$$

Letting $j\to\infty$ and $\epsilon\to 0$, we have

$$-\ln\frac{y^*-V}{y^*-\mu U}\leq(1-\mu)U+\frac{(1-\mu)^2}{6(1+\mu)}U^2,$$

which implies that (3.7) holds. In view of Lemma 1, it follows from (3.6) and (3.7) that $U=V=0$. Thus, (3.2) holds as well. The proof is complete.

4 Numeric Simulations

1.3 gives sufficient conditions under which the positive equilibrium is a global attractor of (1.5)-(1.6). But if (1.8) is not satisfied, the dynamics of the system could be very complicated. In this section, we provide some results of numerical simulations for the dynamics of (1.5)-(1.6).

Let us first consider the case when there is no delay: $k_i = l_i = 0, \ i = 1,2$. For convenience, we choose $r_1 = r_2 = r$ and $\mu_1 = \mu_2 = \mu$. If $\mu = 0.02$ and $r = 1.44 < \frac{3}{2}\frac{1-\mu}{1+\mu}$, then by Theorem 3, we know that (1.2) has a globally attractive positive equilibrium. When r is increased to $r = 2.5$, the global attractivity of the positive equilibrium is destroyed and the system allows periodic solutions with period 2, as is shown in Figure 1. If r is further increased to $r = 2.9$, chaotic dynamics is observed (see Figure 2).

We next consider the case when there are delays. As suggested by Theorem 2-3, the off-diagonal delays $l_i, \ i = 1,2$, play no role in the global attractivity of the positive equilibrium, and this is confirmed by our simulations (fixing $r_i, \ \mu_i$ and k_i for $i = 1,2$ so that the conditions of Theorem 3 are satisfied, but increasing $l_i, \ i = 1,2$ does not change the global attractivity of the positive equilibrium). In such a case, the system always demonstrate global convergence, and the dynamics is very simple, so we give no figure here. But if we fix r_i and l_i and increase the diagonal delays $k_i, \ i = 1,2$, then as the condition (1.8) becomes violated, complicated dynamics can be observed. To this end, we fix $l_1 = l_2 = 1$ and $r_1 = r_2 = 0.72$, and let $k_1 = k_2 = k$. When $k = 1$, condition (1.8) holds, and (1.5) has a global convergent dynamics as claimed in Theorem 3. Now if we increase k to 2 and 4 respectively, the numerical simulations for these two values are shown in Figure 3 and Figure 4 respectively. From these figures, we can see that the diagonal delays do destroy the global attractivity of the positive equilibrium and cause very complicated dynamics. Although it is not clear (at least to us) that what type of dynamics Figure 3 corresponds to, Figure 4 seems to present a chaotic dynamics again (but this time, caused by delay).

Finally, Figure 5 gives the numerics of the case when $r_1 = r_2 = 1, \ \ k_1 = k_2 = 1$ and $l_1 = l_2 = 2$. Here periodic dynamics (with period more than 2) is observed.

To conclude this paper, we would like to point out that the numeric simulations could not be complete, and other types of dynamics are also possible. We leave it for further work (maybe hard though) to theoretically detect and describe those various and complicated dynamics of system (1.5).

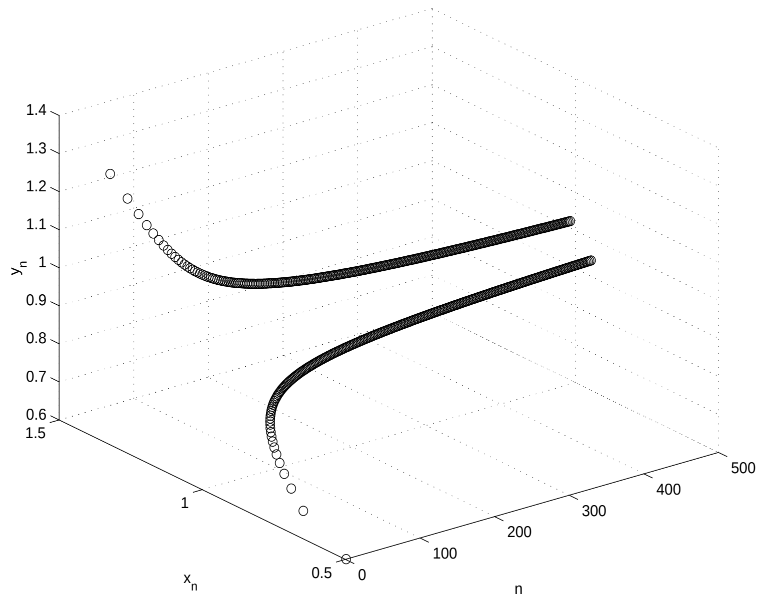

Figure 1 Numeric simulations for (1.5) with $r_1 = r_2 = 2$, $\mu_1 = \mu_2 = 0.02$, $l_i = k_i = 0, i = 1, 2$, and $x_0 = 0.5$, $y_0 = 0.6$.

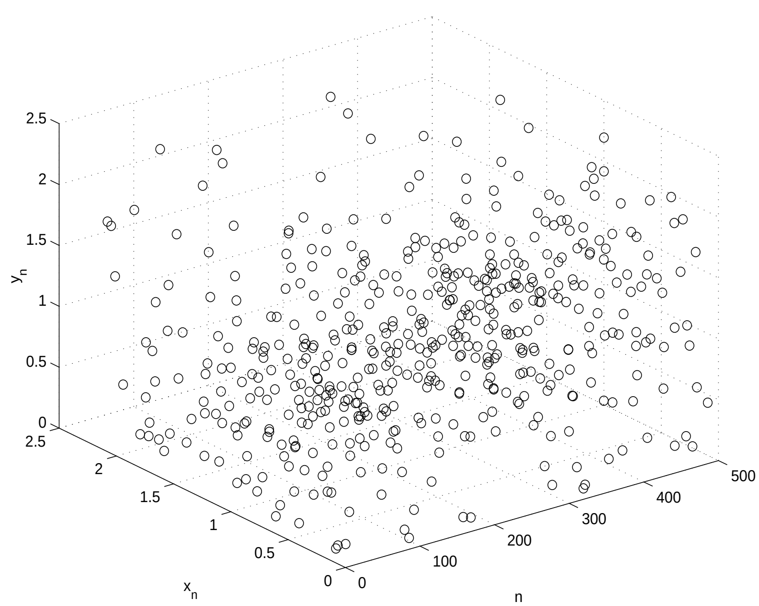

Figure 2 Numeric simulation for (1.5) with $r_1 = r_2 = 2.9$, $\mu_1 = \mu_2 = 0.02$, $l_i = k_i = 0, i = 1, 2$, and $x_0 = 0.5$, $y_0 = 0.6$

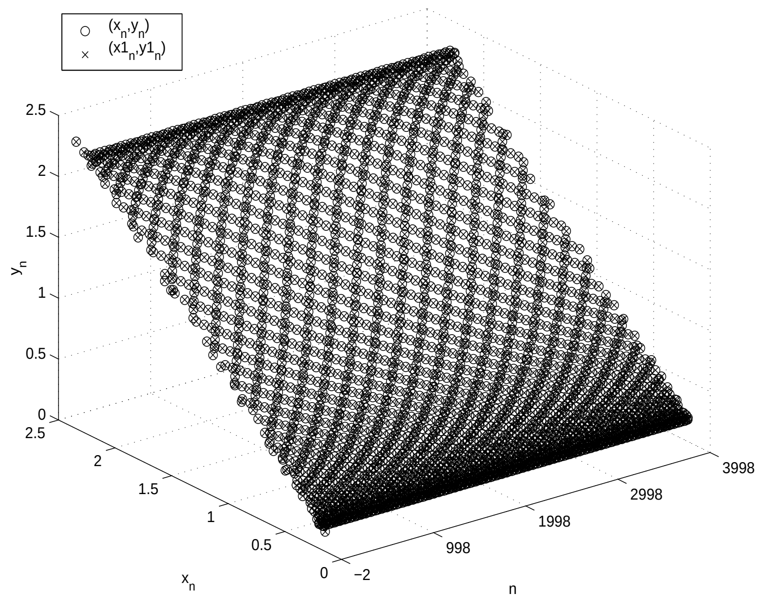

Figure 3 Numeric simulation for (1.5) with $r_1 = r_2 = 0.72$, $\mu_1 = \mu_2 = 0.02$, $l_1 = l_2 = 1$, $k_1 = k_2 = 2$. Two sets of initial data are used: (a) $x_{-2} = y_{-2} = 0.2$, $x_{-1} = y_{-1} = 0.2$, $x_0 = y_0 = 0.5$; (b) $x1_{-2} = y1_{-2} = 0.2$, $x1_{-1} = y1_{-1} = 0.2$, $x1_0 = 0.501$, $y1_0 = 0.5$.

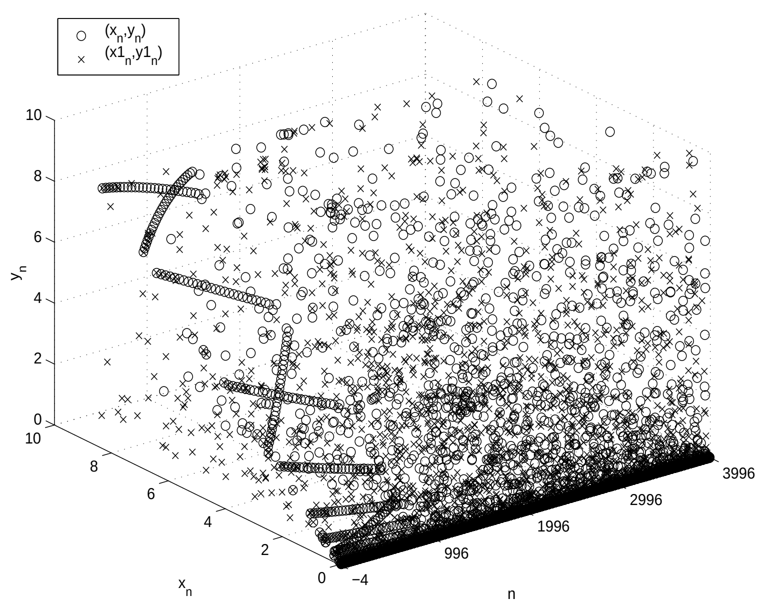

Figure 4 Numeric simulation for (1.5) with $r_1 = r_2 = 0.72$, $\mu_1 = \mu_2 = 0.02$, $l_1 = l_2 = 1$, $k_1 = k_2 = 4$. Two sets of initial data are used: (a) $x_{-4} = y_{-4} = 0.5$, $x_{-3} = y_{-3} = 0.6$, $x_{-2} = y_{-2} = 0.2$, $x_{-1} = y_{-1} = 0.2$, $x_0 = y_0 = 0.5$; (b) $x1_{-4} = y1_{-4} = 0.5$, $x1_{-3} = y1_{-3} = 0.6$, $x1_{-2} = y1_{-2} = 0.2$, $x1_{-1} = y1_{-1} = 0.2$, $x1_0 = 0.501$, $y1_0 = 0.5$.

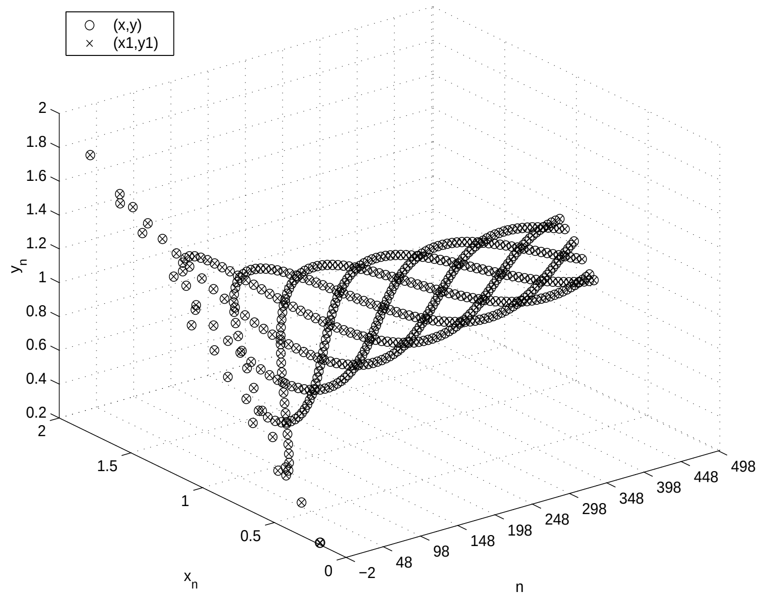

Figure 5 Numeric simulation for (1.5) with $r_1 = r_2 = 1$, $\mu_1 = \mu_2 = 0.02$, $l_1 = l_2 = 2$, $k_1 = k_2 = 1$. Two sets of initial data are used: (a) $x_{-2} = y_{-2} = 0.2$, $x_{-1} = y_{-1} = 0.2, x_0 = y_0 = 0.5$; (b) $x1_{-2} = y1_{-2} = 0.2$, $x1_{-1} = y1_{-1} = 0.2$, $x1_0 = 0.501$, $y1_0 = 0.5$.

References

[1] Diamond, P. *Chaotic behavior of systems of systems of difference equations.* Inter. J. Sys. Sci., **7** (1976), 953–956.

[2] Dohtani, A. *Occurrence of chaos in higher-dimensional discrete-time systems.* SIAM J. Appl. Math., **52** (1992), 1707–1721.

[3] Guckenheimer, J. Oster, G. and Ipacktchi, A. *The dynamics of density dependent population models.* J. Math. Biol., **4** (1977), 101–147.

[4] Hassel, M. P. and Comins, N. H. *Discreet time models for two-species competition.* Theor. Popul. Biol., **9** (1976), 202–221.

[5] Hofbauer, J. Hutson, V. and Jansen, W. *Coexistence for systems governed by difference equations of Lotka-Volterra type.* J. Math. Biol., **25** (1987), 553–570

[6] Hutson, V. and Moran, W. *Persistence of species obeying difference equations.* J. Math. Biol., **15** (1982), 203–213.

[7] Jiang, H. and Rogers, D. *The discrete dynamics of symmetric competition in the plane.* J. Math. Biol., **25** (1987), 573–596.

[8] Levin, S. A. and Goodyear, C. P. *Analysis of an age-structured fishery model.* J. Math. Biol., **9** (1980), 254–274.

[9] Lu, Z. Y. and Wang, W. D. *Permanence and global attractivity for Lotka-Volterra difference systems.* J. Math. Biol., **39** (1999), 269–282.

[10] Marotto, F. R. *Snapback repellers imply chaos in R^n.* J. Math. Anal. Appl., **63** (1978), 199–223.

[11] May, R. M. *Biological populations with nonoverlapping generations: stable points, stable cycles and chaos.* Science, **186** (1974), 645–647.

[12] Morris, H. C. Ryan,E. E. and Dodd, R. K. *Snapback repellers and chaos in a discrete population model with delayed recruitment.* Nonlinear Analysis TMA, **7** (1982), 623–660.

[13] Pounder, J. R. and Rogers, T. D. *The geometry of chaos: dynamics of a nonlinear second order difference equation.* Bull. Math. Biol., **42** (1980), 551–597.

[14] Rogers, T. D. *Chaos in systems in populations biology*, in Progress In Theoretical Biology, Vol. 6, Academic Press, New York, 1981, pp. 91–146.

[15] Saito,Y. Ma, W. B. and Hara, T. A *necessary and sufficient condition for permanence of a Lotka-Volterra discrete system with delays.* J. Math. Anal. Appl., **256** (2001), 162–174

[16] Smith, H. *Planar competitive and cooperative difference equations*. J. Differ. Equations Appl., **3** (1998), 335–357.

[17] So, Joseph W.-H. and Yu, J. S. *Global stability in a logistic equation with piecewise constant arguments.* Hokkaido Math. J., **24** (1995), 269–286.

[18] Wang, W. D. and Lu, Z. Y. *Global stability of discrete models of Lotka-Volterra type.* Nonlinear Anal. Ser B(8): Real World Applications, **35** (1999), 1019–1030.