Population Dynamic Models with Nonlocal Delay on Bounded Domains and Their Numerical Computations *

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Abstract

In this paper, we consider the growth dynamics of a single species population with two age classes and a fixed maturation period living in a bounded spatial domain. A Reaction Diffusion Equation (RDE) model with time delay and nonlocal effect is derived if the mature death and diffusion rates are age independent. We consider and analyse numerical solutions of the models with some birth functions appeared in the well-known Nicholson's blowflies equation. In particular, we report our numerical observations about the occurrences of the periodic waves under certain range of birth rate and death rate parameters.

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1 Introduction

Recently, there has been considerable interest in modelling and analysis of population dynamics with both time delay and diffusion in a spatially varying environment. Of particular concern is the joint impact of the individual motion and the maturation delay on the formulation of the model and on the population dynamics.

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A typical case is the widely used reaction-diffusion equation with delay and local effect (see, Brition [1], Levin [3], Memory [5], Murray [7], and Yoshida [18]), given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + pu(t,x)\left(1 - \frac{u(t-r,x)}{K}\right), \qquad t > 0, \tag{1.1}$$

where $u(t,x)$ is the density of the population of the species at time $t \geq 0$ and location x, D is the diffusion parameter, $p > 0$ is the birth rate parameter and $K > 0$ is the carrying capacity of the environment. $r \geq 0$ is the delay parameter, which models the fact that the regular effect depends on the population at earlier time, $t - r$. The model (1.1) is obtained from the well-known logic ODE model by simply introducing a diffusion term and incorporating a discrete delay in the birth term. This approach already results in an equation with many technical difficulties, see Wu [15]. But in recent years it has been recognised that there are modelling difficulties with this approach. The difficulty is that diffusion and time-delays, even though they are associated with space and time respectively, are not independent of each other, since an individual might not have been at the same location in space at previous times.

Smith [9], Smith and Thieme [10] developed an approach to derive a scalar delay differential equation from the so-called structured model where the population is divided into immature and mature age classes with the time delay being the maturation period. The same idea was also used in So, Wu and Zou [12] to derive a system of delay differential equations for matured population distributed in a patchy environment. So, Wu and Zou in [13] considered the case of a continuous unbounded spatial domain and derived the following non-local reaction-diffusion equation with delay

$$\frac{\partial w}{\partial t} = D_m\frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon\int_{-\infty}^{+\infty} b(w(t-r,y))e^{\frac{-(x-y)^2}{4\alpha}}\,dy, \quad t \geq 0,\ x \in R. \tag{1.2}$$

Here $w(t,x)$ is the total matured population, D_m and d_m are diffusion and death rates for the matured population, respectively, which are age independent. ε is the impact parameter of the death rate for immature and α is a parameter reflecting the effect of the dispersal rate of the immatured population, and $b : [0,\infty) \to [0,\infty)$ is the birth function. Applying the method developed in Wu and Zou [16], existence of travelling wavefronts for (1.2) was studied in [13]. In a more recent work, Liang and Wu [4] considered a species living in a spatially transporting field and derived a model similar to (1.2) but with an advection term accounting for the spatial transport and a spatial translation in the nonlocal delay effect term. Travelling wavefronts were studied both theoretically and numerically in [4].

In this paper, we consider a single species population with two age classes and a fixed maturation period living in a bounded spatial domain. We derive a reaction diffusion equation model with time delay and nonlocal spatial effect when the mature death and diffusion rates are age independent. We also consider and analyse numerical computation of the bounded domain model in the case where the birth functions are the ones appeared in the well-known Nicholson's blowflies equation. In particular, we report our numerical observations about the occurrences of the periodic waves under certain range of birth rate and death rate parameters.

The paper is organised as follows. In Section 2, we derive the reaction diffusion equation models with time delay and nonlocal effect in a one-dimensional bounded domain. Then, the numerical methods are described for solving the nonlocal delay problem in Section 3, and the numerical computation and analysis of the problems are provided in Section 4. Some final remarks are then provided in Section 5.

2 The RDE Models

We consider a single species population in one dimensional spatial domain. Let $u(t,a,x)$ denote the density of the population of the species at the time $t \geq 0$, the age $a \geq 0$, and the spatial location $x \in \Omega = [0, L]$. Let $D(a)$ and $d(a)$ denote the diffusion rate and the death rate, respectively, at age a. Then, the population density function $u(t,a,x)$ satisfies

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u, \quad t > 0, \quad a > 0, \quad x \in \Omega. \tag{2.1}$$

We consider the Neumann boundary conditions:

$$\frac{\partial}{\partial x}u(t,a,0) = 0, \quad \frac{\partial}{\partial x}u(t,a,L) = 0, \quad t \geq 0, \quad a \geq 0. \tag{2.2}$$

Other types of boundary conditions will also be discussed later in the section.

Now, we assume that the population has only two age stages as matured and immatured species, and let $r \geq 0$ be the maturation time for the species and $a_l > 0$ denote the life limit of an individual species. Therefore, $u(t, a_l, x) = 0$ at any time $t > 0$ and any $x \in \Omega$. We integrate the population density $u(t,a,x)$ with respect to age a from r to a_l to obtain the total matured population $w(t,x)$, i.e.,

$$w(t,x) = \int_r^{a_l} u(t,a,x)\,da, \quad x \in \Omega, \quad t \geq 0. \tag{2.3}$$

Since only the mature can reproduce, we have

$$u(t,0,x) = b(w(t,x)) \quad t \geq 0, \quad x \in \Omega, \tag{2.4}$$

where $b : [0,\infty) \to [0,\infty)$ is the birth function.

If the diffusion and death rates for the matured population are age independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in [r, a_l]$, then integrating (2.1) leads to

$$\frac{\partial w}{\partial t} = u(t,r,x) + D_m\frac{\partial^2 w}{\partial x^2} - d_m w. \tag{2.5}$$

In (2.5), we need to eliminate $u(t,r,x)$ to obtain an equation for $w(t,x)$. This can be achieved as follows. Let us fix $s \geq 0$ and define $V^s(t,x) = u(t, t-s, x)$ for $s \leq t \leq s+r$. Then, from (2.1), it follows, for $s \leq t \leq s+r$, that

$$\frac{\partial V^s(t,x)}{\partial t} = D(t-s)\frac{\partial^2 V^s(t,x)}{\partial x^2} - d(t-s)V^s(t,x), \tag{2.6}$$

with

$$V^s(s,x) = b(w(s,x)), \quad x \in \Omega, \tag{2.7}$$

and the corresponding boundary conditions:

$$\frac{\partial}{\partial x} V^s(t,0) = 0, \quad \frac{\partial}{\partial x} V^s(t,L) = 0, \qquad t \geq 0. \tag{2.8}$$

Note that (2.6) is a linear reaction diffusion equation, we can solve (2.6)-(2.8) by the method of separation variables. Let $V^s(t,x) = \Psi(t)\Phi(x)$, from (2.6), it leads to

$$\Psi'(t)\Phi(x) = D(t-s)\Psi(t)\Phi''(x) - d(t-s)\Psi(t)\Phi(x). \tag{2.9}$$

The corresponding eigenvalue problem of (2.6) (2.8) is

$$\frac{d^2\Phi}{dx^2} + \lambda\Phi = 0, \qquad 0 < x < L, \tag{2.10}$$

$$\Phi'(0) = 0, \quad \Phi'(L) = 0. \tag{2.11}$$

We have the following solutions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, \cdots,$$
$$\Phi_n(x) = \cos\sqrt{\lambda_n}x, \qquad n = 0, 1, 2, \cdots.$$

Further, we obtain the following series solution for (2.6) (2.8):

$$V^s(t,x) = \sum_{n=0}^{\infty} a_n(s) e^{-\int_s^t [\lambda_n D(\theta - s) + d(t-\theta)]d\theta} \cos\sqrt{\lambda_n}x. \tag{2.12}$$

With the help of the initial condition (2.7) at $t = s$, it can be derived that

$$\begin{aligned} a_0(s) &= \frac{1}{L}\int_0^L b(w(s,y))\, dy, \\ a_n(s) &= \frac{2}{L}\int_0^L b(w(s,y)) \cos\sqrt{\lambda_n}y\, dy, \quad n = 1, 2, \cdots. \end{aligned}$$

Let D_I and d_I denote the diffusion and death rates of the immatured population, respectively, i.e., $D(a) = D_I(a)$ and $d(a) = d_I(a)$ for $a \in [0, \tau]$. Let

$$\varepsilon = e^{-\int_0^\tau d_I(a)\, da}, \quad \alpha = \int_0^\tau D_I(a)\, da. \tag{2.13}$$

Then, we have

$$\begin{aligned} u(t,r,x) &= V^{t-r}(t,x) \\ &= \varepsilon \sum_{n=0}^{\infty} a_n(t-r) e^{-\lambda_n \alpha} \cos\sqrt{\lambda_n}x \\ &= \frac{\varepsilon}{L}\int_0^L b(w(t-r,y)) \left\{ 1 + \sum_{n=1}^{\infty} \left[\cos\sqrt{\lambda_n}(x-y) + \cos\sqrt{\lambda_n}(x+y) \right] e^{-\lambda_n\alpha} \right\} dy. \end{aligned} \tag{2.14}$$

Therefore we obtain a reaction diffusion equation with time delay for the Neumann boundary condition as follows:

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + F(x, w(t-r, \cdot)), \qquad 0 < x < L, \quad t > 0, \tag{2.15}$$

$$w(t,x) = w_0(t,x), \quad 0 < x < L, \quad t \in [-r,0], \tag{2.16}$$

$$\frac{\partial}{\partial x} w(t,0) = 0, \quad \frac{\partial}{\partial x} w(t,L) = 0, \quad t \geq 0, \tag{2.17}$$

where

$$F(x, w(t-r,\cdot)) = \frac{\varepsilon}{L}\int_0^L b(w(t-r,y))\Big\{1+ \sum_{n=1}^{\infty}\Big[\cos\frac{n\pi}{L}(x-y)+\cos\frac{n\pi}{L}(x+y)\Big]e^{-(\frac{n\pi}{L})^2\alpha}\Big\}\, dy \tag{2.18}$$

and w_0 is an initial function which should be specified.

The homogeneous Neumann boundary condition is also called no flux boundary condition accounting for an isolating boundary. The species in this case can not go passing the boundary of the his living habitat. Similarly, we can consider the problems with Dirichlet boundary conditions representing a hostile environment on the boundary, mixed boundary conditions and periodic boundary conditions.

The model for the Dirichlet problem is

$$\begin{aligned}
\frac{\partial w}{\partial t} &= D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \frac{\varepsilon}{L}\int_0^L b(w(t-r,y))\cdot \\
&\quad \left\{\sum_{n=1}^{\infty}\Big[\cos\frac{n\pi}{L}(x-y)-\cos\frac{n\pi}{L}(x+y)\Big]e^{-(\frac{n\pi}{L})^2\alpha}\right\} dy, \\
&\qquad 0 < x < L, \quad t > 0, \\
w(t,0) &= 0, \quad w(t,L) = 0, \quad t \geq 0, \\
w(t,x) &= w_0(t,x), \quad 0 < x < L, \quad t \in [-r,0].
\end{aligned} \tag{2.19}$$

The model for the mixed boundary problem is

$$\begin{aligned}
\frac{\partial w}{\partial t} &= D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \frac{\varepsilon}{L}\int_0^L b(w(t-r,y))\cdot \\
&\quad \left\{\sum_{n=1}^{\infty}\Big[\cos\frac{(2n-1)\pi}{2L}(x-y)-\cos\frac{(2n-1)\pi}{2L}(x+y)\Big]e^{-(\frac{(2n-1)\pi}{2L})^2\alpha}\right\} dy, \\
&\qquad 0 < x < L, \quad t > 0, \\
w(t,0) &= 0, \quad \tfrac{\partial}{\partial x} w(t,L) = 0, \quad t \geq 0, \\
w(t,x) &= w_0(t,x), \quad 0 < x < L, \quad t \in [-r,0].
\end{aligned} \tag{2.20}$$

The model for the periodic problem is:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= D_m \frac{\partial^2 w}{\partial x^2} - d_m w \\
&\quad + \frac{\varepsilon}{2L}\int_{-L}^L b(w(t-r,y))\left\{1 + 2\sum_{n=1}^{\infty}\Big[\cos\frac{n\pi}{L}(x-y)\Big]e^{-(\frac{n\pi}{L})^2\alpha}\right\} dy, \\
&\qquad -L < x < L, \quad t > 0, \\
w(t,-L) &= w(t,L), \quad \tfrac{\partial}{\partial x} w(t,-L) = \tfrac{\partial}{\partial x} w(t,L), \quad t \geq 0, \\
w(t,x) &= w_0(t,x), \quad -L < x < L, \quad t \in [-r,0].
\end{aligned} \tag{2.21}$$

In the models derived above, ε reflects the impact of the death rate of immature and α represents the effect of the dispersal rate of the immature on the growth rate of matured population. $F(x, w(t-r, \cdot))$ represents the non-local spatial effect with time delay. Therefore, we obtain a general model with nonlocal delay effects on bounded domains. When $\alpha \to 0$ and $\varepsilon \to 1$, that is, as the immature becomes immobile and all immatures live to maturity, then the equation becomes

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + b(w(t-r, x)), \qquad 0 < x < L, \quad t > 0, \tag{2.22}$$

which is local time delay problem on a bounded domain. For this local delay problem with the Dirichlet boundary condition and the birth function $b(w) = pwe^{-aw}$, the existence of positive steady solution and numerical computations were studied by So and Yang [14] and So, Wu and Yang [11]. Also for (2.22) with the same birth function but with the Neumann boundary condition, a thorough study on the dynamics of the model was carried out in Yang and So [17]. In the following sections, we will consider numerical simulations to the nonlocal time delay problems for both the Neumann boundary condition and the Dirichlet boundary condition. The numerical computations are done for two more general birth functions which will be given bellow.

3 Numerical Methods

In this section, we will describe the numerical methods for the reaction diffusion problems with nonlocal delay effects. We consider the model with the Neumann boundary condition.

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + F(x, w(t-r, \cdot)), \qquad 0 < x < L, \quad t > 0, \tag{3.1}$$

where

$$\begin{aligned} F(x, w(t-r, \cdot)) = \frac{\varepsilon}{L} \int_0^L & b(w(t-r, y)) \cdot \\ & \left\{ 1 + \sum_{n=1}^{\infty} \left[\cos\frac{n\pi}{L}(x-y) + \cos\frac{n\pi}{L}(x+y) \right] e^{-(\frac{n\pi}{L})^2 \alpha} \right\} dy, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & w(t,x) = w_0(t,x), \qquad 0 < x < L, \quad t \in [-r, 0], \\ & \tfrac{\partial}{\partial x} w(t, 0) = 0, \quad \tfrac{\partial}{\partial x} w(t, L) = 0, \qquad t \geq 0. \end{aligned} \tag{3.3}$$

Take a uniform spatial partition $\mathcal{T}_h$ for domain $\Omega = [0, L]$ with the nodes $x_i, i = 0, 1, \cdots, I_x + 1$, such that

$$0 = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_{I_x+1} = L$$

with $\Delta x = L/(I_x + 1)$ and $x_i = x_0 + i\Delta x$, let $I_i = [x_{i-1}, x_i]$ be the element. And a uniform time partition is defined as

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_{N_t+1} = T$$

with $\Delta t = T/(N_t + 1)$, $t_n = n\Delta t$. Here, I_x and N_t are positive integers. Δx, Δt are called the spatial discretization step and the time step, respectively. We further define $w^n = w(t_n)$, and let W_i^n denote the approximate value of $w(t_n, x_i)$.

Define the discrete difference operators

$$\delta_x^+ w_i = \frac{w_{i+1} - w_i}{\Delta x}, \qquad \delta_x^- w_i = \frac{w_i - w_{i-1}}{\Delta x}. \tag{3.5}$$

So, the differential operators in (3.1) can be approximated by

$$\frac{\partial w}{\partial t}(t_n) \approx \frac{w^n - w^{n-1}}{\Delta t} + O(\Delta t), \tag{3.6}$$

$$\frac{\partial^2 w}{\partial x^2}(t_n, x_i) \approx \delta_x^-(\delta_x^+ w_i^n) + O((\Delta x)^2). \tag{3.7}$$

Further, for getting the numerical scheme of equation (3.1) on the spatial and time partition, we need treat the nonlocal delay term $F(x, w(t-r,\cdot))$ discretely.

Let $W_i^{n-k(r)}$ be the approximation of $w(t_n - r, x_i)$: $W_i^{n-k(r)}$ is W_i^{n-k} if $r = k\Delta t$ for the integer $k \geq 0$; Otherwise, if $k\Delta t < r < (k+1)\Delta t$, then we define $W_i^{n-k(r)}$ be the linear interpolation of W_i^{n-k} and $W_i^{n-(k+1)}$. High order interpolations can be defined from multilevel values to be used to approximate $W_i^{n-k(r)}$ for increasing the accuracy.

Let $K_M(x,y)$ be the approximation function to the infinite series function

$$K_M(x,y) \approx 1 + \sum_{n=1}^{\infty} \Big[\cos\frac{n\pi}{L}(x-y) + \cos\frac{n\pi}{L}(x+y)\Big] e^{-(\frac{n\pi}{L})^2\alpha} \tag{3.8}$$

with the rate $O(e^{-(\frac{M\pi}{L})^2\alpha})$.

So, we can apply the quadrature techniques to discrete the nonlocal delay term $F(x, w(t-r,\cdot))$ by

$$F_{h,i}^n = F_h(x_i, W_i^{n-k(r)}) \approx \frac{\varepsilon}{L}\int_0^L b(W^{n-k(r)})(y) K_M(x_i, y)\ dy. \tag{3.9}$$

If the Composite Simpson's rule is used, then the error is $O((\Delta x)^4)$. And some fast computation techniques can also be applied to the approximations.

As $W_i^{n-k(r)}$ is defined at the nodes of the spatial partition, we may introduce the interpolation technique to define high order polynomial $\Pi_h W^{n-k(r)}$ for approximating $w(t-r)$, then, the nonlocal delay term can be obtained without using the numerical integration:

$$F_{h,i}^n = \frac{\varepsilon}{L}\int_0^L b(\Pi_h W^{n-k(r)})(y) K_M(x_i, y)\ dy, \tag{3.10}$$

which is expected to obtain much better numerical results.

4 Numerical Computations

In this section, we will numerically study and analyze the solution of the reaction diffusion equation model with nonlocal delayed effects derived in Section 2. In our computations, we will consider two general birth functions, which have been widely used in the well-studied Nicholson's blowflies equation for some special parameters (for example, see, [2], [4], [5], [7], [8], [11], and [13]).

These functions are given by

$$b_1(w) = pwe^{-aw^q}, \tag{4.1}$$

with constants $p > 0$, $q > 0$ and $a > 0$; and

$$b_2(w) = \begin{cases} pw\left(1 - \dfrac{w^q}{K_c^q}\right), & 0 \leq w \leq K_c, \\ 0, & w > K_c, \end{cases} \tag{4.2}$$

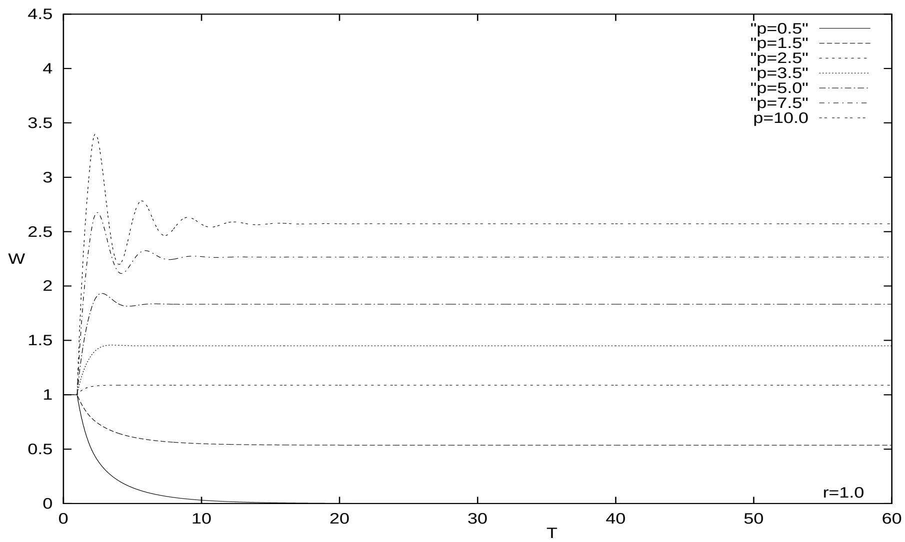

Figure 1: The total matured population at $x = \pi$ under nonlocal delayed effects with the birth function $b_1(w) = pwe^{-aw^q}$. The data are $D_m = 1, d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$ and $r = 1$. We illustrate the effects of the birth rate parameter p from $p = 0.5$ to $p = 10.0$.

with constants $p > 0, K_c > 0$ and $q > 0$.

We solve the reaction diffusion equation with nonlocal delayed effects by using the finite difference method coupled with iterative technique, which is described in Section 3. The numerical results that we report in this section show that biological realistic solutions occur for the reaction diffusion equation models with nonlocal temporally delay effects for a wide range of parameters. Our numerical experiments show that the positive solutions exist under the large range of the biological parameters. Moreover, when the ratio of the birth parameter over the death parameter passes a certain value, our simulations show that a positive periodic solution (periodic wave) occurs.

4.1 Neumann Problems with $b_1(w) = pwe^{-aw^q}$. First, we consider the Neumann problem with nonlocal delayed effects and with the birth function $b_1(w) = pwe^{-aw^q}$. This birth function with $q = 1$ has been widely used in the well-studied Nicholson's blowflies equation. It increases monotonically before reaching the peak, then decays almost exponentially to zero.

Let the domain $\Omega = [0, 2\pi]$. The species satisfies the homogeneous Neumann boundary condition, that is the total matured population $w(t, x)$ has

$$\frac{\partial}{\partial x} w(t, 0) = 0, \quad \frac{\partial}{\partial x} w(t, 2\pi) = 0, \qquad t \geq 0. \tag{4.3}$$

The initial value $w(t, x) = w_0(t, x)$ on $\Omega \times [-r, 0]$ is given as a constant $w_0(t, x) = 1$. We take the uniform spatial and time partition with the step sizes Δx and Δt.

Example 1. Let the diffusion coefficient $D_m = 1$ and the death rate $d_m = 1$ for the mature population, $\alpha = 1$ and $\varepsilon = 1$ for the immature population, and the maturation age (the time delay) $r = 1$. Let $a = 1$ and $q = 1$ in the birth function $b_1(w)$. We then numerically

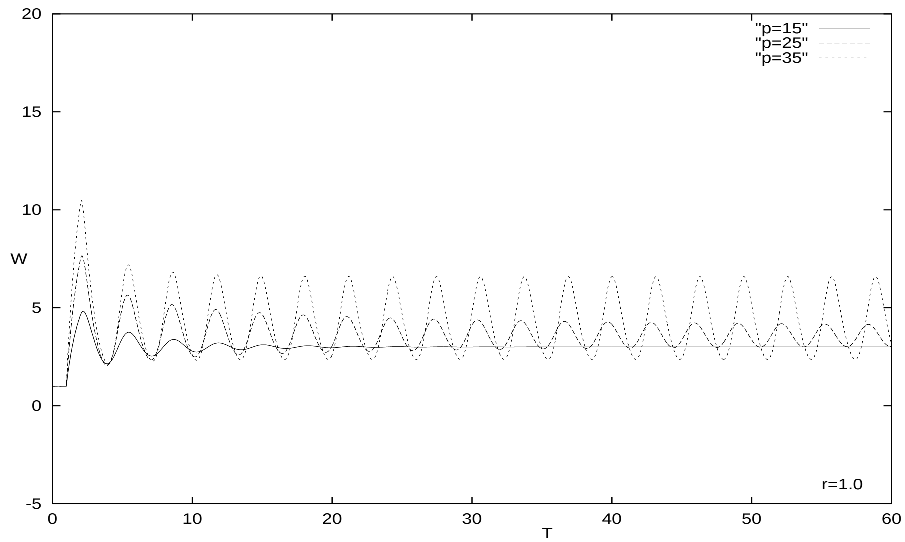

Figure 2: The total matured population at $x = \pi$ under nonlocal delayed effects with the birth function $b_1(w) = pwe^{-aw^q}$. The data are $D_m = 1, d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$. We show the effects of large birth rate parameter $p = 15$, $p = 25$ and $p = 35$ with the time delay $r = 1$.

observe the solutions by varying the birth rate p from $p = 0.5$ to $p = 35$.

The numerical solutions at the middle point $x = \pi$ of the domain are shown in Figure 1 - 3. The x-axis is the t-direction as well the vertical direction represents the value of the total matured population. For the cases with the data in the example 1 the positive solutions exist. In Figure 1, it is clear to see that when p is less than a number ($p = 0.5, 1.5, 2.5, 3.5$) the solution is monotonic as the time increases. When $p = 0.5, 1.5$ the solution converges to a steady value less than the initial value, specially, it is going to zero as the birth rate p is very small ($p \leq 0.5$). In a certain range of $p = 2.5, 3.5$ the population increases and converges to a steady solution. However, when the birth rate parameter goes over a certain value ($p = 5, 7.5, 10.0$) the solutions will still converge to a steady solution but it will be oscillation at the beginning of the time.

In Figure 2, we further increase the birth rate p from $p = 15$ to $p = 35$. We can see clearly that the total matured population for large birth rate $p = 35$ is a periodic wave. In this test the time delay is $r = 1$. Increase time delay to $r = 2$ and use all same other data, the periodic waves appear clearly in Figure 3 for $p = 15, p = 25$ and $p = 35$. Comparing the solutions in Figure 2 and 3, it can be seen that the large delay leads to the occurrence of the periodic waves ($p = 15, p = 25$) and also increase the period length of the periodic waves ($p = 35$). The peak value of the periodic waves are also affected by the birth rate parameter in Figure 3. The peak is getting large as the time delay r becomes large. And the three dimensional surface of the periodic wave is shown in Figure 4.

4.2 Neumann Problems with $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$. Now, we consider the numerical

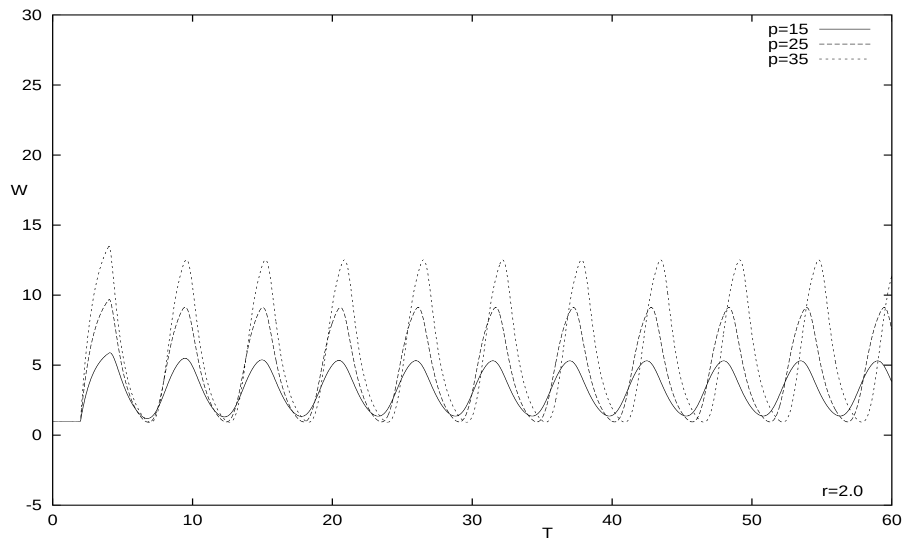

Figure 3: The total matured population at $x = \pi$ under nonlocal delayed effects with the birth function $b_1(w) = pwe^{-aw^q}$. The data are $D_m = 1, d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, and $q = 1$. The change under the effects of the large time delay $r = 2$ with the birth rate parameter $p = 15$, $p = 25$ and $p = 35$.

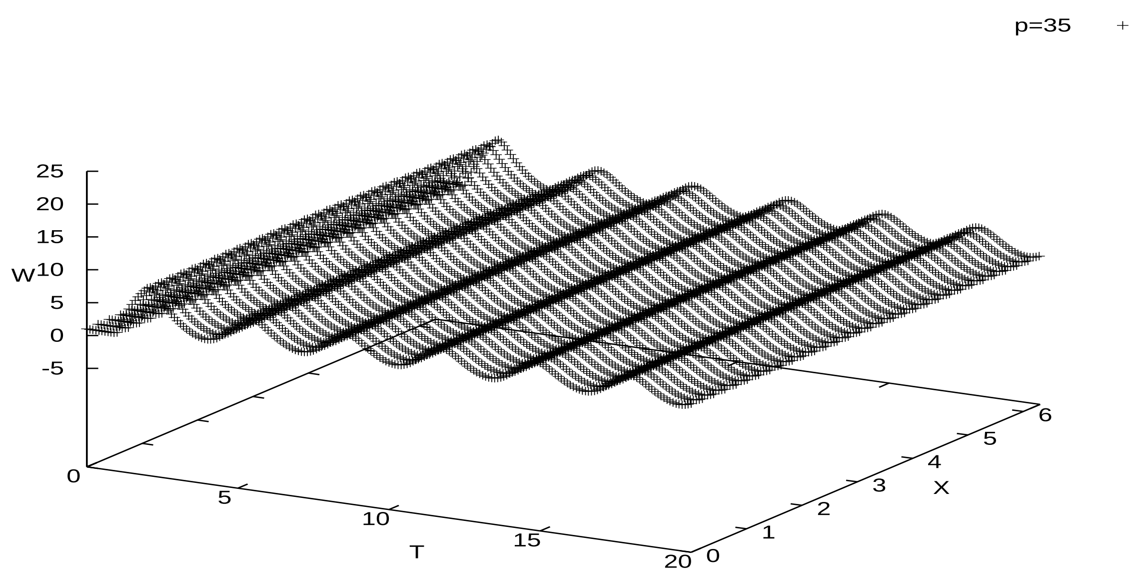

Figure 4: The three dimensional surface of the periodic wave with birth function $b_1(w) = pwe^{-aw^q}$, while the birth rate parameter $p = 35$ and other parameters $D_m = 1$, $d_m = 1$, $\alpha = 1$, $q = 1$, $\varepsilon = 1$, $a = 1$, $r = 1$.

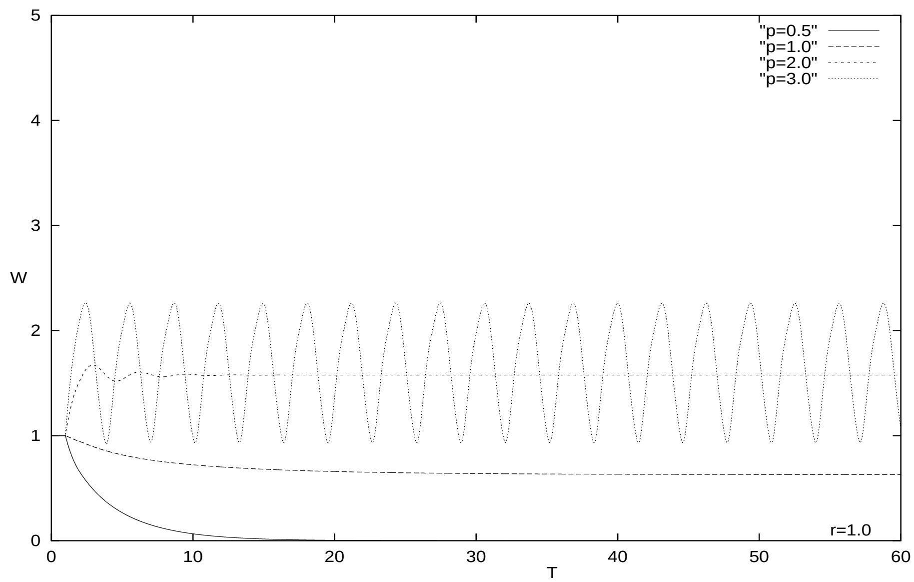

Figure 5: The shape of the solution of the total matured population at $x = \pi$ with birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$, while the birth rate parameter $p = 0.5, 1, 2, 3$ is varied. Other data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $q = 2$, $K_c = 2$, and$r = 1$.

solution with birth function:

$$b_2(w) = \begin{cases} pw\left(1 - \dfrac{w^q}{K_c^q}\right), & 0 \le w \le K_c, \\ 0, & w > K_c, \end{cases} \tag{4.4}$$

with constants $p > 0, K_c > 0$ and $q > 0$.

In this part, we show the effects of the varying parameter q and the carrying capacity parameter K_c of the birth function on the solution of the matured population. Meanwhile, we also consider the effect of varying the birth rate parameter p.

Example 2. Let $\alpha = 1, \varepsilon = 1$ for the immature population. Fix the diffusion constant $D_m = 1$ and the death rate constant $d_m = 1$. We will observe numerically the effects of the birth rate parameter p, and the carrying capacity parameter K_c, the parameter q and the time delay r to the total matured population $w(t, x)$. (a) $q = 2$, $K_c = 2$ and $r = 1$ and vary the birth rate parameter p from $p = 0.5$ to $p = 3$; (b) Take $p = 5$, $q = 2$ and $r = 1$, choose the carrying capacity parameter $K_c = 2$ and $K_c = 4$; (c) Let $K_c = 2$, $p = 5$ and $r = 1$, change the parameter q from $q = 0.5$ to $q = 2$; (d) $q = 2$, $K_c = 2$ and $p = 5$. Vary the time delay $r = 1$ and $r = 2$.

The numerical solutions of the matured population at the middle point $x = \pi$ are shown in Figure 5 - 8. In Figure 5, we can see that the solution of the total matured population is monotonic as the time increases and converges to a steady value when p is small ($p = 0.5, 1.0, 2.0$). But, when the birth rate parameter goes over a certain value ($p = 0.3$), the total matured population appears a periodic wave. The effect of the carrying capacity K_c of the environment to the matured population are shown in Figure 6. Increasing the

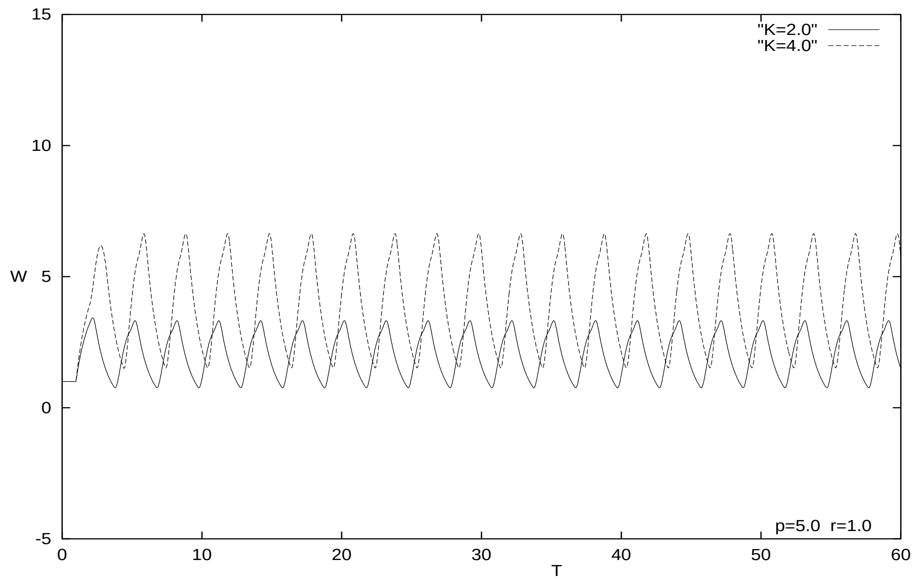

Figure 6: The effect of the carrying parameter K_c to the matured population with the birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$, while data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $q = 2$, $r = 1$ and $p = 5$. Take $K_c = 2.0$ and $K_c = 4.0$.

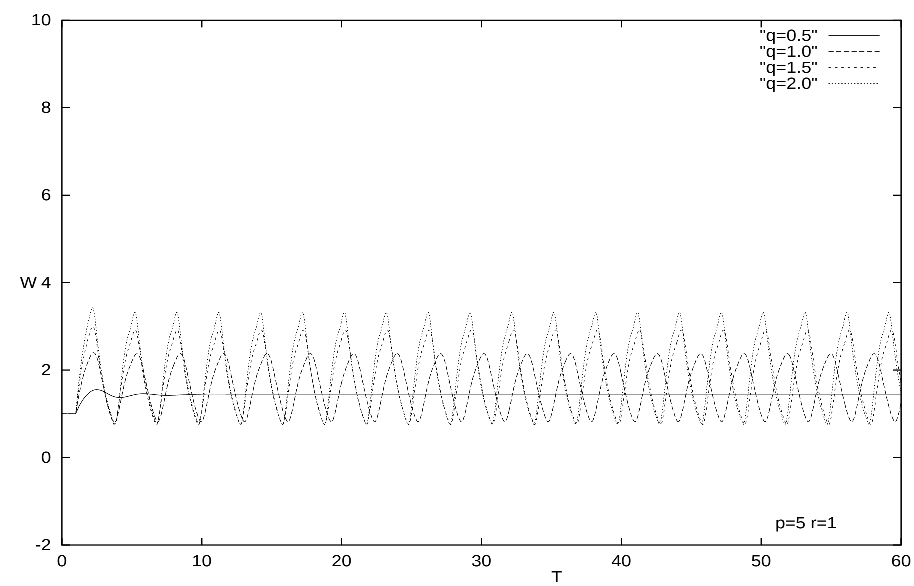

Figure 7: The shape of total matured population at $x = \pi$ with the birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$, while the parameter $q = 0.5, 1.0, 1.5, 2.0$ is varied. Other data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 2$, $r = 1$ and $p = 5$.

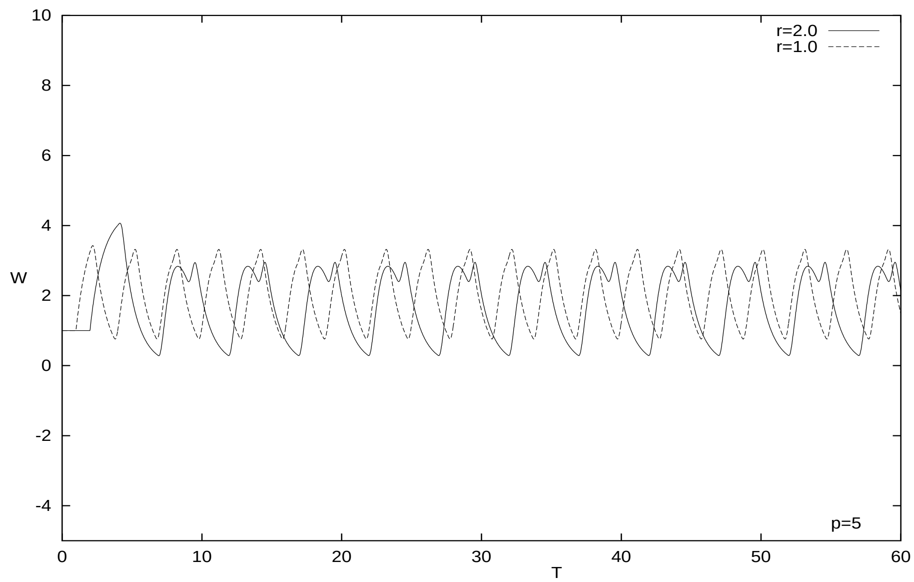

Figure 8: The effect of the time delay on the total matured population with birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$. The data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 2, q = 2$, and $p = 5$. The time delay is chosen as $r = 1.0$ and $r = 2.0$.

carrying capacity parameter K_c will lead to the increasing peak height of the periodic waves as well as the small change of the period of the periodic waves ($K_c = 2.0, 4.0$). Similar results are obtained for considering the effect of the parameter q in Figure 7. As varying the value $q = 0.5, 1.0, 1.5, 2.0$, the period and the peak height of the periodic wave change. Figure 8 shows the computation of the cases with two different time delay $r = 1$ and $r = 2$. The values of the time delay affect the periodic wave solutions, it can be seen that the period and the shape of the periodic wave have changed.

4.3 Dirichlet Problems with Nonlocal Delayed Effects. In this part, we will consider the numerical computation for the Dirichlet problems with nonlocal delayed effects for both the birth function $b_1(w) = pwe^{-aw^q}$ and $b_2(w) = pw\left(1 - \dfrac{w^q}{K_c^q}\right)$. We show the effects of the diffusion rate D_m of the matured population and the carrying capacity parameter K_c as well as the time delay on the matured population.

Example 3. We consider the Dirichlet problem with the birth function $b_1(w) = pwe^{-aw^q}$ and show the effect of the diffusion D_m on the solution of the matured population. Take the birth rate $p = 80$, the death rate $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, and $q = 1$. Let the time delay $r = 1$. We numerically compute the solution with different diffusion values $D_m = 0.1, 1.0, 10$.

For this Dirichlet problem, the total matured population also has the periodic wave solution when the birth rate parameter are in a certain range. The numerical computation results for the problem with the birth function $b_1(w) = pwe^{-aw^q}$ are shown in Figure 9 - 10. The plot at $x = \pi$ shows the effect of the diffusion parameter D_m on the matured population in Figure 9. The three dimensional surface for $p = 60$ is shown in Figure 10.

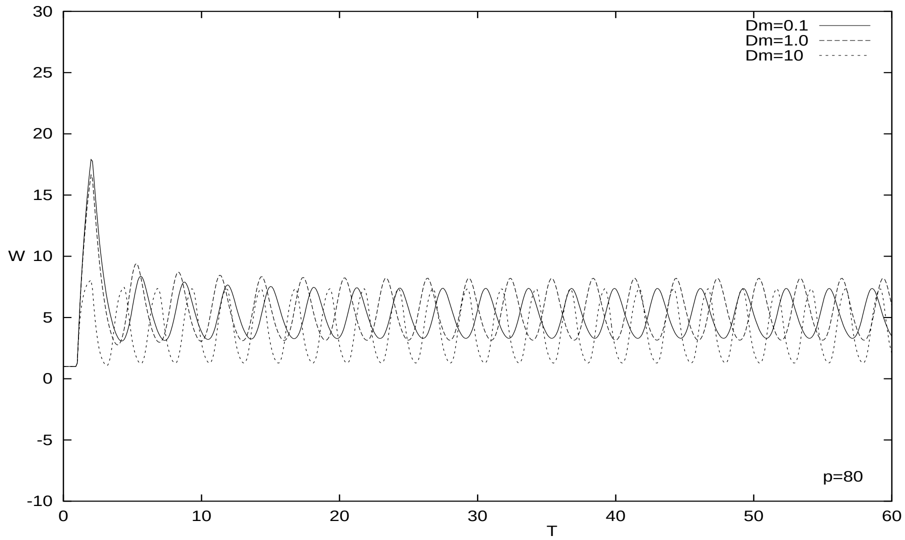

Figure 9: The total matured population at $x = \pi$ under nonlocal delayed effects with the birth function $b_1(w) = pwe^{-aw^q}$. The data are $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$, $p = 80$, and the time delay $r = 1$. We show the effects of the diffusion parameter $D_m = 0.1, 1.0, 10$.

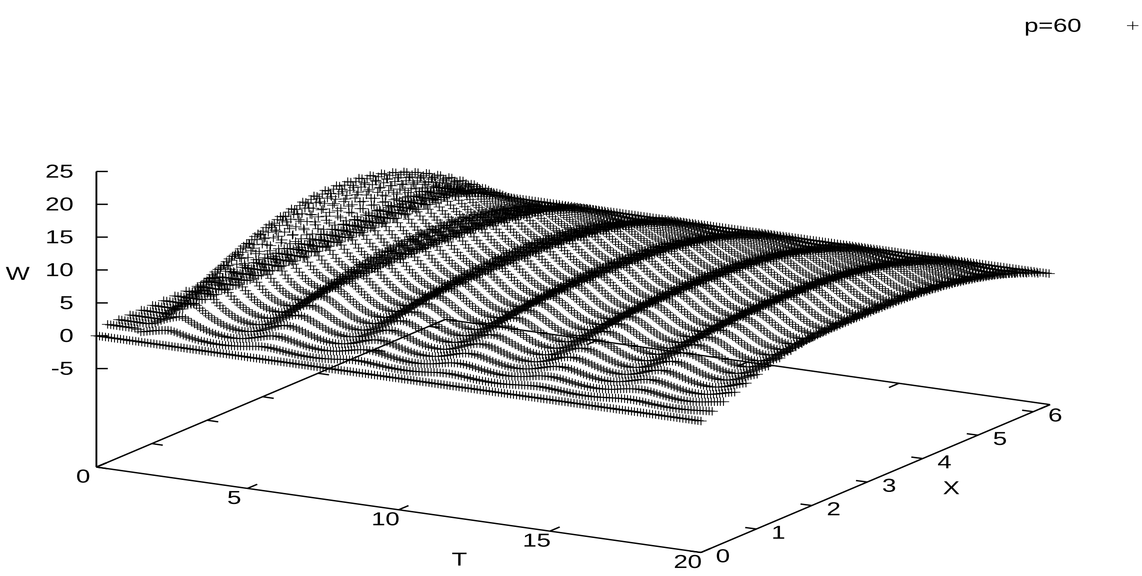

Figure 10: The three dimensional surface of the periodic wave with birth function $b_1(w) = pwe^{-aw^q}$, while the birth rate parameter $p = 60$, and other parameters $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $a = 1$, $q = 1$ and $r = 1$.

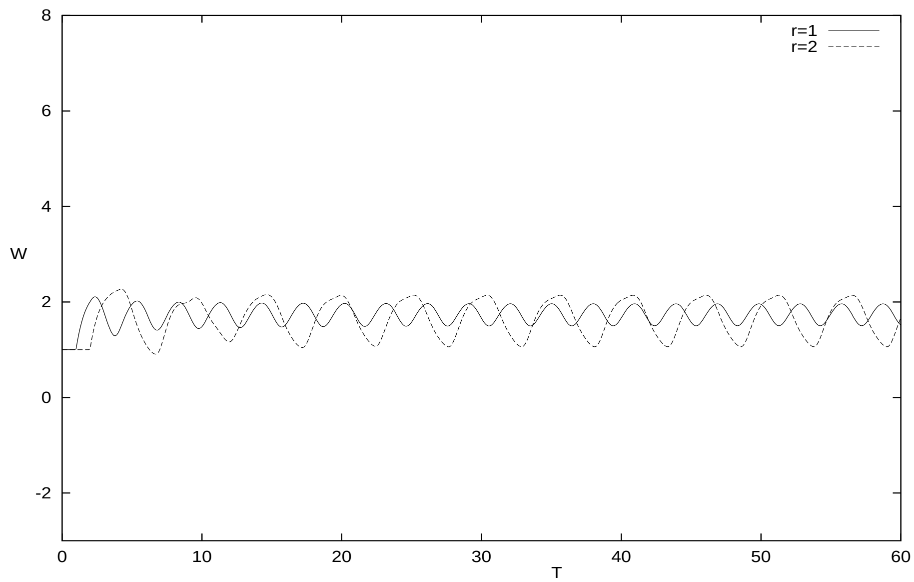

Figure 11: The effect of the time delay on the total matured population with birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$. The data are $D_m = 1$, $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 2, q = 2$, and $p = 4$. The time delay is chosen as $r = 1.0$ and $r = 2.0$.

Example 4. Consider the Dirichlet problem with the birth function $b_2(w) = pw\left(1 - \frac{w^q}{K_c^q}\right)$. we will focus on the effect of the time delay r and the diffusion D_m on the total matured population. The data in this computation are the death rate $d_m = 1$i, $q = 1$, the carrying capacity parameter $K_c = 2$, $\alpha = 1$ and $\varepsilon = 1$. (a) Choose $D_m = 1$ and $p = 4$, change the time delay $r = 1$ and $r = 2$; (b) Let $r = 1$ and $p = 60$, vary the diffusion parameter $D_m = 0.1, 1.0, 10$.

For this Dirichlet's case with $b_2(w) = pw\left(1 - \frac{w^q}{K_c^q}\right)$, the numerical solutions of the total matured population at $x = \pi$ are given in Figure 11 - 12. It is very clear to see that for this case the values of the time delay will affect the periodic wave solutions (see Figure 11). It not only increases the period of the wave, but also change the the shape of the periodic wave for this birth function $b_2(w)$. The plot at $x = \pi$ shows the effect of the diffusion parameter D_m on the population in Figure 11. For this birth function, the effect of the diffusion parameter on the shape of the periodic wave is much sensitive in Figure 12.

5 Conclusion

In this paper, we develop a reaction diffusion equation (RDE) model for the growth dynamics of a single species population living in a bounded spatial domain. The model is derived from an age structured population model and contains a time delay and nonlocal effect term, in which the fixed maturation period is considered as the time delay. The model can be used to study the behaviour of the mature population.

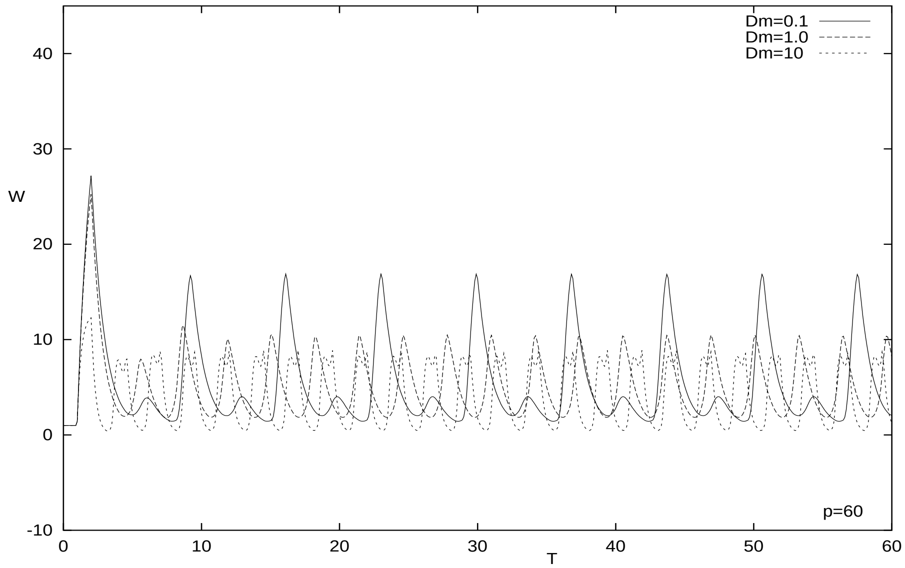

Figure 12: The effect of the diffusion parameter on the total matured population with the birth function $b_2(w) = pw(1 - \frac{w^q}{K_c^q})$, while the parameter $D_m = 0.1, 1.0, 10$ is varied. Other data are $d_m = 1$, $\alpha = 1$, $\varepsilon = 1$, $K_c = 2, q = 2$, $r = 1$ and $p = 60$.

Our numerical analyses for the solutions of the model with two widely used birth functions are reported in Section 4. When the ratio of the birth rate parameter p over the death rate parameter d_m is in a certain range, the solution of the mature population is positive and converges to a steady solution in t-direction. Outside of this range, numerical simulations suggest possible occurrence of the positive periodic wave solution in t-direction. Furthermore, numerical results show that the value of the time delay will affect the period and the peak height of the wave, even affect the shape of the periodic wave. The further theoretical investigation of these positive periodic waves of the problems will be the next step work. However, the theoretical analysis for finding the relation between the period value of wave and the value of the time delay would be a challenging task.

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