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GLOBAL ATTRACTIVITY IN A PREDATOR–PREY SYSTEM WITH PURE DELAYS

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Abstract We consider a delay predator–prey system without instantaneous negative feedback and establish some conditions for global attractivity of the positive equilibrium of the system which generalize and improve some of the existing ones. When the system is decoupled, one of the main results reduces to the well-known Wright 3/2 stability condition for the delayed logistic equation.

Keywords: predator–prey system; pure delay; global attractivity

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1. Introduction

We consider the global attractivity of the positive equilibrium of a predator–prey system with delays modelled by

$$\left.\begin{aligned}\dot{N}_1(t) &= N_1(t)[a_1 - b_1 N_1(t-\tau_1) - c_1 N_2(t-\sigma_1)],\\ \dot{N}_2(t) &= N_2(t)[-a_2 + c_2 N_1(t-\sigma_2) - b_2 N_2(t-\tau_2)],\end{aligned}\right\} \tag{1.1}$$

with initial conditions

$$N_i(t) = \phi_i(t) \geqslant 0, \quad t \in [-\Delta, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, \tag{1.2}$$

where $a_i, b_i > 0$, $c_i \geqslant 0$, $\tau_i, \sigma_i \geqslant 0$ for $i = 1, 2$ and $\Delta = \max\{\tau_1, \tau_2, \sigma_1, \sigma_2\}$.

When the predator species is absent, the prey species is governed by the well-known delay logistic equation

$$\left.\begin{aligned}\dot{N}_1(t) &= N_1(t)[a_1 - b_1 N_1(t-\tau_1)],\\ N(s) \geqslant 0 \quad &\text{for } s \in [-\tau_1, 0], \qquad N(0) > 0.\end{aligned}\right\} \tag{1.3}$$

For (1.3), a well-known result (usually referred as Wright's 3/2 criterion) is that if $a_1\tau_1 \leqslant \frac{3}{2}$, then the positive equilibrium a_1/b_1 is globally attractive.

On the other hand, if all delays are zero in (1.1), then system (1.1) simplifies to the following autonomous system of ordinary differential equations:

$$\left.\begin{aligned}\dot{N}_1(t) &= N_1(t)[a_1 - b_1N_1(t) - c_1N_2(t)],\\ \dot{N}_2(t) &= N_2(t)[-a_2 + c_2N_1(t) - b_2N_2(t)].\end{aligned}\right\} \tag{1.4}$$

It is well known that all positive solutions $N(t) = (N_1(t), N_2(t))$ of (1.4) satisfy $N(t) \to N^* = (N_1^*, N_2^*)$ as $t \to \infty$ if and only if

$$a_1c_2 - a_2b_1 > 0, \tag{A1}$$

where

$$N_1^* = \frac{a_1b_2 + a_2c_1}{b_1b_2 + c_1c_2}, \qquad N_2^* = \frac{a_1c_2 - a_2b_1}{b_1b_2 + c_1c_2}. \tag{1.5}$$

From this fundamental result, one naturally expects that under (A1) N^* remains globally attractive for (1.1), (1.2) if the delays are sufficiently small. This expectation was confirmed recently by He [**7**]. Indeed, by constructing a Lyapunov functional, He established the following theorem.

Theorem 1.1. *Suppose that (A1) holds. Then the positive equilibrium N^* for (1.1) is globally attractive, provided that*

$$(1 + M_1^2)\tau_1 + \frac{c_1}{b_2}\left\{\tau_1 + \left[1 + \frac{c_2}{b_1}(1 + M_2^2)\right]\sigma_1 + M_1^2\sigma_2 + M_2^2\tau_2\right\} < 2, \tag{1.6}$$

and

$$(1 + M_2^2)\tau_2 + \frac{c_2}{b_1}\left\{\tau_2 + \left[1 + \frac{c_1}{b_2}(1 + M_1^2)\right]\sigma_2 + M_1^2\tau_1 + M_2^2\sigma_1\right\} < 2, \tag{1.7}$$

where

$$M_1 = \frac{a_1}{b_1}\mathrm{e}^{a_1\tau_1} \quad \text{and} \quad M_2 = \frac{-a_2 + c_2M_1}{b_2}\mathrm{e}^{(-a_2+c_2M_1)\tau_2}. \tag{1.8}$$

Hofbauer and So [**9**] studied a general Lotka–Volterra system allowing distributed delays but *with instantaneous negative feedback*, which, in the case of $n = 2$ and in the context of predator–prey, includes the following system:

$$\left.\begin{aligned}\dot{N}_1(t) &= N_1(t)[a_1 - b_1N_1(t) - c_1N_2(t - \sigma_1)],\\ \dot{N}_2(t) &= N_2(t)[-a_2 + c_2N_1(t - \sigma_2) - b_2N_2(t)],\end{aligned}\right\} \tag{1.9}$$

The main theorem of [**9**] can be stated, as below, in terms of (1.9).

Theorem 1.2. *Suppose that (A1) holds. Then the positive equilibrium N^* for (1.9) with (1.2) is globally attractive for all σ_1 and σ_2 if and only if $b_1b_2 - c_1c_2 = 0$ or*

$$b_1b_2 - c_1c_2 > 0. \tag{DD}$$

Theorem 1.2 is proved by constructing a Lyapunov functional, taking advantage of the fact that there is no delay in the negative feedback terms $b_1N_1(t)$ and $b_2N_2(t)$ (i.e. the system has instantaneous negative feedbacks).

From Theorem 1.2, we see that under the diagonal dominating condition (DD), the off-diagonal delays do not affect the global attractivity of N^* (assuming (A1)). This suggests that one only needs to worry about the diagonal delays in this context. He [**6**] made an attempt to partly address this problem by considering *one* diagonal delay. In fact, He considered the system

$$\left.\begin{aligned}\dot{N}_1(t) &= N_1(t)[a_1 - b_1N_1(t-\tau) - c_1N_2(t-\sigma_1)],\\ \dot{N}_2(t) &= N_2(t)[-a_2 + c_2N_1(t-\sigma_2) - b_2N_2(t)]\end{aligned}\right\} \tag{1.10}$$

and established the following result.

Theorem 1.3. *Assume that (A1) holds, and that*

$$\frac{c_1c_2}{b_1b_2} < \frac{1 - a_1\tau e^{a_1\tau}}{1 + a_1\tau e^{a_1\tau}}. \tag{1.11}$$

Then the equilibrium N^* is globally attractive for (1.10) with $\sigma_1 = \sigma_2$.

Obviously, (1.11) implies (DD). Note that (1.11) is equivalent to

$$a_1\tau e^{a_1\tau} < \frac{b_1b_2 - c_1c_2}{b_1b_2 + c_1c_2}, \tag{1.12}$$

which coincides with (DD) when $\tau = 0$. Therefore, under (DD), (1.12) gives an estimate for the smallness of τ with which N^* remains globally attractive for (1.10) with $\sigma_1 = \sigma_2$.

Observe that if the capture rate $c_1 = 0$, the prey species again is governed by (1.3). As mentioned before, when $a_1\tau_1 \leqslant \frac{3}{2}$, every positive solution $N_1(t)$ of (1.3) tends to $N_1^* = a_1/b_1$, and thus the equation for N_2 in (1.1) can be considered as an asymptotically autonomous equation with the limiting equation

$$\dot{N}_2(t) = N_2(t)[a - b_2N_2(t-\tau_2)], \tag{1.13}$$

where $a = -a_2 + (c_2a_1)/b_1 > 0$ under (A1). By the theory of asymptotically autonomous systems (see, for example, [**2**]) and Wright's criterion, one knows that the N_2 component of the solution of (1.1) converges to N_2^* as $t \to \infty$, provided that $a\tau_2 \leqslant \frac{3}{2}$, which holds when $\tau_2 = 0$.

With the above observation in mind, we feel that Theorems 1.1–1.3 are not satisfactory at least in the following sense. The restrictions (1.6) and (1.7) in Theorem 1.1 and (1.11) in Theorem 1.3 for smallness of delays do not reduce to Wright's 3/2 criterion when the system (1.1) is decoupled by letting $c_1 = 0$. Moreover, Theorem 1.3 was only for a special case of (1.1) (i.e. $\tau_2 = 0$ and $\sigma_1 = \sigma_2$), and even in such a special case, as observed above, (1.12) can be improved.

Motivated by the above dissatisfaction, and encouraged by the authors' recent work [**27, 29**], where 3/2-type criteria were obtained for the delayed *competitive system* of

Lotka–Volterra type without instantaneous negative feedback, we will establish some criteria of 3/2 type for the global attractivity of the positive equilibrium N^*. Note that, owing to the lack of instantaneous negative feedback, the global attractivity of systems 'without instantaneous negative feedback' (or 'of pure-delay type') becomes much more difficult and has been studied by Gopalsamy [**3**], Gopalsamy and He [**5**], He [**6**–**8**], Kuang [**12**, **13**], Kuang and Smith [**14**, **15**], Smith [**19**], So *et al.* [**23**] and Tang and Zou [**28**]. Also note that 3/2-type stability criteria for various *scalar* delay-differential equations are available in [**1**, **10**–**12**, **16**–**18**, **20**–**22**, **24**–**26**, **31**–**34**].

The rest of the paper is organized as follows. In § 2, we give the main results. In § 3, we establish some preliminary lemmas, which address the persistence and dissipativity of system (1.8) and therefore, are of some interest and importance themselves. In § 4, by combing these lemmas with the 'sandwiching' technique and using some subtle techniques of integration and inequality, we give the proofs of the main theorems.

2. Main results

Theorem 2.1. *Assume that (A1) and (DD) hold, and that*

$$b_1(a_1b_2 + a_2c_1)\tau \leqslant \tfrac{3}{2}(b_1b_2 - c_1c_2) + \frac{c_1c_2(b_1b_2 - c_1c_2)}{2(b_1b_2 + c_1c_2)}. \tag{2.1}$$

Then the positive equilibrium $N^* = (N_1^*, N_2^*)$ of (1.10) is a global attractor.

It is easily seen that, by letting $c_1 = 0$, Theorem 2.1 reproduces Wright's 3/2 result for the autonomous delayed logistic equation (1.3). Note that the above 3/2-type condition (2.1) is established for (1.10), where only one diagonal delay is present. In the case when both diagonal delays are present, i.e. system (1.1), we are unable to obtain a similar result by our method. The main difficulty is that in the case $\tau_2 \neq 0$ we cannot determine the two important inequalities (4.6) and (4.7) from (4.4), but these play a key role in the proof of Theorem 2.1. However, the following theorem allows *small* $\tau_2 > 0$, which is along the lines of Theorem 1.2.

Theorem 2.2. *Let*

$$M_1 = \frac{a_1}{b_1}\exp(a_1\tau_1 + \mathrm{e}^{-a_1\tau_1} - 1), \tag{2.2}$$

$$M_2 = \frac{-a_2 + c_2M_1}{b_2}\exp[(-a_2 + c_2M_1)\tau_2 + \mathrm{e}^{-(-a_2+c_2M_1)\tau_2} - 1], \tag{2.3}$$

and

$$B_i = \begin{cases} \dfrac{[2 - (M_ib_i\tau_i)^2]}{[2 + (M_ib_i\tau_i)^2]} & \text{if } M_ib_i\tau_i \leqslant 1, \\[2ex] \dfrac{[3 - 2(M_ib_i\tau_i)]}{[1 + 2(M_ib_i\tau_i)]} & \text{if } M_ib_i\tau_i > 1 \end{cases} \tag{2.4}$$

for $i = 1, 2$. Assume that (A1) and (DD) hold, and that

$$\frac{c_1c_2}{b_1b_2} < B_1B_2. \tag{2.5}$$

Then the positive equilibrium N^* for (1.1) is a global attractor.

When $\tau_2 = 0$, $B_2 = 1$, and we thereby have the following result for (1.10).

Corollary 2.3. *Assume that (A1) and (DD) hold and that*

$$\frac{c_1c_2}{b_1b_2} < \begin{cases} \dfrac{[2-(\tau M_1 b_1)^2]}{[2+(\tau M_1 b_1)^2]} & \text{if } \tau M_1 b_1 \leqslant 1, \\[2ex] \dfrac{[3-2(\tau M_1 b_1)]}{[1+2(\tau M_1 b_1)]} & \text{if } \tau M_1 b_1 > 1, \end{cases} \tag{2.6}$$

where M_1 is defined by (2.2). Then the positive equilibrium N^* for (1.10) is a global attractor.

Remark 2.4. In view of the proof of [**7**, Theorem 1.1], if the M_1 and M_2 in (1.6) and (1.7) are replaced by (2.2) and (2.3), respectively, the conclusion in Theorem 1.1 still holds.

Remark 2.5. Theorems 2.2 and 1.1 are complementary. In Theorem 1.1, the condition (DD) on the coefficients of (1.1) is not needed, but the restrictions on the off-diagonal delays are added, whereas Theorem 2.2 is contrary to Theorem 1.1.

Remark 2.6. When $\tau_1 = \tau_2 = 0$, $B_1 = B_2 = 0$ and (2.5) reduces to (DD). Thus, in such a special case of $n = 2$ and in the predator–prey context, Theorem 1.2 is slightly less restrictive than Theorem 2.2, with the difference being between the use of 'non-negative' and 'positive' for the term $b_1b_2 - c_1c_2$. However, as stated in the title and in § 1, dealing with positive diagonal delays τ_1 and τ_2 is the primary goal of this work, which Theorem 1.2 fails to acheive.

Remark 2.7. In condition (2.6),

$$\tau M_1 b_1 = a_1\tau \exp(a_1\tau + \mathrm{e}^{-a_1\tau} - 1) < a_1\tau \mathrm{e}^{(a_1\tau)^2/2}.$$

Hence, condition (2.6) improves on (1.11) greatly.

3. Preliminary lemmas

In this section, we give some lemmas which will be used in § 4 in the proofs of the main theorems. The first one is from [**29**].

Lemma 3.1. *Let $a > 0$ and $0 < \mu < 1$. Then the system of inequalities*

$$\left.\begin{aligned} y &\leqslant (a+\mu x)\exp\left[\frac{1-\mu}{a}x - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}x^2\right] - a, \\ x &\leqslant a - (a-\mu y)\exp\left[-\frac{1-\mu}{a}y - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}y^2\right] \end{aligned}\right\} \tag{3.1}$$

has a unique solution: $(x, y) = (0, 0)$ in the region $D = \{(x, y) : 0 \leqslant x < a,\ 0 \leqslant y < a/\mu\}$.

Lemma 3.2. *Assume that (A1) holds and let $(N_1(t), N_2(t))$ be the solution of (1.1) and (1.2). Then eventually*

$$0 < N_i(t) \leqslant M_i, \quad i = 1, 2, \tag{3.2}$$

where M_1 and M_2 are defined by (2.2) and (2.3), respectively.

Proof. From (1.1) and (1.2), it is easy to see that $N_i(t) > 0$ for $t \geqslant 0$ and $i = 1, 2$. Hence,

$$\dot{N}_1(t) \leqslant N_1(t)[a_1 - b_1 N_1(t - \tau_1)] \leqslant a_1 N_1(t), \quad t \geqslant 0. \tag{3.3}$$

If $N_1(t) \leqslant a_1/b_1$ eventually, then the first inequality in (3.2) holds naturally for large t and $i = 1$. If $N_1(t) \geqslant a_1/b_1$ eventually, then it follows from (3.3) that $\lim_{t\to\infty} N_1(t) = a_1/b_1$, and so (3.2) holds for large t and $i = 1$. In what follows, we consider only the case when $N_1(t)$ oscillates on a_1/b_1. Let t^* be an arbitrary local left maximum point of $N_1(t)$ such that $N_1(t^*) > a_1/b_1$. Then $\dot{N}_1(t^*) \geqslant 0$, and it follows from (3.3) that there exists $\xi \in [t^* - \tau_1, t^*]$ such that $N_1(\xi) = a_1/b_1$. For $t \in [\xi, t^*]$, integrating (3.3) from $t - \tau_1$ to ξ, we get

$$-\ln \frac{N_1(t - \tau_1)}{N_1(\xi)} \leqslant a_1(\xi + \tau_1 - t), \quad \xi \leqslant t \leqslant t^*.$$

Thus,

$$N_1(t - \tau_1) \geqslant \frac{a_1}{b_1} \exp[-a_1(\xi + \tau_1 - t)], \quad \xi \leqslant t \leqslant t^*.$$

Substituting this into the first inequality in (3.3), we obtain

$$\frac{\dot{N}_1(t)}{N_1(t)} \leqslant a_1\{1 - \exp[-a_1(\xi + \tau_1 - t)]\}, \quad \xi \leqslant t \leqslant t^*. \tag{3.4}$$

Integrating (3.4) from ξ to t^*, we have

$$\begin{aligned} \ln \frac{b_1 N_1(t^*)}{a_1} &\leqslant a_1 \int_{\xi}^{t^*} \{1 - \exp[-a_1(\xi + \tau_1 - t)]\}\,\mathrm{d}t \\ &= a_1(t^* - \xi) + \mathrm{e}^{-a_1\tau_1} - \exp[-a_1(\xi + \tau_1 - t^*)] \\ &= a_1(t^* - \xi) - \exp[-a_1(\xi + \tau_1 - t^*)] + \mathrm{e}^{-a_1\tau_1} \\ &\leqslant a_1\tau_1 - 1 + \mathrm{e}^{-a_1\tau_1}. \end{aligned}$$

Here we have used the fact that the function $f(x) = x - \mathrm{e}^{x - a\tau_1}$ is increasing in the interval $[0, a\tau_1]$ and hence $f(x) \leqslant f(a\tau_1) = a\tau_1 - 1$ for $x \in [0, a\tau_1]$. The above inequality implies that

$$N_1(t^*) \leqslant \frac{a_1}{b_1} \exp(a_1\tau_1 + \mathrm{e}^{-a_1\tau_1} - 1).$$

It follows that, for large t,

$$N_1(t) \leqslant \frac{a_1}{b_1} \exp(a_1\tau_1 + \mathrm{e}^{-a_1\tau_1} - 1) = M_1.$$

Choose $T > 0$ such that $N_1(t) \leqslant M_1$ for $t \geqslant T$. Then from (1.1), we have

$$\dot{N}_2(t) \leqslant N_2(t)[-a_2 + c_2 M_1 - b_2 N_2(t - \tau_2)], \quad t \geqslant T + \Delta. \tag{3.5}$$

Note that

$$c_2 M_1 - a_2 = c_2 \frac{a_1}{b_1} \exp(a_1\tau_1 + \mathrm{e}^{-a_1\tau_1} - 1) - a_2 > \frac{a_1 c_2 - a_2 b_1}{b_1} > 0.$$

Hence, similarly, from (3.5) we eventually have

$$N_2(t) \leqslant \frac{-a_2 + c_2 M_1}{b_2} \exp[(-a_2 + c_2 M_1)\tau_2 + \exp(-(-a_2 + c_2 M_1)\tau_2) - 1] = M_2.$$

The proof is complete. □

The following lemma is a corollary of [**30**, Theorem 2.1].

Lemma 3.3. *Assume that (A1) holds and let* $(N_1(t), N_2(t))$ *be the solution of (1.1) and (1.2). Then*

$$0 < \liminf_{t\to\infty} N_i(t) \leqslant \limsup_{t\to\infty} N_i(t) < \infty, \quad i = 1, 2. \tag{3.6}$$

4. Proofs of the main results

Proof of Theorem 2.1. By the transformation

$$x_i(t) = N_i(t) - N_i^*, \quad i = 1, 2,$$

system (1.10) becomes

$$\left.\begin{aligned} \dot{x}_1(t) &= (N_1^* + x_1(t))[-b_1 x_1(t - \tau) - c_1 x_2(t - \sigma_1)], \\ \dot{x}_2(t) &= (N_2^* + x_2(t))[c_2 x_1(t - \sigma_2) - b_2 x_2(t)]. \end{aligned}\right\} \tag{4.1}$$

Clearly, the global attractivity of N^* of system (1.10) is equivalent to that of $(0, 0)$ for (4.1), meaning that

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, 2, \tag{4.2}$$

for all solutions $x(t) = (x_1(t), x_2(t))$ with $x_1(t) > -N_1^*$ and $x_2(t) > -N_2^*$ for $t \geqslant 0$. We have two cases to consider in order to prove (4.2).

Case 1. $b_1 x_1(t - \tau) + c_1 x_2(t - \sigma_1)$ or $c_2 x_1(t - \sigma_2) - b_2 x_2(t)$ is non-oscillatory. It is harmless to assume that $b_1 x_1(t - \tau) + c_1 x_2(t - \sigma_1)$ is non-oscillatory. Then, $\dot{x}_1(t)$ is sign-definite eventually, which implies that $x_1(t)$ is monotonous eventually. By Lemma 3.3, we have $x_1(t) \to \alpha_1$ as $t \to \infty$ and $N_1^* + \alpha_1 > 0$. On the other hand, using the boundedness of $x_1(t)$ and $x_2(t)$, we can conclude from (4.1) that both $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are also bounded on $[0, \infty)$, which implies that $x_1(t)$ and $x_2(t)$ are uniformly continuous on $[0, \infty)$. It follows immediately that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are also uniformly continuous on $[0,\infty)$. Therefore, by [**4**, Lemma 1.2.3], $\dot{x}_1(t)\to 0$ as $t\to\infty$. Hence, from (4.1), we obtain

$$b_1\alpha_1 + c_1 x_2(t-\sigma_1)\to 0 \quad \text{as } t\to\infty,$$

which implies that the limit $\alpha_2=\lim_{t\to\infty} x_2(t)$ exists. Analogously to the above proof, we have $\dot{x}_2(t)\to 0$ as $t\to\infty$. Hence, from (4.1) and Lemma 3.3, we have

$$b_1\alpha_1 + c_1\alpha_2 = 0, \qquad c_2\alpha_1 - b_2\alpha_2 = 0,$$

which imply that $\alpha_1=\alpha_2=0$, i.e. (4.2) holds.

Case 2. Both $b_1x_1(t-\tau)+c_1x_2(t-\sigma_1)$ and $c_2x_1(t-\sigma_2)-b_2x_2(t)$ are oscillatory. Then there exist two infinity sequences $\{s_n\}$ and $\{t_n\}$ such that

$$b_1x_1(s_n-\tau)+c_1x_2(s_n-\sigma_1)=0, \quad c_2x_1(t_n-\sigma_2)-b_2x_2(t_n)=0, \quad n=1,2,\dots, \tag{4.3}$$

$$x_2(t_{2n-1})\leqslant x_2(t)\leqslant x_2(t_{2n}) \quad \text{for } t_{2n-1}\leqslant t\leqslant t_{2n},\ n=1,2,\dots, \tag{4.4}$$

and

$$\liminf_{n\to\infty} x_2(t_{2n-1}) = \liminf_{t\to\infty} x_2(t) \leqslant \limsup_{t\to\infty} x_2(t) = \limsup_{n\to\infty} x_2(t_{2n}). \tag{4.5}$$

Set

$$-v=\liminf_{t\to\infty} x_1(t) \quad \text{and} \quad u=\limsup_{t\to\infty} x_1(t).$$

Then from (4.3)–(4.5), we have

$$\limsup_{t\to\infty} x_2(t) = \limsup_{n\to\infty} x(t_{2n}) = \frac{c_2}{b_2}\limsup_{n\to\infty} x_1(t_{2n}-\sigma_2) \leqslant \frac{c_2}{b_2}u \tag{4.6}$$

and

$$\liminf_{t\to\infty} x_2(t) = \liminf_{n\to\infty} x(t_{2n-1}) = \frac{c_2}{b_2}\liminf_{n\to\infty} x_1(t_{2n-1}-\sigma_2) \geqslant -\frac{c_2}{b_2}v. \tag{4.7}$$

Hence,

$$0=\lim_{n\to\infty}[b_1x_1(s_n-\tau)+c_1x_2(s_n-\sigma_1)] \leqslant b_1u + c_1\limsup_{t\to\infty} x_2(t) \leqslant \left(b_1+\frac{c_1c_2}{b_2}\right)u$$

and

$$0=\lim_{n\to\infty}[b_1x_1(s_n-\tau)+c_1x_2(s_n-\sigma_1)] \geqslant -b_1v + c_1\liminf_{t\to\infty} x_2(t) \geqslant -\left(b_1+\frac{c_1c_2}{b_2}\right)v.$$

Thus, in view of Lemma 3.3 and the above results, we have

$$-N_1^* < -v \leqslant 0 \leqslant u < \infty. \tag{4.8}$$

Set $\mu = c_1c_2/b_1b_2$. Then $0<\mu<1$. In what follows, we show that v and u satisfy the inequalities

$$N_1^* + u \leqslant (N_1^*+\mu v)\exp\left[\frac{1-\mu}{N_1^*}v - \frac{(1-\mu)^2(1+2\mu)}{6{N_1^*}^2(1+\mu)}v^2\right] \tag{4.9}$$

and

$$N_1^* - v \geqslant (N_1^* - \mu u) \exp\left[-\frac{1-\mu}{N_1^*}u - \frac{(1-\mu)^2(1+2\mu)}{6N_1^{*2}(1+\mu)}u^2\right]. \tag{4.10}$$

For the sake of simplicity, we set

$$A = \frac{3(1-\mu)}{2N_1^*(1+\mu)} + \frac{\mu(1-\mu)}{N_1^*(1+\mu)^2} = \frac{(1-\mu)(3+5\mu)}{2N_1^*(1+\mu)^2}.$$

Then (2.1) implies $b_1\tau \leqslant A$. Let $\varepsilon > 0$ be sufficiently small such that $v_1 \equiv v + \varepsilon < N_1^*$. Choose $T > 0$ such that

$$-v_1 < x_1(t) < u + \varepsilon \equiv u_1 \quad \text{and} \quad -\frac{c_2}{b_2}v_1 < x_2(t) < \frac{c_2}{b_2}u_1, \quad t \geqslant T - \Delta. \tag{4.11}$$

Set $v_2 = (1+\mu)v_1$ and $u_2 = (1+\mu)u_1$. Then, from (4.1), we have

$$\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant b_1[-x_1(t-\tau) + \mu v_1] \leqslant b_1 v_2, \quad t \geqslant T \tag{4.12}$$

and

$$-\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant b_1[x_1(t-\tau) + \mu u_1] \leqslant b_1 u_2, \qquad t \geqslant T. \tag{4.13}$$

First, we prove that (4.9) holds. If $u \leqslant \mu v$, then (4.9) obviously holds. Therefore, we will prove (4.9) only in the case when $u > \mu v$. For simplicity, it is harmless to assume that $u > \mu v_1$. Thus, we cannot have $x_1(t) \leqslant \mu v_1$ eventually. On the other hand, if $x_1(t) \geqslant \mu v_1$ eventually, then it follows from the first inequality in (4.12) that $x_1(t)$ is non-increasing and that $u = \lim_{t\to\infty} x_1(t) = \mu v_1$. This is also impossible. Therefore, it follows that $x_1(t)$ oscillates about μv_1.

Let $\{p_n\}$ be an increasing sequence such that $p_n \geqslant T + \Delta$, $\dot{x}_1(p_n) = 0$, $x_1(p_n) \geqslant \mu v_1$, $\lim_{n\to\infty} p_n = \infty$ and $\lim_{n\to\infty} x_1(p_n) = u$. By (4.12), there exists $\xi_n \in [p_n - \tau, p_n]$ such that $x_1(\xi_n) = \mu v_1$. For $t \in [\xi_n, p_n]$, integrating (4.12) from $t - \tau$ to ξ_n we get

$$-\ln\frac{N_1^* + x_1(t-\tau)}{N_1^* + x_1(\xi_n)} \leqslant b_1 v_2(\xi_n + \tau - t), \quad \xi_n \leqslant t \leqslant p_n.$$

Thus,

$$x_1(t-\tau) \geqslant -N_1^* + (N_1^* + \mu v_1)\exp[-b_1 v_2(\xi_n + \tau - t)], \quad \xi_n \leqslant t \leqslant p_n.$$

Substituting this into the first inequality in (4.12), we obtain

$$\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant (N_1^* + \mu v_1)b_1[1 - \exp(-b_1 v_2(\xi_n + \tau - t))], \quad \xi_n \leqslant t \leqslant p_n.$$

Combining this with (4.12), we have

$$\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant \min\{b_1 v_2,\ (N_1^* + \mu v_1)b_1[1 - \exp(-b_1 v_2(\xi_n + \tau - t))]\}, \quad \xi_n \leqslant t \leqslant p_n. \tag{4.14}$$

Analogously to the proof in [**29**], we can prove (4.9) by (4.14) and the fact that $b_1\tau \leqslant A$.

Next, we will prove that (4.10) holds as well. If $v = 0$, then (4.10) holds naturally. In what follows, we assume that $v > 0$. Then, from (4.9), we have

$$\left.\begin{aligned} u &< N_1^*(1+\mu)\mathrm{e}^{1-\mu} - N_1^* < 2N_1^*, \\ \mu u &< \mu\left[(N_1^*+\mu v)\exp\left(\frac{(1-\mu)v}{N_1^*}\right) - N_1^*\right] < v < N_1^*. \end{aligned}\right\} \tag{4.15}$$

Thus we may assume, without loss of generality, that $v > \mu u_1$. In view of this and (4.13), we can show that neither $x_1(t) \geqslant -\mu u_1$ eventually nor $x_1(t) \leqslant -\mu u_1$ eventually. Therefore, $x_1(t)$ oscillates about $-\mu u_1$.

Let $\{q_n\}$ be an increasing sequence such that $q_n \geqslant T + \Delta$, $\dot{x}_1(q_n) = 0$, $x_1(q_n) \leqslant -\mu u_1$, $\lim_{n\to\infty} q_n = \infty$ and $\lim_{n\to\infty} x_1(q_n) = -v$. By (4.13), there exists $\eta_n \in [q_n - \tau, q_n]$ such that $x_1(\eta_n) = -\mu u_1$. For $t \in [\eta_n, q_n]$, by (4.13), we have

$$x_1(t-\tau) \leqslant (N_1^* - \mu u_1)\exp[b_1 u_2(\eta_n + \tau - t)] - N_1^*, \quad \eta_n \leqslant t \leqslant q_n.$$

Substituting this into the first inequality in (4.13), we obtain

$$-\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant (N_1^* - \mu u_1) b_1 [\exp(b_1 u_2(\eta_n + \tau - t)) - 1], \quad \eta_n \leqslant t \leqslant q_n.$$

Combining this with (4.13), we have

$$-\frac{\dot{x}_1(t)}{N_1^* + x_1(t)} \leqslant \min\{b_1 u_2,\ (N_1^* - \mu u_1) b_1 [\exp(b_1 u_2(\eta_n + \tau - t)) - 1]\}, \quad \eta_n \leqslant t \leqslant q_n. \tag{4.16}$$

Analogously to the proof in [**29**], we can prove (4.10) by (4.16) and the fact that $b_1\tau \leqslant A$. In view of Lemma 3.1, it follows from (4.9) and (4.10) that $u = v = 0$. Thus, (4.2) holds. The proof is complete. □

Proof of Theorem 2.2. By the transformation

$$x_i(t) = N_i(t) - N_i^*, \quad i = 1, 2,$$

system (1.1) becomes

$$\left.\begin{aligned} \dot{x}_1(t) &= (N_1^* + x_1(t))[-b_1 x_1(t-\tau_1) - c_1 x_2(t-\sigma_1)], \\ \dot{x}_2(t) &= (N_2^* + x_2(t))[c_2 x_1(t-\sigma_2) - b_2 x_2(t-\tau_2)]. \end{aligned}\right\} \tag{4.17}$$

Let $(x_1(t), x_2(t))$ be any solution of (4.17) with $N_i^* + x_i(t) > 0$ for $t \geqslant 0$ and $i = 1, 2$. By Lemma 3.2, there exists $T > 0$ such that

$$N_i^* + x_i(t) \leqslant M_i, \quad t \geqslant T,\ i = 1, 2. \tag{4.18}$$

We have two cases to consider in order to prove (4.2).

Case 1. $b_1 x_1(t-\tau_1) + c_1 x_2(t-\sigma_1)$ or $c_2 x_1(t-\sigma_2) - b_2 x_2(t-\tau_2)$ is non-oscillatory. In this case, by a similar proof to that of case 1 in Theorem 2.1, we can show that (4.2) holds.

Case 2. Both $b_1x_1(t-\tau_1)+c_1x_2(t-\sigma_1)$ and $c_2x_1(t-\sigma_2)-b_2x_2(t-\tau_2)$ are oscillatory. Set

$$U_i = \limsup_{t\to\infty} |x_i(t)|, \quad i=1,2.$$

By Lemma 3.2, $0 \leqslant U_i < \infty$, $i=1,2,\dots,n$. It suffices to prove that $U_1 = U_2 = 0$. To this end, assume that $U_1 > 0$ and $U_2 > 0$. Hence, by (4.17), for any given sufficiently small $\varepsilon > 0$, there exist two sequences $\{t_{in}\}$, $i=1,2$ with $t_{in} - \Delta > T$ such that

$$\left.\begin{array}{c} t_{in} \to \infty, \quad |x_i(t_{in})| \to U_i \quad \text{as } n\to\infty, \quad |x_i(t_{in})| > U_i - \varepsilon, \\ |\dot{x}_i(t_{in})| = 0, \quad |x_i(t)| < U_i + \varepsilon \quad \text{for } t \geqslant t_1, \end{array}\right\} \quad i=1,2. \tag{4.19}$$

where $t_1 = \min\{t_{i1} : i=1,2\}$. We can assume that $|x_i(t_{in})| = x_i(t_{in})$ (if necessary, we use $-x_i(t)$ instead of $x_i(t)$ and $-b_i$, $-c_i$ instead of b_i, c_i for $i=1,2$). Then, by (4.17), we have $0 = b_1x_1(t_{1n}-\tau_1) + c_1x_2(t_{1n}-\sigma_1)$, which yields

$$x_1(t_{1n}-\tau_1) \leqslant \frac{c_1}{b_1}(U_2+\varepsilon) \equiv \beta_1.$$

Set

$$b_{12} = \begin{cases} \dfrac{[2+(M_1b_1\tau_1)^2]c_1}{[2-(M_1b_1\tau_1)^2]b_1} & \text{if } M_1b_1\tau_1 \leqslant 1, \\ \dfrac{[1+2(M_1b_1\tau_1)]c_1}{[3-2(M_1b_1\tau_1)]b_1} & \text{if } M_1b_1\tau_1 > 1, \end{cases}$$

and

$$b_{21} = \begin{cases} \dfrac{[2+(M_2b_2\tau_2)^2]c_2}{[2-(M_2b_2\tau_2)^2]b_2} & \text{if } M_2b_2\tau_2 \leqslant 1, \\ \dfrac{[1+2(M_2b_2\tau_2)]c_2}{[3-2(M_2b_2\tau_2)]b_2} & \text{if } M_2b_2\tau_2 > 1. \end{cases}$$

Then, by (2.5), $b_{12}b_{21} < 1$. In what follows, we show that

$$x_1(t_{1n}) \leqslant b_{12}(U_2+\varepsilon) + \begin{cases} \dfrac{2\varepsilon(M_1b_1\tau_1)^2}{[2-(M_1b_1\tau_1)^2]} & \text{if } M_1b_1\tau_1 \leqslant 1, \\ \dfrac{2\varepsilon(2M_1b_1\tau_1-1)}{3-2M_1b_1\tau_1} & \text{if } M_1b_1\tau_1 > 1. \end{cases} \tag{4.20}$$

If $x_1(t_{1n}) \leqslant \beta_1$, then (4.20) obviously holds. If $x_1(t_{1n}) > \beta_1$, then there exists $\xi_{1n} \in [t_{1n}-\tau_1, t_{1n}]$ such that $x_1(\xi_{1n}) = \beta_1$. From (4.17) we have

$$\begin{aligned} \dot{x}_1(t) &\leqslant (N_1^* + x_1(t))b_1[-x_1(t-\tau_1)+\beta_1] \\ &\leqslant M_1b_1[(U_1+\varepsilon)+\beta_1], \quad t \geqslant T_2 = t_1 + \Delta. \end{aligned} \tag{4.21}$$

By (4.21), we have

$$\beta_1 - x_1(t-\tau_1) \leqslant M_1b_1[(U_1+\varepsilon)+\beta_1](\xi_{1n}+\tau_1-t), \quad \xi_{1n} \leqslant t \leqslant t_{1n}.$$

Substituting this into the first inequality in (4.21), we obtain

$$\dot{x}_1(t) \leqslant (M_1 b_1)^2[(U_1+\varepsilon)+\beta_1](\xi_{1n}+\tau_1-t), \quad \xi_{1n} \leqslant t \leqslant t_{1n}.$$

Combining this and (4.21), we have

$$\dot{x}_1(t) \leqslant M_1 b_1[(U_1+\varepsilon)+\beta_1]\min\{1, M_1 b_1(\xi_{1n}+\tau_1-t),\}, \quad \xi_{1n} \leqslant t \leqslant t_{1n}. \tag{4.22}$$

We consider the following three subcases.

Case 2.1. $M_1 b_1 \tau_1 \leqslant 1$. In this case, by (4.23) we have

$$\begin{aligned} x_1(t_{1n}) - x_1(\xi_{1n}) &\leqslant [(U_1+\varepsilon)+\beta_1](M_1 b_1)^2 \int_{\xi_{1n}}^{t_{1n}} (\xi_{1n}+\tau_1-t)\,\mathrm{d}t \\ &= [(U_1+\varepsilon)+\beta_1](M_1 b_1)^2[\tau_1(t_{1n}-\xi_{1n}) - \tfrac{1}{2}(t_{1n}-\xi_{1n})^2] \\ &\leqslant \tfrac{1}{2}(M_1 b_1 \tau_1)^2[(U_1+\varepsilon)+\beta_1] \\ &\leqslant \tfrac{1}{2}(M_1 b_1 \tau_1)^2[x_1(t_{1n})+\beta_1+2\varepsilon]. \end{aligned}$$

Case 2.2. $M_1 b_1 \tau_1 > 1$ and $M_1 b_1(t_{1n}-\xi_{1n}) \leqslant 1$. In this case, by (4.23) we have

$$\begin{aligned} x_1(t_{1n}) - x_1(\xi_{1n}) &\leqslant [(U_1+\varepsilon)+\beta_1](M_1 b_1)^2 \int_{\xi_{1n}}^{t_{1n}} (\xi_{1n}+\tau_1-t)\,\mathrm{d}t \\ &= [(U_1+\varepsilon)+\beta_1](M_1 b_1)^2[\tau_1(t_{1n}-\xi_{1n}) - \tfrac{1}{2}(t_{1n}-\xi_{1n})^2] \\ &\leqslant \tfrac{1}{2}(2M_1 b_1 \tau_1 - 1)[(U_1+\varepsilon)+\beta_1] \\ &\leqslant \tfrac{1}{2}(2M_1 b_1 \tau_1 - 1)[x_1(t_{1n})+\beta_1+2\varepsilon]. \end{aligned}$$

Case 2.3. $M_1 b_1 \tau_1 > 1$ and $M_1 b_1(t_{1n}-\xi_{1n}) > 1$. In this case, let $\eta_{1n} \in [\xi_{1n}, t_{1n}]$ be such that $M_1 b_1(t_{1n}-\eta_{1n}) = 1$. Then by (4.23) we have

$$\begin{aligned} x_1(t_{1n}) - x_1(\xi_{1n}) &\leqslant [(U_1+\varepsilon)+\beta_1]M_1 b_1\left[\eta_{1n}-\xi_{1n} + M_1 b_1 \int_{\eta_{1n}}^{t_{1n}} (\xi_{1n}+\tau_1-t)\,\mathrm{d}t\right] \\ &= [(U_1+\varepsilon)+\beta_1][(M_1 b_1)^2\tau_1(t_{1n}-\eta_{1n}) - \tfrac{1}{2}(M_1 b_1)^2(t_{1n}-\eta_{1n})^2] \\ &= \tfrac{1}{2}(2M_1 b_1 \tau_1 - 1)[(U_1+\varepsilon)+\beta_1] \\ &\leqslant \tfrac{1}{2}(2M_1 b_1 \tau_1 - 1)[x_1(t_{1n})+\beta_1+2\varepsilon]. \end{aligned}$$

Combining Cases 2.1–2.3, we have

$$x_1(t_{1n}) \leqslant \begin{cases} \dfrac{[2+(M_1 b_1 \tau_1)^2]c_1}{[2-(M_1 b_1 \tau_1)^2]b_1}(U_2+\varepsilon) + \dfrac{2\varepsilon(M_1 b_1 \tau_1)^2}{2-(M_1 b_1 \tau_1)^2} & \text{if } M_1 b_1 \tau_1 \leqslant 1, \\ \dfrac{[1+2(M_1 b_1 \tau_1)]c_1}{[3-2(M_1 b_1 \tau_1)]b_1}(U_2+\varepsilon) + \dfrac{2\varepsilon(2M_1 b_1 \tau_1 - 1)}{3-2(M_1 b_1 \tau_1)} & \text{if } M_1 b_1 \tau_1 > 1. \end{cases}$$

This shows that (4.21) is true. Letting $n \to \infty$ and $\varepsilon \to 0$ in (4.21), we obtain

$$U_1 \leqslant b_{12} U_2. \tag{4.23}$$

Similarly, we have

$$U_2 \leqslant b_{21} U_1. \tag{4.24}$$

By (4.23) and (4.24), we have

$$U_1 \leqslant b_{12} b_{21} U_1 < U_1 \quad \text{and} \quad U_2 \leqslant b_{12} b_{21} U_2 < U_2.$$

This is a contradiction. The proof is complete. □

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