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SPATIAL SPREADING WAVE IN A DIFFUSIVE SIR MODEL WITH DELAYED INFECTION FORCE REFLECTING HOST PRECAUTION

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Abstract. In this paper, we first propose a diffusive SIR model with the incidence rate function revised to a general nonlinear incidence rate function with delay that reflects the impact of behavior changes due to the precaution of the host. The main concern is the existence/non-existence of traveling wave solutions. When $R_0 \leq 1$, we prove that the model does not allow a traveling wave for any speed. When $R_0 > 1$ however, we show that there exists a minimal wave speed $c_* > 0$ in the sense that for every $c \geq c_*$, the model has a traveling wave with speed c, while the model does not have a traveling wave with any speed $c \in (0, c_*)$. We derive an equation that implicitly determines the final size of the model. Finally, for some examples, we numerically explore the impact of some model parameters; particularly, and interestingly, we show that the parameters involved in the behavior change term can cause multiple outbreaks. These novel results are mathematically interesting and practically significant because they can help us better understand the role and impact of some non-pharmaceutical interventions during epidemics.

1. Introduction.

A classic and widely-referred-to SIR infectious disease model is the system of ordinary differential equations

$$\begin{cases} S'(t) = -\beta S(t)I(t), \\ I'(t) = \beta S(t)I(t) - \gamma I(t), \\ R'(t) = \gamma I(t), \end{cases} \tag{1}$$

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which is a special case of a more general framework in the pioneering work Kermack–McKendrick [11]. Here $S(t)$, $I(t)$, and $R(t)$ denote the populations of the susceptible, infected, and recovered compartments, respectively, at time t. The parameter β is the transmission rate, and γ is the recovery rate. Since there is no demography in this model, it purely represents the transmission dynamics of an infectious disease for which the recovered individuals would carry lifetime immunity to the disease. The dynamics of (1) is fully known: (i) $S(t)$ will decrease from $S(0) = S_0$ to a positive value S_∞, resulting in a final size S_∞ for the epidemic which is implicitly determined by an equation involving β, γ, and S_0; (ii) $I(t)$ will tend to 0 (disease will eventually die out), but the pattern of its dying out depends on the value $S_0 = S(0)$: If $\beta S_0/\gamma < 1$, $I(t)$ directly decreases to 0; if $\beta S_0/\gamma > 1$, there will be an outbreak in the sense that $I(t)$ first increases to certain value until $S(t)$ is lowered to γ/β, and then, it starts to decrease to 0.

To reflect the nature of spatial spreading of an SIR type disease, Hosono and Ilyas [7] incorporated the spatial diffusion into (1), leading to the following reaction diffusion system:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - \beta S(x,t)I(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)I(x,t) - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3 \dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t). \end{cases} \tag{2}$$

Here, the positive constants d_1, d_2, and d_3 are the diffusion coefficients of the susceptible, infected, and recovered individuals. The main concern of (2) is the existence of traveling wave solutions which are a special type of solutions of the form $(S(x+ct), I(x+ct), R(x+ct))$ that connect two homogeneous steady states. Observe that the model (2) has infinitely many homogeneous disease-free equilibria; however, by the results on (1), the traveling wave connecting the two disease-free equilibria $(S_{-\infty}, 0, 0)$ and $(S_\infty, 0, 0)$ is most meaningful and significant. Using the shooting technique and invariant manifold theory, Hosono and Ilyas proved that if the basic reproduction number $\beta S_{-\infty}/\gamma > 1$, then for every $c \geq c_* = 2\sqrt{d_2(\beta S_{-\infty} - \gamma)}$, there exists such a traveling wave solution with speed c. Here, $S_{-\infty}$ plays the role of initial value of the susceptible population in the sense the initial time is pushed to $t = -\infty$ rather than $t = 0$,

In 2012, Wang et al. [19] considered a diffusive model obtained by replacing the mass action incidence rate $\beta S(x,t)I(x,t)$ in (2) with standard incidence rate $\beta S(x,t)I(x,t)/[S(x,t) + I(x,t)]$. Based on the Schauder fixed point theorem and the bilateral Laplace transform, they proved that this system has a traveling wave solution $(S(x+ct), I(x+ct))$ connecting two disease-free equilibria $(S_{-\infty}, 0)$ and $(S_\infty, 0)$ for $c > c_* = 2\sqrt{d_2(\beta - \gamma)}$, provided that the basic reproduction number $\beta/\gamma > 1$, and it has no such traveling wave solution with speed c for $0 < c < c_*$ or if $\beta/\gamma \leq 1$. We notice that they did not explore whether a traveling wave solution when the speed $c = c_*$ exists. Later, Zhou et al. [26] discussed the case of $c = c_*$ for this model and proved the existence of traveling wave solutions with the minimal speed $c = c_*$ when $\beta/\gamma > 1$.

In the context of traveling wave solutions of the diffusive SIR models, in addition to the mass action incidence rate and standard incidence rate in the R-D systems in [7, 19], some other researchers have further investigated the existence of traveling waves for some reaction-diffusion disease models *with delay* in the general incidence rate term [1,9,16,18,22,25,27]. Among these works, Bai and Zhang [1] considered a diffusive *vector-borne disease* model obtained by replacing $\beta S(x,t)I(x,t)$ in (2) with

$$\beta S(x,t)\int_0^r k(\theta)g(I(x,t-\theta))d\theta$$

where the kernel $k(\theta)$ is the probability of an infected vector becomes infectious after θ units of time since infection. They proved that when the basic reproduction number $R_0 > 1$, there exists a c^* that for each $c > c^*$ the diffusive model has a traveling wave solutions with speed c. More recently, Hu and Zou [9] investigated a more general incidence rate function

$$f(S(x,t))\int_0^\infty k(\theta)g(I(x,t-\theta))d\theta$$

with infinite delay, and they proved that when the basic reproduction number R_0 of the model is less than 1, the model has no traveling wave solution; and, when $R_0 > 1$, there exists a $c^* > 0$ such that for every $c > c^*$, the model has a traveling wave with speed c, and for any $c \in (0, c^*)$, the model has no traveling wave with speed c. However, both [1] and [9] did not consider the existence of traveling wave with *minimal speed* $c = c^*$, which is generally much more challenging.

Note that an *incident rate* function can generally be rewritten as $f(t)\cdot S(t)$ in which the function $f(t)$ is called the *infection force*. Obviously, $f(t) = \beta I(t)$ corresponds to the mass action incidence rate, and $f(t) = \beta I(t)/[S(t)+I(t)]$ corresponds to the standard incidence rate. Recently, Cheng and Zou [3] revisited the notion of *infection force* from a new angle. Note that susceptible individuals tend to become more cautious when the disease becomes more severe, and hence adopt more measures to protect themselves. On the other hand, the public health agency may implement some control measures, such as physical distancing, mask wearing, travel restrictions, isolation, and vaccination. Because of these non-pharmaceutical interventions, among the biologically/epidemiologically susceptible individuals, only a fraction of them are remain socially active and can possibly be infected. With this consideration, Cheng and Zou [3–5] introduced the notion of *practically susceptible population* expressed as $S_p(t) = P(L(t))S(t)$ where $L(t)$ is a measure of the severity of the epidemic. Replacing $S(t)$ by $S_p(t)$ in an incidence rate of the form $f(I(t))\cdot S(t)$ gives

$$f(I(t))S_p(t) = [f(I(t))P(L(t))]\cdot S(t), \tag{3}$$

which results in a new infection force $f(I(t)P(L(t))$. Here, the fraction term $P(L)$ is non-increasing in the severity level (prevalence) L and satisfies $P(L)\in[0,1]$ for all $L \geq 0$. When $L(t) = I(t)$ and $f(I) = \beta I$, and $P(L)$ takes some particular forms of decreasing functions, those infection force functions adopted in [2,6,11,13–15,17,23] can all be obtained.

In reality, it may be more reasonable to measure the severity of an epidemic by a weighted sum of infected cases in a past period of time, meaning that

$$L(t) = \int_0^\tau w(\theta)I(t-\theta)d\theta, \tag{4}$$

where the constant $\tau > 0$ specifies a length of time interval one wishes to look at, and the weight function $w(\theta)$ satisfies $w(\theta) \geq 0$ and $\int_0^\tau w(\theta)d\theta = 1$. When the infections are reported only in discrete times, $w(\theta)$ is taken as $w(\theta) = \Sigma_{i=0}^{n} k_i \delta(\tau_i)$. In this case, the severity level $L(t)$ becomes the following discrete weighted sum:

$$L(t) = \Sigma_{i=0}^{n} k_i I(t - \tau_i),$$

where $0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n \leq \tau$ and $k_i \geq 0, i = 0, 1, ..., n$ are the discrete weights satisfying $\Sigma_{i=0}^{n} k_i = 1$.

Such severity measurements together with the aforementioned fraction function reflecting the impact of non-pharmaceutical interventions leads to the following general incidence rate function

$$H_p(t) = f(I(t))S_p(t) = f(I(t))P\left(L(t)\right)S(t) = f(I(t))P\left(\int_0^{\tau} w(\theta)I(t-\theta)d\theta\right)S(t), \tag{5}$$

where the original non-decreasing infection force function $f(t) = f(I(t))$ is revised to

$$f_p(t) = f(I(t))P(L(t)) = f(I(t))P\left(\int_0^{\tau} w(\theta)I(t-\theta)d\theta\right)$$

Now, if one wants to know the impact of the non-pharmaceutical interventions on the spatial spread of an infectious disease of SIR type with a general incidence rate function $f(I(t))$, it is then reasonable to modify the diffusive SIR model (2) to the following new diffusive model:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - f(I(x,t))P\left(L(x,t)\right)S(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + f(I(x,t))P\left(L(x,t)\right)S(x,t) - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3 \dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t). \end{cases} \tag{6}$$

Here, the variables and parameters have the same meanings as those in (2) and

$$L(x,t) = \int_0^{\tau} w(\theta)I(x, t-\theta)d\theta, \tag{7}$$

in which the weight function $w(\theta)$ can be taken as a discrete or continuous form. Based on their background/meanings, the functions $f(I)$ and $P(L)$ are assumed to satisfy the following conditions.

Assumption 1.1. *The original infection force function $f(I)$ satisfies*

- *(i)* $f(0) = 0$ *and* $f(I) > 0$ *for* $I > 0$*;* f *is non-decreasing and Lipschitz continuous in* $[0, S_0]$*, i.e., there is* $L_1 > 0$ *such that* $|f(I_1) - f(I_2)| \leq L_1 |I_1 - I_2|$ *for any* $I_1, I_2 \in [0, S_0]$*;*
- *(ii)* $f'(0^+) := \lim_{I \to 0^+} \frac{f(I)}{I}$ *exists and* $f(I) \leq f'(0^+)I$ *for any* $I \in [0, S_0]$*;*
- *(iii) there exist* $a > 1$*,* $b > 0$*, and* $k > 0$ *such that* $f(I) \geq f'(0^+)I - kI^a$ *for any* $I \in [0, b]$.

Remark 1.2. Assumption (i) is of biological background, and assumptions (ii) and (iii) are mainly used to construct upper and lower solutions, see Lemmas 4.4, 4.5, 4.6, and 5.1. The infection force functions $f(I) = \beta I$ and $f(I) = \beta I/(1 + \alpha I)$ both satisfy all conditions in this assumption.

Assumption 1.3. *The fraction function $P(\cdot)$ satisfies*

- *(i)* $P : [0, +\infty) \to \mathbb{R}$ *is non-increasing and* $P(L) \in [0, 1]$ *for all* $L \geq 0$*;*

(ii) *$P(L)$ is differentiable and $P'(L)$ is bounded on $[0, +\infty)$, i.e., there is $P'_M > 0$ such that $|P'(L)| \le P'_M$ for all $L \ge 0$.*

Remark 1.4. Assumption (ii) is mainly used to construct a lower solution, see Lemma 4.5. There are many functions satisfying this assumption, and below are just some examples:

$$P_1(L) = \frac{1}{1+\alpha L},\ P_2(L) = \frac{1}{1+\alpha L^2},\ P_3(L) = e^{-mL},$$

where all parameters are positive.

In this paper, we focus on the phenomenon of wave propagation of (6) that account for spatial spread of an infectious disease, more specifically, existence of positive traveling wave solutions of (6). Noting that $R(x,t)$ does not appear in the first two equations, we only need to consider the following sub-system:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - f(I(x,t))P\left(L(x,t)\right)S(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + f(I(x,t))P\left(L(x,t)\right)S(x,t) - \gamma I(x,t). \end{cases} \tag{8}$$

Since this model focuses on the transmission dynamics and ignores the demographic structure (i.e., no recruitment and no deaths), this model has infinitely many disease-free equilibria. We are interested traveling waves connecting two disease free equilibria $(S_0, 0)$ and $(S_\infty, 0)$, with S_0 being the initial size of the susceptible population and S_∞ representing the final size of susceptible population. Such a traveling wave solution characterizes the spatial spreading/invasion and the wave speed accounts for the speed of spatial spread of the disease, as well as the severity of an epidemic reflected by the final size, and is thus of theoretical and practical significance.

The method we use in this paper will be a combination of those in [9,19]: The Schauder fixed point theorem, and the bilateral Laplace transform. By these methods, we can determine the basic reproduction number to be $R_0 := f'(0_0^+)P(0)S_0/\gamma$ and identify $c_* := 2\sqrt{d_2(f'(0^+)P(0)S_0 - \gamma)}$ to be the minimal wave speed when $R_0 > 1$ in the sense stated in Theorem 2.1 and Theorem 2.4. Moreover, by constructing a suitable auxiliary function, we show that the I component of the traveling wave, $I(y) = I(x+ct)$, has an upper bound $2c\left(S_0 - S_\infty\right)/(c + \sqrt{c^2 + 4d_2\gamma})$ that not only depends on the wave speed c but also on other parameters in system (8), such as the diffusion parameter d_2 and the recovery rate γ of infected individuals. This is a better estimate compared to the upper bound $S_0 - S_\infty$ given in [1,9,19]. Such a bound can help estimate the prevalence of the disease and understand how the the diffusion rate of $I(t)$, recovery rate γ, and the wave speed affect the magnitude of the epidemic.

Existence of the *minimal speed wave* (traveling wave with minimal wave speed $c = c_*$) is of particular importance because typically an infectious disease begins in a spatially compact region and the corresponding solution evolves toward the traveling wave with a *minimal-speed wave*. However, the proof of existence of the *minimal-speed wave* turns out to be more challenging for (8). This is because although the original infection force $f(t) = f(I(t))$ is non-decreasing in I, the non-parmaceuticaly mediated new infection force function given by (5) can be non-monotone. Therefore, the commonly-used limiting arguments used in [18,20,24] cannot be applied for (8).

We will achieve this task by constructing an appropriate invariant convex set for the traveling wave mapping.

The rest of this paper is organized as follows. In Section 2, we state the main theorems on existence/non-existence of traveling wave solutions of (8). To prove them, we introduce the related differential and integral operators and present some of their important properties in Section 3. In Section 4, we transform the existence problem of traveling wave solutions into an existence problem of a fixed point and prove the existence for the case of $c > c_*$ by applying the Schauder fixed point theorem. We prove the existence of minimal speed wave in Section 5. In Section 6, we prove the non-existence by contradiction in two cases. Combining the results in Sections 4-6, we then complete the proof of the two main theorems. We also derive estimates for the disease prevalence and an equation for the final size that indicate how the final size is affected by the fraction function. In Section 7, we demonstrate the main theorems by choosing $f(I) = \beta I$ and some particular fraction functions for $P(L)$, as well as a special weight function $w(\theta)$. With these chosen functions, we perform rich numerical simulations and shed more light on the disease dynamics; particularly, we observe the occurrence of traveling waves with multiple peaks that account for multiple outbreaks.

2. **Main results.**

We are interested in traveling wave solutions of (8), specifically those of the form $(S(x+ct), I(x+ct))$. Let $y = x + ct$, with $c > 0$ representing the wave speed. Then, the special solution $(S(y), I(y))$ of (8) satisfies

$$\begin{cases} cS'(y) = d_1 S''(y) - f(I(y))P\left(L(y)\right)S(y), & \text{(9a)} \\ cI'(y) = d_2 I''(y) + f(I(y))P\left(L(y)\right)S(y) - \gamma I(y), & \text{(9b)} \end{cases}$$

where $L(y) = \int_0^\tau w(\theta) I(y - c\theta)d\theta$.

Considering the biological background of the model, we expect that the susceptible individuals decrease from the initial population S_0 to a smaller value $S_\infty \in (0, S_0)$, and infected individuals which started with very small values will also eventually die out. Naturally, we hope to find the special solution of (9) with the form $(S(y), I(y))$ satisfying (9) and the following boundary conditions:

$$\begin{aligned} &S(-\infty) := \lim_{y\to-\infty} S(y) = S_0,\ S(\infty) := \lim_{y\to+\infty} S(y) = S_\infty < S_0, \\ &I(\pm\infty) := \lim_{y\to\pm\infty} I(y) = 0. \end{aligned} \tag{10}$$

We are now in the position to state our main results as follows.

Theorem 2.1. *Assume that assumptions 1.1 and 1.3 hold. Let* $R_0 = f'(0^+)P(0) \times S_0/\gamma > 1$ *and*

$$c_* = 2\sqrt{d_2(f'(0^+)P(0)S_0 - \gamma)}.$$

Then, for each $c \geq c_*$, *system* (9)-(10) *has a non-trivial and non-negative solution* $(S(y), I(y))$. *Moreover,* $(S(y), I(y))$ *has the following properties:*

(i) $S(y)$ *is non-increasing;*

(ii) $0 < I(y) \leq \dfrac{2c\left(S_0 - S_\infty\right)}{c + \sqrt{c^2 + 4d_2\gamma}}$ $(< S_0)$ *for all* $y \in \mathbb{R}$;

(iii) the final size S_∞ *satisfies the integral equalities*

$$\gamma \int_{-\infty}^{+\infty} I(y)dy = \int_{-\infty}^{+\infty} f(I(y))P\left(\int_0^\tau w(\theta)I(y-c\theta)d\theta\right)S(y)dy = c\left(S_0 - S_\infty\right). \tag{11}$$

Remark 2.2. The conclusion in Theorem 2.1-(iii) offers an equation relating the final size S_∞ to the infection force function $f(\cdot)$, the precaution faction function $P(\cdot)$ as well as the severity measurement reflected by the weight function $w(\theta)$. This together with the estimate in Theorem 2.1-(ii), provides a basis for numerically computing the final size when the aforementioned three functions are given. Of course, it can also help us analytically explore, to certain extent, the impact of some model parameters. For example, one can see that anything that can increase (decrease) the left-hand side in (11) will reduce (increase) the final size S_∞.

Remark 2.3. Denote

$$\hat{f}(I) = \begin{cases} f(I), \ I \in [0, S_0]; \\ f(S_0), \ I > S_0. \end{cases}$$

If system (9) with $f(I)$ replaced by $\hat{f}(I)$ has a non-trivial and non-negative traveling wave solution $(S(y), I(y))$ satisfying (i) and (ii), then clearly this solution is also a traveling wave solution of system (9) with the original $f(I)$. Then, we can prove Theorem 2.1 under the assumption $f(I) \equiv f(S_0)$ for $I > S_0$. Thus, by Assumption 1.1 we conclude that $f(I)$ is non-decreasing and Lipschitz continuous for $I \geq 0$ and

$$f(I) \leq f(S_0), \ f(I) \leq f'(0^+)I, \text{ for any } I \geq 0.$$

Theorem 2.4. *Assume that Assumptions 1.1-1.3 hold. Then, system (9)-(10) has no non-trivial and non-negative traveling wave solution $(S(y), I(y))$ satisfying the boundary conditions (10) if $R_0 \leq 1$, or $R_0 > 1$ but $0 < c < c_*$.*

Clearly the above results show that whether or not such travelling wave solutions for (8) exist totally depends on R_0 and c_*. Also, when $f(I) = \beta I$ and the fraction function $P(L) \equiv 1$, (8) reduces to (2), and hence the results in [7] is included in the above main results as special cases. In Section 7, as examples, we will choose some particular fraction functions and forms of the severity level function $L(t)$ to further illustrate the above main results.

3. Preliminaries.

In this section we present some preliminary results, which will be used to prove the existence theorem. Let $\rho > 0$ and

$$L_\rho = \left\{ \varphi : \mathbb{R} \to \mathbb{R} : \sup_{y \in \mathbb{R}} |\varphi(y)| e^{-\rho|y|} < \infty \right\},$$

$$B_\rho = \left\{ \varphi \in C(\mathbb{R}, \mathbb{R}) : \sup_{y \in \mathbb{R}} |\varphi(y)| e^{-\rho|y|} < \infty \right\}, \ |\varphi|_\rho = \sup_{y \in \mathbb{R}} |\varphi(y)| e^{-\rho|y|}.$$

Then, it is not difficult to verify that $(B_\rho, |\cdot|_\rho)$ is a Banach space. Next, we introduce the usual product space

$$B_\rho \times B_\rho = \{\Phi = (\phi, \varphi) : \phi, \varphi \in B_\rho\}, \ |\Phi|_\rho = \max\{|\phi|_\rho, |\varphi|_\rho\}$$

and $(B_\rho \times B_\rho, |\cdot|_\rho)$ is also a Banach space.

For any $\zeta_i > 0$ $(i = 1, 2)$, we define the second-order differential operator

$$\Delta_i \varphi := -d_i \varphi'' + c\varphi' + \zeta_i \varphi, \text{ for } \varphi \in C^2(\mathbb{R}, \mathbb{R}).$$

Note that the equation $-d_i\lambda^2 + c\lambda + \zeta_i = 0$ always has two real roots denoted by λ_i^+, λ_i^- with

$$\lambda_i^+ = \frac{c + \sqrt{c^2 + 4d_i\zeta_i}}{2d_i}, \ \lambda_i^- = \frac{c - \sqrt{c^2 + 4d_i\zeta_i}}{2d_i}.$$

Clearly, $-\lambda_i^- < \lambda_i^+$, $i = 1, 2$.

Choose $\rho > 0$ such that $0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}$ and define the integral operator Δ_i^{-1} by

$$\left(\Delta_i^{-1}\varphi\right)(y) := \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)}\left(\int_{-\infty}^{y} e^{\lambda_i^-(y-\xi)}\varphi(\xi)d\xi + \int_{y}^{\infty} e^{\lambda_i^+(y-\xi)}\varphi(\xi)d\xi\right), \ \forall \varphi \in L_\rho, \ i = 1, 2.$$

Then, the asymptotical properties of $\varphi \in L_p$ ensure that this integral operator is well-defined. Furthermore, some calculations show that

$$\left(\Delta_i^{-1}\varphi\right)'(y) = \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)}\left(\lambda_i^- \int_{-\infty}^{y} e^{\lambda_i^-(y-\xi)}\varphi(\xi)d\xi + \lambda_i^+ \int_{y}^{\infty} e^{\lambda_i^+(y-\xi)}\varphi(\xi)d\xi\right), \tag{12a}$$

$$\left(\Delta_i^{-1}\varphi\right)''(y) = \frac{1}{d_i(\lambda_i^+ - \lambda_i^-)}\left(\left(\lambda_i^-\right)^2 \int_{-\infty}^{y} e^{\lambda_i^-(y-\xi)}\varphi(\xi)d\xi + \left(\lambda_i^+\right)^2 \int_{y}^{\infty} e^{\lambda_i^+(y-\xi)}\varphi(\xi)d\xi\right) - \frac{1}{d_i}\varphi(x). \tag{12b}$$

Also, it is easy to see that for any $\rho_0 < \rho$, there holds

$$B_{\rho_0} \subset B_\rho \subset L_\rho.$$

This, together with Lemma 3.1 in [9], immediately leads to the following lemma.

Lemma 3.1. *For any $0 < \rho_0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}$, there holds*

(i) $\Delta_i^{-1} : B_\rho \to B_\rho$ is a bounded linear operator;
(ii) The operator $\Delta_i^{-1} : B_{\rho_0} \to B_\rho$ is compact.

Furthermore, applying Lemma 3.1 [19], we find the operators Δ_i and Δ_i^{-1} satisfy the relations stated in the following lemma.

Lemma 3.2. *For $\varphi \in B_\rho$ with $0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}$, one has*

$$\Delta_i\left(\Delta_i^{-1}\varphi\right) = \varphi, \ i = 1, 2. \tag{13}$$

Moreover, if $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ and $\varphi, \varphi', \varphi'' \in B_\rho$, then there holds

$$\Delta_i^{-1}\left(\Delta_i\varphi\right) = \varphi. \tag{14}$$

In fact, by a direct calculation, we can show that equality (13) is still true for $\varphi \in L_\rho$. This lemma implies that Δ_i^{-1} is the inverse operator of Δ_i in some sense. In fact, the next lemma concludes more than what (14) were established in Lemma 3.2 [9] and Lemma 2.1 [8].

Lemma 3.3. *Let $0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}$ and assume that $\varphi \in L_\rho$ satisfies*

(i) $\varphi', \varphi'' \in L_\rho$ and φ'' is continuous on $\mathbb{R}\backslash\{y_k\}$, where $\{y_k\}$ is a finite increasing sequence;

(ii) $\varphi\left(y_k^+\right), \varphi\left(y_k^-\right), \varphi'\left(y_k^+\right)$, and $\varphi'\left(y_k^-\right)$ exist.

Then, $\Delta_i^{-1}\left(\Delta_i\varphi\right) \in C(\mathbb{R}, \mathbb{R})$ $(i = 1, 2)$ and

$$\left[\Delta_i^{-1}\left(\Delta_i\varphi\right)\right](y) = \varphi(y) + \frac{\sum_{y<y_j}\left(b_j - \lambda_i^- a_j\right)\mathrm{e}^{\lambda_i^+(y-y_j)} + \sum_{y>y_j}\left(b_j - \lambda_i^+ a_j\right)\mathrm{e}^{\lambda_i^-(y-y_j)}}{\lambda_i^+ - \lambda_i^-} \tag{15}$$

for any $y \notin \{y_j\}$, *where* $a_j = \varphi\left(y_j^+\right) - \varphi\left(y_j^-\right)$ *and* $b_j = \varphi'\left(y_j^+\right) - \varphi'\left(y_j^-\right)$. *Moreover, if* $\varphi \in C(\mathbb{R}, \mathbb{R})$, (15) *becomes*

$$\left[\Delta_i^{-1}\left(\Delta_i \varphi\right)\right](y) = \varphi(y) + \frac{\sum_{y<y_j} b_j e^{\lambda_i^+(y-y_j)} + \sum_{y>y_j} b_j e^{\lambda_i^-(y-y_j)}}{\lambda_i^+ - \lambda_i^-}. \tag{16}$$

The linearization of (9b) at $(S_0, 0)$ is

$$cI'(y) = d_2 I''(y) + f'(0^+)P(0)S_0 I(y) - \gamma I(y)$$

and the corresponding characteristic equation is

$$K(c, \lambda) := -d_2\lambda^2 + c\lambda + \gamma - f'(0^+)P(0)S_0 = 0, \tag{17}$$

which is a quadratic equation of λ parameterized by c. Denoting

$$c_* := 2\sqrt{d_2(f'(0^+)P(0)S_0 - \gamma)},$$

then the quadratic property of $K(c, \lambda)$ immediately lead to the following lemma.

Lemma 3.4. *Assume that* $R_0 > 1$. *Then,*

(i) if $0 < c < c_*$, $K(c, \lambda)$ *remains negative for all* $\lambda > 0$; *and, at* $c = c_*$, *the equation* $K(c, \lambda) = 0$ *has a unique but double positive root* λ_*;

(ii) if $c > c_*$, *the equation* $K(c, \lambda) = 0$ *has two different positive roots* λ_1, λ_2 *with* $\lambda_1 < \lambda_2$; *moreover,* $K(c, \lambda) > 0$ *if* $\lambda_1 < \lambda < \lambda_2$, *and* $K(c, \lambda) < 0$ *for* $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$.

4. **Existence of traveling waves with speed** $c > c_*$**.** In this section, we always assume that $R_0 > 1$ and $c > c_*$ to prove Theorem 2.1 using the Schauder fixed point theorem. To this end, we first show that the traveling wave solution is actually a fixed point of certain operator and prove the continuity and compactness of this operator.

Define the functions

$$S_+(y) :\equiv S_0, \quad S_-(y) := \begin{cases} S_0(1 - M_1 e^{\varepsilon_1 y}), & y < \xi_1, \\ 0, & y \geq \xi_1, \end{cases} \tag{18}$$

$$I_+(y) := e^{\lambda_1 y}, \quad I_-(y) := \begin{cases} e^{\lambda_1 y}\left(1 - M_2 e^{\varepsilon_2 y}\right), & y < \xi_2, \\ 0, & y \geq \xi_2, \end{cases} \tag{19}$$

where $\xi_1 := \frac{1}{\varepsilon_1} \ln \frac{1}{M_1}$ and $\xi_2 := \frac{1}{\varepsilon_2} \ln \frac{1}{M_2}$, and $\varepsilon_1, \varepsilon_2, M_1$, and M_2 are positive constants to be determined later.

Now, we let $\zeta_i \geqslant d_i\lambda_1^2 + c\lambda_1 + 1$ so that $\lambda_1 < -\lambda_i^-$, $i = 1, 2$. We also choose ρ_0 and ρ such that

$$\lambda_1 < \rho_0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}.$$

Let

$$\Omega := \{(S, I) \in B_\rho \times B_\rho : S_- \leq S \leq S_+, I_- \leq I \leq I_+\},$$

which is bounded, closed, and convex in $B_\rho \times B_\rho$. Define the operators F_1 and F_2 on Ω by

$$\begin{aligned} F_1(S, I) &:= \zeta_1 S(y) - f(I(y))P\left(L(y)\right)S(y), \\ F_2(S, I) &:= \zeta_2 I(y) + f(I(y))P\left(L(y)\right)S(y) - \gamma I(y), \end{aligned} \quad (S, I) \in \Omega. \tag{20}$$

For F_1 and F_2, one has the following lemma.

Lemma 4.1. *The operator* $F_i : \Omega \to B_\rho$ *is continuous for* $i = 1, 2$.

Proof. First, for any $(S,I)\in\Omega$, we see that

$$|F_1(S,I)| \le \zeta_1 S_0 + f'(0^+)P(0)S_0 e^{\lambda_1 y},$$
$$|F_2(S,I)| \le \left(\zeta_2 + f'(0^+)P(0)S_0 + \gamma\right) e^{\lambda_1 y},$$

which implies that $F_i(S,I)\in B_\rho$ as $\lambda_1<\rho$, and hence $F_i(\Omega)\subset B_\rho$.

Next, we show the continuity of F_i. For any $(S_1,I_1),(S_2,I_2)\in\Omega$, there holds

$$\begin{aligned}
&|f(I_1(y))g\left(P\left(L_1(y)\right)S_1(y)\right) - f(I_2(y))g\left(P\left(L_2(y)\right)S_2(y)\right)|\\
&\le f(S_0)P(0)\left|S_1(y)-S_2(y)\right| + P(0)S_0\left|f(I_1(y))-f(I_2(y))\right|\\
&\quad + f(S_0)S_0\left|P\left(L_1(y)\right)-P\left(L_2(y)\right)\right|\\
&\le f(S_0)P(0)\left|S_1(y)-S_2(y)\right| + P(0)S_0L_1\left|I_1(y)-I_2(y)\right|\\
&\quad + f(S_0)S_0P'_M\int_0^\tau w(\theta)\left|I_1(y-c\theta)-I_2(y-c\theta)\right|d\theta.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left|F_1(S_1,I_1)-F_1(S_2,I_2)\right|e^{-\rho|y|}\\
&\le \left(\zeta_1+f(S_0)P(0)\right)\left|S_1(y)-S_2(y)\right|e^{-\rho|y|} + P(0)S_0L_1\left|I_1(y)-I_2(y)\right|e^{-\rho|y|}\\
&\quad + f(S_0)S_0P'_M\int_0^\tau w(\theta)\left|I_1(y-c\theta)-I_2(y-c\theta)\right|e^{-\rho|y-c\theta|}e^{\rho(|y-c\theta|-|y|)}d\theta\\
&\le \left(\zeta_1+f(S_0)P(0)\right)\left|S_1-S_2\right|_\rho\\
&\quad + \left(P(0)S_0L_1 + f(S_0)S_0P'_M\int_0^\tau w(\theta)e^{\rho c\theta}d\theta\right)\left|I_1-I_2\right|_\rho,
\end{aligned}$$

implying that $F_1:\Omega\to B_\rho$ is continuous.

By a similar argument, we have

$$\begin{aligned}
&\left|F_2(S_1,I_1)-F_2(S_2,I_2)\right|e^{-\rho|y|}\\
&\le f(S_0)P(0)\left|S_1-S_2\right|_\rho\\
&\quad + \left(\zeta_2+\gamma+P(0)S_0L_1 + f(S_0)S_0P'_M\int_0^\tau w(\theta)e^{\rho c\theta}d\theta\right)\left|I_1-I_2\right|_\rho,
\end{aligned}$$

which shows $F_2:\Omega\to B_\rho$ is also continuous, and the proof is completed. □

Remark 4.2. If we replace ρ by ρ_0, this conclusion still holds because $\lambda_1<\rho_0$.

Now, we define the operator $\mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2):\Omega\to B_\rho\times B_\rho$ by

$$\mathcal{G}_i(S,I) = \Delta_i^{-1}\left(F_i(S,I)\right). \tag{21}$$

From Lemma 4.1, it is known that $F_i(\Omega)\subset B_\rho$, and thus, $\mathcal{G}_i(\Omega)\subset B_\rho$ by Lemma 3.1. Therefore, $\mathcal{G}_i$ is well-defined in Ω. Now, applying Lemmas 3.1 and 4.1, one has the following lemma for $\mathcal{G}$.

Lemma 4.3. *The operator $\mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2):\Omega\to B_\rho\times B_\rho$ is continuous and compact.*

Proof. From Lemmas 3.1 (*i*) and 4.1, it is easily seen that $\mathcal{G}_i$ is continuous from Ω to B_ρ. Meanwhile, $\mathcal{G}_i:\Omega\to B_\rho$ is compact by Lemmas 3.1 (*ii*) and 4.1. The proof is completed. □

If the operator $\mathcal{G}$ has a fixed point in Ω, i.e., there exists $(S,I)\in\Omega$ satisfying

$$\begin{cases} S=\mathcal{G}_1(S,I),\\ I=\mathcal{G}_2(S,I),\end{cases}$$

then from Lemma 13 we obtain that

$$\begin{cases} \Delta_1 S = F_1(S, I), \\ \Delta_2 I = F_2(S, I), \end{cases}$$

implying that $(S(y), I(y))$ is a solution of system (9). Furthermore, if we can verify that this solution also satisfies the boundary conditions (10), then it is a traveling wave solution that we look for. In what follows, we proceed to prove the existence of a fixed point for the mapping $\mathcal{G}$ that satisfies (10).

From Lemma 4.3, $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \Omega \to B_\rho \times B_\rho$ is continuous and compact. In order to apply the Schauder fixed point theorem to $\mathcal{G}$, it remains to show the invariance of the set Ω under the mapping $\mathcal{G}$. To this end, we need to establish two auxiliary results.

Lemma 4.4. *For any ε_1 and M_1 satisfying*

$$\varepsilon_1 \in \left(0, \min\left\{\lambda_1, \frac{c}{d_1}\right\}\right) \text{ and } M_1 \geq \max\left\{1, \frac{f'(0^+)P(0)}{c\varepsilon_1 - d_1\varepsilon_1^2}\right\}, \tag{22}$$

the function $S_-(y)$ defined in (18) satisfies

$$-d_1 S_-''(y) + cS_-'(y) + f'(0^+)P(0)S_0 I0^+(y) \leq 0 \text{ for } y < \xi_1 = \frac{1}{\varepsilon_1}\ln\frac{1}{M_1}. \tag{23}$$

Proof. Let ε_1 and M_1 satisfy (22). Then, $c\varepsilon_1 - d_1\varepsilon_1^2 > 0$ and $\xi_1 \leq 0$. Thus, for $y < \xi_1 \leq 0$,

$$\begin{aligned} &- d_1 S_-''(y) + cS_-'(y) + f'(0^+)P(0)S_0 I_+(y) \\ &= -\left(c\varepsilon_1 - d_1\varepsilon_1^2\right) S_0 M_1 e^{\varepsilon_1 y} + f'(0^+)P(0)S_0 e^{\lambda_1 y} \\ &\leq \left[-\left(c\varepsilon_1 - d_1\varepsilon_1^2\right) M_1 + f'(0^+)P(0)\right] S_0 e^{\varepsilon_1 y} \\ &\leq 0. \end{aligned}$$

So, (23) holds and the proof is completed. □

Lemma 4.5. *For any ε_2 and M_2 satisfying*

$$\begin{cases} \varepsilon_2 \in (0, \min\{\varepsilon_1, \lambda_1, (a-1)\lambda_1, \lambda_2 - \lambda_1\}), \\ M_2 \geq \max\left\{e^{-\varepsilon_2\xi_1}, e^{-\frac{\varepsilon_2 \ln b}{\lambda_1}}, \frac{f'(0^+)P_M' S_0 \int_0^\tau w(\theta)e^{-\lambda_1 c\theta}d\theta + f'(0^+)P(0)S_0 M_1 + P(0)S_0 k}{K(c, \lambda_1+\varepsilon_2)}\right\}, \end{cases} \tag{24}$$

the function $I_-(y)$ defined in (19) satisfies

$$\begin{aligned} &- d_2 I_-''(y) + cI_-'(y) + \gamma I_-(y) - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c\theta)d\theta\right) S_-(y) \\ &\leq 0, \; y \neq \xi_2 = \frac{1}{\varepsilon_2}\ln\frac{1}{M_2}. \end{aligned} \tag{25}$$

Proof. For any $y > \xi_2$, (25) clearly holds true because $I_-(y) = 0$. For the case of $y < \xi_2$, let ε_2 and M_2 satisfy (24). Then, $K(c, \lambda_1 + \varepsilon_2) > 0$ and (by the definition of ξ_1 and ξ_2)

$$\xi_2 \leq \xi_1, \; e^{\lambda_1 \xi_2} \leq b.$$

Then, by the choice of ε_2 and M_2, there holds

$$I_-(y) = e^{\lambda_1 y}\left(1 - M_2 e^{\varepsilon_2 y}\right) \leq e^{\lambda_1 y} \leq e^{\lambda_1 \xi_2} \leq b, \text{ for } y < \xi_2,$$

which together with Assumption 1.1 implies

$$f(I_-(y)) \geq f'(0^+)I_-(y) - kI_-^a(y), \text{ for } y < \xi_2.$$

Thus, we can deduce that for all $y < \xi_2$,

$$
\begin{aligned}
& -d_2 I_-''(y) + cI_-'(y) + \gamma I_-(y) - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y-c\theta)d\theta\right)S_-(y) \\
&= -d_2\left[\lambda_1^2 - (\lambda_1+\varepsilon_2)^2 M_2 e^{\varepsilon_2 y}\right]e^{\lambda_1 y} + c\left[\lambda_1 - (\lambda_1+\varepsilon_2)M_2 e^{\varepsilon_2 y}\right]e^{\lambda_1 y} \\
&\quad + \gamma\left(1 - M_2 e^{\varepsilon_2 y}\right)e^{\lambda_1 y} - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y-c\theta)d\theta\right)S_-(y) \\
&\le -K(c,\lambda_1+\varepsilon_2)M_2 e^{(\lambda_1+\varepsilon_2)y} + f'(0^+)P(0)S_0 e^{\lambda_1 y} - f'(0^+)P(0)S_0 M_2 e^{(\lambda_1+\varepsilon_2)y} \\
&\quad - \left(f'(0^+)I_-(y) - kI_-^a(y)\right)P\left(\int_0^\tau w(\theta)I_+(y-c\theta)d\theta\right)S_-(y) \\
&\le -K(c,\lambda_1+\varepsilon_2)M_2 e^{(\lambda_1+\varepsilon_2)y} \\
&\quad + f'(0^+)\left(P(0)S_0 - P\left(\int_0^\tau w(\theta)I_+(y-c\theta)d\theta\right)S_-(y)\right)e^{\lambda_1 y} \\
&\quad - f'(0^+)\left[P(0)S_0 - P\left(\int_0^\tau w(\theta)I_+(y-c\theta)d\theta\right)S_-(y)\right]M_2 e^{(\lambda_1+\varepsilon_2)y} + P(0)S_0 k e^{a\lambda_1 y} \\
&\le -K(c,\lambda_1+\varepsilon_2)M_2 e^{(\lambda_1+\varepsilon_2)y} + f'(0^+)P_M' S_0\int_0^\tau w(\theta)e^{-\lambda_1 c\theta}d\theta e^{\lambda_1 y}e^{\lambda_1 y} \\
&\quad + f'(0^+)P(0)S_0 M_1 e^{(\lambda_1+\varepsilon_1)y} + P(0)S_0 k e^{(\lambda_1+\varepsilon_2)y} \\
&\le \Big(-K(c,\lambda_1+\varepsilon_2)M_2 + f'(0^+)P_M' S_0\int_0^\tau w(\theta)e^{-\lambda_1 c\theta}d\theta \\
&\quad + f'(0^+)P(0)S_0 M_1 + P(0)S_0 k\Big)e^{(\lambda_1+\varepsilon_2)y} \\
&\le 0.
\end{aligned}
$$

This completes the proof. □

Now we take

$$\zeta_1 = \max\{d_1\lambda_1^2 + c\lambda_1 + 1, f(S_0)P(0)\},\ \zeta_2 = \max\{d_2\lambda_1^2 + c\lambda_1 + 1, \gamma\}.$$

Then, we are ready to prove the invariance of the set Ω under the mapping $\mathcal{G}$.

Lemma 4.6. *The operator $\mathcal{G}$ maps Ω to Ω, i.e., for any $(S,I)\in\Omega$, we have*

$$S_- \le \mathcal{G}_1(S,I) \le S_+,\ I_- \le \mathcal{G}_2(S,I) \le I_+.$$

Proof. Since S_- and I_- satisfy all the conditions of Lemma 3.3 and

$$S_-'(\xi_1^+) = 0,\ S_-'(\xi_1^-) < 0,\ I_-'(\xi_2^+) = 0,\ I_-'(\xi_2^-) < 0,$$

(16) indeed implies that

$$\Delta_1^{-1}\left(\Delta_1(S_-)\right) \ge S_-,\ \Delta_2^{-1}\left(\Delta_2(I_-)\right) \ge I_-.$$

On the other hand, by Lemma 3.2, one has

$$\Delta_1^{-1}\left(\Delta_1(S_+)\right) = S_+,\ \Delta_2^{-1}\left(\Delta_2(I_+)\right) = I_+.$$

Since Δ_i^{-1} is a linear and positive operator, we can prove that

$$\Delta_1(S_-) \le F_1(S,I) \le \Delta_1(S_+),\ \Delta_2(I_-) \le F_2(S,I) \le \Delta_2(I_+),\ \text{for any } (S,I)\in\Omega.$$

It is easy to see that

$$F_1(S,I) = \zeta_1 S - f(I)P(L)S \le \zeta_1 S_+ = \Delta_1(S_+).$$

Now we show that $F_1(S,I) \geq \Delta_1(S_-)$. For any $y \geq \xi_1$, since $\zeta_1 > f(S_0)P(0)$, there holds

$$F_1(S,I) \geq (\zeta_1 - f(S_0)P(0))\, S(y) \geq 0 = \Delta_1(S_-(y)).$$

For $y < \xi_1$, observe that $f(I)P(L)S \leq f'(0^+)P(0)S_0 I_+(y)$. This together with (23) implies $F_1(S,I) \geq \Delta_1(S_-(y))$. Hence, $\Delta_1(S_-) \leq F_1(S,I) \leq \Delta_1(S_+)$.

Noting that

$$F_2(S,I) = \zeta_2 I + f(I)P(L)S - \gamma I \leq (\zeta_2 - \gamma) I_+ + f'(0^+)P(0)S_0 I_+,$$

we can easily verify that $-d_2 I''_+(y) + cI'_+(y) + (\gamma - f'(0^+)P(0)S_0)I_+(y) \geq 0$ by the characteristic equation (17), and hence $F_2(S,I) \leq \Delta_2(I_+)$.

Finally, we verify $F_2(S,I) \geq \Delta_2(I_-)$. This inequality obviously holds if $y \geq \xi_2$ because $\zeta_2 > \gamma$. For $y < \xi_2$, since

$$F_2(S,I) \geq (\zeta_2 - \gamma)\, I_-(y) + f(I_-(y))P\left(\int_0^\tau w(\theta) I_+(y - c\theta)d\theta\right) S_-(y),$$

by (25) in Lemma 4.5, we see that

$$\begin{aligned} &- d_2 I''_-(y) + cI'_-(y) + \gamma I_-(y) \\ &- f(I_-(y))P\left(\int_0^\tau w(\theta) I_+(y - c\theta)d\theta\right) S_-(y) \leq 0, \text{ for } y < \xi_2. \end{aligned}$$

Thus, we conclude that $F_2(S,I) \geq \Delta_2(I_-)$ for all $y \in \mathbb{R}$. The proof is complete. □

With all the above preparations, we are now in the position to prove Theorem 2.1 for the case of $c > c_*$, leaving the case of $c = c_*$ for the next section.

Proof of Theorem 2.1 in the case of $c > c_*$. Note that Lemmas 4.3 and 4.6 show that $\mathcal{G}$ is continuous and compact from the convex closed set Ω into Ω. Then, by the Schauder fixed point theorem, the map $\mathcal{G}$ has a fixed point (S,I) in Ω, which is a solution of system (9). We are left to verify the boundary conditions (10) for this solution.

Since $S_- \leq S \leq S_+$, $I_- \leq I \leq I_+$, by (18) and (19), we have

$$S(-\infty) := \lim_{y\to-\infty} S(y) = S_0, \quad I(-\infty) := \lim_{y\to-\infty} I(y) = 0 \quad \lim_{y\to-\infty} I(y)e^{-\lambda_1 y} = 1,$$

and the last limit implies that $I(y)$ is non-trivial. In fact, we can further prove that $I(y) > 0$ for all $y \in \mathbb{R}$. Suppose on the contrary that there exists y_0 such that $I(y_0) = 0$. Then, $I(y)$ reaches its minimum value in an interval (a,b) that contains y_0 and ξ_2. Note that (9b) implies that

$$-d_2 I''(y) + cI'(y) + \gamma I(y) \geqslant 0, \text{ for all } y \in \mathbb{R}.$$

This together with the elliptic strong extremum principle implies that $I(y) \equiv 0$ for $y \in [a,b]$, which is a contradiction to the fact that $I(y) > 0$ for $y \in [a, \xi_2)$.

Note that for a function h with $\lim_{y\to-\infty} h(y)$ existing, by the L'Hôpital's rule and (12a), there holds

$$\lim_{y\to-\infty} (\Delta_i^{-1} h)'(y) = 0.$$

From Lemma 4.1, we see that $F_1(S,I), F_2(S,I) \in B_\rho$. On the other hand we have that $\lim_{y\to-\infty} F_1(S(y), I(y)) = \zeta_1 S_0$ and $\lim_{y\to-\infty} F_2(S(y), I(y)) = 0$. Therefore,

$$S'(-\infty) := \lim_{y\to-\infty} S'(y) = \lim_{y\to-\infty} \left[\Delta_1^{-1}\left(F_1(S,I)\right)\right]'(y) = 0,$$

$$I'(-\infty) := \lim_{y\to-\infty} I'(y) = \lim_{y\to-\infty} \left[\Delta_2^{-1}\left(F_2(S,1)\right)\right]'(y) = 0,$$

which together with (9) further lead to

$$S''(-\infty) := \lim_{y\to-\infty} S''(y) = 0, I''(-\infty) := \lim_{y\to-\infty} I''(y) = 0.$$

Summarizing the above discussion, we obtain the asymptotic behavior of S and I for $y \to -\infty$ as stated below:

$$S(-\infty) = S_0,\ S'(-\infty) = S''(-\infty) = 0,\ I(-\infty) = I'(-\infty) = I''(-\infty) = 0. \tag{26}$$

In what follows, we study the asymptotic behavior when y tends to ∞. First, integrating (9a) from $-\infty$ to y gives

$$d_1 S'(y) - c\left(S(y) - S_0\right) = \int_{-\infty}^{y} f(I(y))P\left(L(y)\right)S(y)dy. \tag{27}$$

This implies the integral on the right-hand side of (27) is bounded. Otherwise, $S'(y)$ is unbounded as y tends to ∞, thus $S(y) \to \infty$ as $y \to \infty$, a contradiction. Hence, it follows from (27) that $S'(y)$ is bounded on $\mathbb{R}$, and so is $S''(y)$ on $\mathbb{R}$ by (9a) and the fact $f(I(y))P\left(L(y)\right)S(y) \le f(S_0)P(0)S_0$.

On the other hand, we derive from (9a) that

$$\left(\mathrm{e}^{-cy/d_1} S'(y)\right)' = \frac{\mathrm{e}^{-cy/d_1}}{d_1} f(I(y))P\left(L(y)\right)S(y).$$

Now, integrating the above equation from y to ∞ yields

$$\mathrm{e}^{-cy/d_1} S'(y) = -\int_{y}^{\infty} \frac{\mathrm{e}^{-cx/d_1}}{d_1} f(I(x))P\left(L(x)\right)S(x)\mathrm{d}x, \tag{28}$$

so $S'(y) \le 0$ and $S'(y) \not\equiv 0$, i.e., $S(y)$ is non-increasing, and hence $S_\infty := \lim_{y\to\infty} S(y)$ exists and $S_\infty < S_0$. Now, applying L'Hôpital's rule to (28) leads to $S'(\infty) := \lim_{y\to\infty} S'(y) = 0$. It then follows from (9b) that $S''(\infty) := \lim_{y\to\infty} S''(y) = 0$. Letting $y \to \infty$ in (27) yields

$$\int_{-\infty}^{\infty} f(I(y))P\left(L(y)\right)S(y)dy = c\left(S_0 - S_\infty\right). \tag{29}$$

Observe that (9b) can be rewritten as

$$-d_2 I''(y) + cI'(y) + \gamma I(y) = f(I(y))P\left(L(y)\right)S(y) := \varphi(y), \tag{30}$$

from which we obtain

$$\begin{aligned} I(y) =& C_1 \mathrm{e}^{\lambda^+ y} + C_2 \mathrm{e}^{\lambda^- y} \\ &+ \frac{1}{d_2\left(\lambda^+ - \lambda^-\right)}\left(\int_{-\infty}^{y} \mathrm{e}^{\lambda^-(y-\xi)}\varphi(\xi)\mathrm{d}\xi + \int_{y}^{\infty} \mathrm{e}^{\lambda^+(y-\xi)}\varphi(\xi)\mathrm{d}\xi\right), \end{aligned} \tag{31}$$

where C_1 and C_2 are constants and

$$\lambda^- = \frac{c - \sqrt{c^2 + 4d_2\gamma}}{2d_2} < 0, \quad \lambda^+ = \frac{c + \sqrt{c^2 + 4d_2\gamma}}{2d_2} > 0.$$

Clearly, the integral in (31) is well-defined because

$$\frac{1}{d_2\left(\lambda^+ - \lambda^-\right)}\left|\int_{-\infty}^{y} \mathrm{e}^{\lambda^-(y-\xi)}\varphi(\xi)\mathrm{d}\xi + \int_{y}^{\infty} \mathrm{e}^{\lambda^+(y-\xi)}\varphi(\xi)\mathrm{d}\xi\right| \le \frac{f(S_0)P(0)S_0}{\gamma}.$$

Note that $C_2 = 0$ since $\lim_{y\to-\infty} I(y) = 0$, and $C_1 = 0$ since $\lim_{y\to\infty} I(y) = 0$. Consequently, (31) becomes

$$I(y) = \frac{1}{d_2\left(\lambda^+ - \lambda^-\right)}\left(\int_{-\infty}^{y} \mathrm{e}^{\lambda^-(y-\xi)}\varphi(\xi)\mathrm{d}\xi + \int_{y}^{\infty} \mathrm{e}^{\lambda^+(y-\xi)}\varphi(\xi)\mathrm{d}\xi\right).$$

Now integrating the above equation and applying Fubini's theorem give that

$$\int_{-\infty}^{\infty} I(y)dy = \frac{1}{\gamma}\int_{-\infty}^{\infty} f(I(y))P\left(L(y)\right)S(y)dy.$$

This, together with (29) and (7), results in

$$\gamma\int_{-\infty}^{+\infty} I(y)dy = \int_{-\infty}^{+\infty} f(I(y))P\left(\int_0^{\tau} w(\theta)I(y-c\theta)d\theta\right)S(y)dy = c\left(S_0 - S_\infty\right), \tag{32}$$

proving (11) in Theorem 2.1-(iii).

Next, we prove $I(\infty)=0$. Note that

$$\begin{aligned}|I'(y)| &= \frac{1}{d_2\left(\lambda^+-\lambda^-\right)}\left|\lambda^-\int_{-\infty}^{y} \mathrm{e}^{\lambda^-(y-\xi)}\varphi(\xi)\mathrm{d}\xi + \lambda^+\int_{y}^{\infty} \mathrm{e}^{\lambda^+(y-\xi)}\varphi(\xi)\mathrm{d}\xi\right| \\ &\le \frac{\lambda^+ f(S_0)P(0)S_0}{\gamma}.\end{aligned}$$

This shows that $I'(y)$ is bounded on $\mathbb{R}$. On the other hand, $I(y)$ is integrable on $\mathbb{R}$, and hence, $I(\infty) := \lim_{y\to\infty} I(y) = 0$. Now, integrating (30) from $-\infty$ to y leads to

$$-d_2 I'(y) + cI(y) + \gamma\int_{-\infty}^{y} I(\xi)\mathrm{d}\xi = \int_{-\infty}^{y}\varphi(\xi)\mathrm{d}\xi,$$

implying that $I'(\infty) := \lim_{y\to\infty} I'(y) = 0$ by (32). Also, it follows from (9b) that $I''(\infty) := \lim_{y\to+\infty} I''(y) \to 0$. Therefore, we conclude that

$$S_\infty < S_0,\ S'(\infty) = S''(\infty) = 0,\ I(\infty) = I'(\infty) = I''(\infty) = 0. \tag{33}$$

Finally, it remains to prove the estimate in Theorem 2.1-(ii). To this end, we define

$$J(y) := I(y) + \frac{2\gamma}{c+\sqrt{c^2+4d_2\gamma}}\int_{-\infty}^{y} I(\xi)\mathrm{d}\xi,\ y\in\mathbb{R}.$$

This function is well-defined since the function I is integrable on $\mathbb{R}$. By some simple calculations, we obtain

$$J'(y) = I'(y) + \frac{2\gamma}{c+\sqrt{c^2+4d_2\gamma}}I(y),\ J''(y) = I''(y) + \frac{2\gamma}{c+\sqrt{c^2+4d_2\gamma}}I'(y). \tag{34}$$

Then, it follows from (32) and (33) that

$$\begin{aligned}&J(\infty) := \lim_{y\to\infty} J(y) = \frac{2\gamma}{c+\sqrt{c^2+4d_2\gamma}}\int_{-\infty}^{\infty} I(x)\mathrm{d}x = \frac{2c(S_0-S_\infty)}{c+\sqrt{c^2+4d_2\gamma}}, \\ &\lim_{y\to\infty} J'(y) = 0.\end{aligned}$$

Let $\alpha = (c+\sqrt{c^2+4d_2\gamma})/2$. From (30) and (34), it follows that

$$-d_2 J''(y) + \alpha J'(y) = f(I(y))P\left(L(y)\right)S(y),$$

from which we deduce that

$$-\left(J'(y)\mathrm{e}^{-\frac{\alpha}{d_2}y}\right)' = \frac{1}{d_2} f(I(y))P\left(L(y)\right)S(y)\mathrm{e}^{-\frac{\alpha}{d_2}y}.$$

Integrating this equation from y to ∞ gives

$$J'(y)\mathrm{e}^{-\frac{\alpha}{d_2}y} = \frac{1}{d_2}\int_{y}^{\infty} \mathrm{e}^{-\frac{\alpha}{d_2}\xi} f(I(\xi))P\left(L(\xi)\right)S(\xi)\mathrm{d}\xi \ge 0,$$

which means the function J is non-decreasing, and, therefore,

$$I(y) \leq J(y) \leq J(\infty) = \frac{2c(S_0 - S_\infty)}{c + \sqrt{c^2 + 4d_2\gamma}} \text{ for all } y \in \mathbb{R},$$

confirming the estimate in Theorem 2.1-(ii). The proof is completed. □

5. **Traveling waves with minimal speed** $c = c_*$**.** In this section, we aim to prove Theorem 2.1 for the case of $c = c_*$. The commonly used method is the limiting argument, which is usually applied to proving the existence of a traveling wave with critical speed for monotone systems [18,20,24]. For our disease model, we can also extract a sequence $c_n \to c_*$ such that the traveling wave solutions (S_n, I_n) converges to (S, I) as $n \to \infty$ and (S, I) satisfies (9). The main difficulty in the limiting process is to show the limit (S, I) satisfies the boundary condition (10) at $\pm\infty$. The monotonicity of $S(y)$ infers that $S(-\infty)$ exists, however there is no guarantee that $S(-\infty) = S_0$ because the limiting process $n \to \infty$ and the process of $y \to \infty$ may not be commutable. Moreover, $I(\pm\infty)$ do not necessarily exist, and even if they do, they may not necessarily be equal to zero.

Instead of using the above mentioned approaches to prove the existence of traveling waves with minimal speed, here we will follow [26] to still employ the Schauder fixed theorem for the same operator $\mathcal{G}$, but confined to a subset Γ different from the subset Ω in Section 4. This subset is obtained by carefully constructing a pair of upper-lower solutions for this critical case, which are *suitable modifications* of S_-, S_+, I_+, and I_- in Section 4 as below:

$$S_+(y) :\equiv S_0, \qquad S_-(y) := \begin{cases} S_0\left(1 - \dfrac{1}{\delta}e^{\delta y}\right), & y < y_1, \\ 0, & y \geq y_1, \end{cases}$$

$$I_+(y) := \begin{cases} -K_1 y e^{\lambda_* y}, & y < y_2, \\ M, & y \geq y_2, \end{cases} \quad I_-(y) := \begin{cases} \left(-K_1 y - K_2(-y)^{\frac{1}{2}}\right) e^{\lambda_* y}, & y < y_3, \\ 0, & y \geq y_3, \end{cases} \tag{35}$$

Here, λ_* is the positive double root of $K(c_*, \lambda) = 0$ as stated Lemma 3.4-(i), and the other parameters are given by $y_1 = \ln\delta/\delta$, $y_2 = -1/\lambda_*$, $K_1 = eM\lambda_*$, $M = f(S_0)P(0)S_0/\gamma$, and $y_3 = -(K_2/K_1)^2$, where $\delta > 0$ and $K_2 > 0$ will be determined in Lemma 5.1. Observe that the function $g(y) := -K_1 y e^{\lambda_* y}$ has a unique maximum point $y = -1/\lambda_*$, and $g(y) \leq M$ for all $y \in \mathbb{R}$. Then, $I_-(y) \leq I_+(y) \leq M$ for all $y \in \mathbb{R}$. So, $S_+(y)$ and $I_+(y)$ both are bounded on $\mathbb{R}$. Thus, for any $0 < \rho_0 < \rho < \min\left\{-\lambda_1^-, -\lambda_2^-\right\}$, the set

$$\Gamma := \{(S, I) \in B_\rho \times B_\rho : S_- \leq S \leq S_+, I_- \leq I \leq I_+\}$$

is clearly bounded, closed, and convex in $B_\rho \times B_\rho$.

Note that for any $(S, I) \in \Gamma$, there hold

$$|F_1(S, I)| \leq \zeta_1 S_0 + f(M)P(0)S_0,$$
$$|F_2(S, I)| \leq (\zeta_2 + \gamma) M + f(M)P(0)S_0.$$

So, $F_i(\Gamma) \subset B_\rho$ $(i = 1, 2)$. Using a similar argument to that in Lemmas 4.1 and 4.3, we conclude that the operator $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \Gamma \to B_\rho \times B_\rho$ is continuous and compact (here F_i and $\mathcal{G}_i$ are given by (20) and (21) with $c = c_*$, respectively). The key point of the existence proof now is to show the invariance of Γ, i.e., $\mathcal{G}(\Gamma) \subset \Gamma$.

For this, we choose

$$\zeta_1 \geq f(S_0)P(0),\ \zeta_2 \geq \gamma.$$

Then, we get the following result.

Lemma 5.1. *The operator $\mathcal{G}$ maps Γ to Γ, i.e., for any $(S, I) \in \Gamma$, we have*

$$S_- \leq \mathcal{G}_1(S, I) \leq S_+,\ I_- \leq \mathcal{G}_2(S, I) \leq I_+.$$

Proof. Since S_- and I_- satisfy all conditions of Lemma 3.3 and

$$S'_-(y_1^+) = 0,\ S'_-(y_1^-) \leq 0,\ I'_+(y_2^+) = 0,\ I'_+(y_2^-) \geq 0,\ I'_-(y_3^+) = 0,\ I'_-(y_3^-) \leq 0,$$

then (16) implies that

$$\Delta_1^{-1}(\Delta_1(S_-)) \geq S_-,\ \Delta_2^{-1}(\Delta_2(I_+)) \leq I_+,\ \Delta_2^{-1}(\Delta_2(I_-)) \geq I_-.$$

On the other hand, applying Lemma 3.2 leads to $\Delta_1^{-1}(\Delta_1(S_+)) = S_+$. From the linearity and positivity of Δ_i^{-1}, we just need to prove that for any $(S, I) \in \Gamma$, there hold

$$\Delta_1(S_-) \leq F_1(S, I) \leq \Delta_1(S_+),\ \Delta_2(I_-) \leq F_2(S, I) \leq \Delta_2(I_+).$$

Step 1. $\Delta_1(S_-) \leq F_1(S, I) \leq \Delta_1(S_+)$.

First, we have

$$F_1(S, I) = \zeta_1 S - f(I)P(L)S \leq \zeta_1 S_+ = \Delta_1(S_+).$$

Next, we show that $F_1(S, I) \geq \Delta_1(S_-)$. Since $\zeta_1 > f(S_0)P(0)$, there holds, for any $y \geq y_1$,

$$F_1(S, I) \geq (\zeta_1 - f(S_0)P(0))\, S(y) \geq 0 = \Delta_1(S_-(y)) \text{ for all } y \geq y_1.$$

For $y < y_1$, it suffices to show

$$-d_1 S''_-(y) + c_* S'_-(y) + f'(0^+)P(0)S_0 I_+(y) \leq 0$$

since $F_1(S, I) \geq \zeta_1 S_- - f'(0^+)P(0)S_0 I_+(y)$. Clearly, there is a positive δ_* such that $\frac{\ln \delta_*}{\delta_*} = -\frac{1}{\lambda_*}$ and $\frac{\ln \delta}{\delta} \leq -\frac{1}{\lambda_*}$ for any $\delta \leq \delta_*$, implying that $y_1 \leq y_2$ for any $\delta \leq \delta_*$. Taking $\delta \leq \delta_*$, we then obtain

$$\begin{aligned} &- d_1 S''_-(y) + c_* S'_-(y) + f'(0^+)P(0)S_0 I_+(y) \\ &= d_1 S_0 \delta e^{\delta y} - c_* S_0 e^{\delta y} - f'(0^+)P(0)S_0 K_1 y e^{\lambda_* y} \\ &= \left[S_0(d_1\delta - c_*) - f'(0^+)P(0)S_0 K_1 y e^{(\lambda_* - \delta)y}\right] e^{\delta y}, \end{aligned} \tag{36}$$

which is negative for sufficiently small $\delta \in \left(0, \min\left\{\frac{c_*}{d_1}, \lambda_*, \delta_*\right\}\right)$ because

$$\lim_{y\to-\infty} f'(0^+)P(0)S_0 K_1 y e^{(\lambda_*-\delta)y} = 0,\ \lim_{\delta\to 0^+} S_0(d_1\delta - c_*) = -S_0 c_*,$$

$$\lim_{\delta\to 0^+} y_1 = \lim_{\delta\to 0^+} \frac{ln\delta}{\delta} = -\infty.$$

Step 2. $\Delta_2(I_-) \leq F_2(S, I) \leq \Delta_2(I_+)$.

Let us show $F_2(S, I) \leq \Delta_2(I_+)$ first. Since $\zeta_2 \geq \gamma$, we infer that, for $y \geq y_2$,

$$\begin{aligned} F_2(S, I) - \Delta_2(I_+) &= \zeta_2 I - \gamma I + f(I)P(L)S - \Delta_2(I_+) \\ &\leq (\zeta_2 - \gamma)I_+ + f(S_0)P(0)S_0 - \Delta_2(I_+) \\ &= f(S_0)P(0)S_0 - \gamma M \\ &= 0. \end{aligned}$$

For $y < y_2$, there holds

$$\begin{aligned} F_2(S,I) - \Delta_2(I_+) &\le (\zeta_2 - \gamma)I_+(y) + f'(0^+)P(0)S_0 I_+(y) - \Delta_2(I_+(y)) \\ &= d_2 I_+''(y) - c_* I_+'(y) - \left(\gamma - f'(0^+)P(0)S_0\right) I_+(y) \\ &= \left[-d_2\lambda_*^2 + c_*\lambda_* + \left(\gamma - f'(0^+)P(0)S_0\right)\right] K_1 y e^{\lambda_* y} \\ &\quad + (c_* - 2d_2\lambda_*)K_1 e^{\lambda_* y} \\ &= 0. \end{aligned}$$

So, $F_2(S,I) \le \Delta_2(I_+)$. Then, we verify $F_2(S,I) \ge \Delta_2(I_-)$. Using $\zeta_2 \ge \gamma$ again, we deduce that

$$F_2(S,I) \ge (\zeta_2 - \gamma)\, I_-(y) + f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) S_-(y).$$

So it follows that $F_2(S,I) \ge 0 = \Delta_2(I_-)$ for $y \ge y_3$. For $y < y_3$, if there holds

$$-d_2 I_-''(y) + c_* I_-'(y) + \gamma I_-(y) - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) S_-(y) \le 0, \tag{37}$$

then $F_2(S,I) \ge \Delta_2(I_-)$ for $y < y_3$.

By virtue of $\lim_{y\to-\infty} I_-(y) = 0$, there exists $\zeta_1 < 0$ such that $I_-(y) \le \min\{b, S_0\}$ for $y \le \zeta_1$. From Assumption 1.1, we have

$$f'(0^+)I_-(y) - kI_-^a(y) \le f(I_-(y)) \le f'(0^+)I_-(y), \text{ for any } y \le \zeta_1.$$

Moreover, it is easily seen that

$$\lim_{y\to-\infty} \frac{P(0) - P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right)}{\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta} = -P'(0).$$

Hence, there exist $\zeta_2 < -\frac{1}{\lambda_*} < 0$ and $A_1 > 0$ such that

$$P(0) - P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) \le A_1(-y + c_*\tau)e^{\lambda_* y}, \text{ for } y \le \zeta_2.$$

Now, choosing $K_2 \ge \max\left\{K_1\sqrt{-\frac{ln\delta}{\delta}}, K_1\sqrt{-\min\{\zeta_1,\zeta_2\}}\right\}$, then

$$y_3 \le y_1, \ y_3 \le \min\{\zeta_1, \zeta_2\} < y_2.$$

Thus, we obtain that, for $y < y_3$,

$$\begin{aligned} &- d_2 I_-''(y) + c_* I_-'(y) + \gamma I_-(y) - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) S_-(y) \\ &= -\frac{d_2}{4}K_2(-y)^{-\frac{3}{2}}e^{\lambda_* y} + f'(0^+)P(0)S_0 I_-(y) \\ &\quad - f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) S_-(y) \\ &\le -\frac{d_2}{4}K_2(-y)^{-\frac{3}{2}}e^{\lambda_* y} + P(0)S_0\left(f'(0^+)I_-(y) - f(I_-(y))\right) \\ &\quad + f(I_-(y))S_0\left(P(0) - P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right)\right) \\ &\quad + f(I_-(y))P\left(\int_0^\tau w(\theta)I_+(y - c_*\theta)d\theta\right) S_0\frac{1}{\delta}e^{\delta y} \end{aligned}$$

$$
\begin{aligned}
&\le -\frac{d_2}{4}K_2(-y)^{-\frac{3}{2}}e^{\lambda_* y} + P(0)S_0 k I_-^a(y) + f'(0^+)I_-(y)S_0 A_1(-y+c_*\tau)e^{\lambda_* y} \\
&\quad + f'(0^+)I_-(y)P(0)S_0\frac{1}{\delta}e^{\delta y} \\
&\le -\frac{d_2}{4}K_2(-y)^{-\frac{3}{2}}e^{\lambda_* y} + P(0)S_0 k K_1^a(-y)^a e^{\lambda_* a y} \\
&\quad + f'(0^+)K_1 S_0 A_1(-y+c_*\tau)(-y)e^{2\lambda_* y} \\
&\quad + f'(0^+)K_1 P(0)S_0\frac{1}{\delta}(-y)e^{(\lambda_*+\delta)y} \\
&= (-y)^{-\frac{3}{2}}e^{\lambda_* y}\left[-\frac{d_2}{4}K_2 + P(0)S_0 k K_1^a(-y)^{a+\frac{3}{2}}e^{\lambda_*(a-1)y}\right. \\
&\quad + f'(0^+)K_1 S_0 A_1(-y+c_*\tau)(-y)^{\frac{5}{2}}e^{\lambda_* y} \\
&\quad \left. + f'(0^+)K_1 P(0)S_0\frac{1}{\delta}(-y)^{\frac{5}{2}}e^{\delta y}\right].
\end{aligned}
$$

Define

$$
\begin{aligned}
h(y) :=& P(0)S_0 k K_1^a(-y)^{a+\frac{3}{2}}e^{\lambda_*(a-1)y} + f'(0^+)K_1 S_0 A_1(-y+c_*\tau)(-y)^{\frac{5}{2}}e^{\lambda_* y} \\
&+ f'(0^+)K_1 P(0)S_0\frac{1}{\delta}(-y)^{\frac{5}{2}}e^{\delta y}.
\end{aligned}
$$

In view of the fact that $h(y)$ is continuous on $(-\infty,0]$, $\lim_{y\to-\infty}h(y)=0$, and $h(0)=0$, there exists $M_h>0$ independent of K_2 such that $h(y)\le M_h$ for $y\le 0$ and so $-\frac{d_2}{4}K_2+h(y)\le 0$ for all $y\le 0$ with $K_2\ge\frac{4M_h}{d_2}$.

Now, letting

$$
K_2 = \max\left\{K_1\sqrt{-\frac{ln\delta}{\delta}},\, K_1\sqrt{-\min\{\zeta_1,\zeta_2\}},\, \frac{4M_h}{d_2}\right\},
$$

we then see that (37) holds true for $y<y_3$ from the above discussion, and hence $F_2(S,I)\ge\Delta_2(I_-)$.

Combining the above, the proof of this lemma is completed. □

From Lemma 5.1, $\mathcal{G}$ is continuous and compact from the convex closed set Γ into itself. Hence, applying the Schauder fixed point theorem again, we infer the map $\mathcal{G}$ has a fixed point (S_*,I_*) in Γ, which is a solution of system (9) with $c=c_*$. By a similar argument to that at the end of Section 4, we conclude that (S_*,I_*) satisfies the boundary conditions

$$
\begin{aligned}
&S_*(-\infty) := \lim_{y\to-\infty}S_*(y)=S_0,\ S_*(\infty) := \lim_{y\to+\infty}S_*(y)=S_\infty<S_0, \\
&I_*(\pm\infty) := \lim_{y\to\pm\infty}I_*(y)=0.
\end{aligned}
$$

and there holds

$$
\begin{aligned}
\gamma\int_{-\infty}^{+\infty}I_*(y)dy &= \int_{-\infty}^{+\infty}f(I_*(y))P\left(\int_0^\tau w(\theta)I_*(y-c_*\theta)d\theta\right)S_*(y)dy \\
&= c_*\left(S_0-S_\infty\right).
\end{aligned} \tag{38}
$$

Moreover, $S_*(y)$ is non-increasing, and $I_*(y)$ is strictly positive and satisfies

$$
I_*(y)\le\frac{2c_*\left(S_0-S_\infty\right)}{c_*+\sqrt{c_*^2+4d_2\gamma}}\quad\text{for all}\ \ y\in\mathbb{R}. \tag{39}
$$

As a result, the existence of the traveling wave stated in Theorem 2.1 is also proved for the case of $c = c_*$.

6. **Non-existence of traveling waves.** In this section, we will show that system (8) has no traveling wave solution (S, I) connecting $(S_0, 0)$ and $(S_\infty, 0)$ if $R_0 \le 1$ or $R_0 > 1$, but $0 < c < c_*$.

Suppose on the contrary system (8) has a non-trivial and a non-negative traveling wave solution (S, I) satisfying the boundary conditions (10). Then, (S, I) satisfies (9). By the theory of the second-order linear ODEs, it follows from (9) that

$$S(y) = \frac{1}{d_1\left(\delta_1^+ - \delta_1^-\right)}\left(\int_{-\infty}^{y} \mathrm{e}^{\delta_1^-(y-\xi)}(\gamma S(\xi) - \varphi(\xi))\mathrm{d}\xi \right. \\ \left. + \int_{y}^{\infty} \mathrm{e}^{\delta_1^+(y-\xi)}(\gamma S(\xi) - \varphi(\xi))\mathrm{d}\xi\right), \tag{40}$$

$$I(y) = \frac{1}{d_2\left(\delta_2^+ - \delta_2^-\right)}\left(\int_{-\infty}^{y} \mathrm{e}^{\delta_2^-(y-\xi)}\varphi(\xi)\mathrm{d}\xi + \int_{y}^{\infty} \mathrm{e}^{\delta_2^+(y-\xi)}\varphi(\xi)\mathrm{d}\xi\right), \tag{41}$$

where $\varphi(\xi) = f(I(\xi))P\left(L(\xi)\right)S(\xi)$ and $\delta_i^\pm$ are the roots of $-d_i\delta^2 + c\delta + \gamma = 0$ with $\delta_i^+ > 0$, $\delta_i^- < 0$.

Differentiating (40) with respect to y gives

$$S'(y) = \frac{1}{d_1\left(\delta_1^+ - \delta_1^-\right)}\left(\delta_1^- \int_{-\infty}^{y} \mathrm{e}^{\delta_1^-(y-\xi)}(\gamma S(\xi) - \varphi(\xi))\mathrm{d}\xi \right. \\ \left. + \delta_1^+ \int_{y}^{\infty} \mathrm{e}^{\delta_1^+(y-\xi)}(\gamma S(\xi) - \varphi(\xi))\mathrm{d}\xi\right),$$

which implies $\lim_{y\to\pm\infty} S'(y) = 0$. Similarly, we have $\lim_{y\to\pm\infty} I'(y) = 0$. So, it follows from (9) that $\lim_{y\to\pm\infty} S''(y) = \lim_{y\to\pm\infty} I''(y) = 0$. That is,

$$\lim_{y\to\pm\infty} S'(y) = \lim_{y\to\pm\infty} S''(y) = 0, \quad \lim_{y\to\pm\infty} I'(y) = \lim_{y\to\pm\infty} I''(y) = 0. \tag{42}$$

Then, we can show that $I(y)$ is non-trivial. In fact, if $I(y) \equiv 0$, (9a) and (42) imply that $S_\infty = S_0$, and then $(S(y), I(y)) \equiv (S_0, 0)$, which is a contradiction. Thus, we can prove that $I(y) > 0$ for $y \in \mathbb{R}$ by a similar discussion to that in Section 4.

Moreover, similar to the discussion in Section 4, we can also derive that $S(y)$ is non-increasing and

$$\int_{-\infty}^{\infty} f(I(y))P\left(L(y)\right)S(y)dy = c\left(S_0 - S_\infty\right).$$

So, from (41) and using Fubini's theorem, we see that I is integrable on $\mathbb{R}$, and

$$\int_{-\infty}^{\infty} I(y)dy = \frac{1}{\gamma}\int_{-\infty}^{\infty} f(I(y))P\left(L(y)\right)S(y)dy. \tag{43}$$

With the above preparation, we can present the proof of Theorem 2.4 below.

Proof of Theorem 2.4. Assume that system (8) has a non-trivial and non-negative traveling wave solution (S, I) satisfying the boundary conditions. Then, we can derive contradictions in two cases as follows.

Case 1. $R_0 = f'(0^+)P(0)S_0/\gamma \le 1$.

By a similar argument to that at the end of Section 4, we can deduce that

$$I(y) \le \frac{2c(S_0 - S_\infty)}{c + \sqrt{c^2 + 4d_2\gamma}} < S_0 - S_\infty \le S_0 \text{ for all } y \in \mathbb{R}.$$

Then, by Assumption 1.1, we infer that

$$f(I(y))P\left(L(y)\right)S(y) \le f'(0^+)P(0)S_0 I(y),$$

This together with (9b) implies that

$$-\left(I'(y)e^{-\frac{c}{d_2}y}\right)' \le \left(f'(0^+)P(0)S_0-\gamma\right)I(y)e^{-\frac{c}{d_2}y} \le 0,$$

when $R_0 \le 1$. So, $I'(y)e^{-\frac{c}{d_2}y}$ is non-decreasing on $\mathbb{R}$. Since $I'(\infty)=0$, it follows that $I'(y)\le 0$ for $y\in\mathbb{R}$. Then, combining $I(\pm\infty)=0$ yields $I(y)\equiv 0$, which contradicts with the fact $I(y)$ is non-trival. Thus, (8) has no non-trivial non-negative traveling wave solutions if $R_0\le 1$.

Case 2. $R_0 = f'(0^+)P(0)S_0/\gamma > 1$, but $0<c<c_*$.

Since

$$\lim_{y\to-\infty}\frac{f(I(y))}{I(y)}P\left(L(y)\right)S(y)-\gamma = f'(0^+)P(0)S_0-\gamma,$$

there exists $\xi_0<0$ such that, for $y\le\xi_0$,

$$\frac{f(I(y))}{I(y)}P\left(L(y)\right)S(y)-\gamma > \frac{f'(0^+)P(0)S_0-\gamma}{2} := \eta.$$

Then, from (9b),

$$-d_2 I''(y)+cI'(y)\ge \eta I(y)\ge 0,\ \text{for } y\le\xi_0.$$

Integrating the above inequalities from $-\infty$ to y and letting $\mathscr{G}(y):=\int_{-\infty}^{y}I(\xi)d\xi$, we obtain that

$$-d_2 I'(y)+cI(y)\ge\eta\mathscr{G}(y)\ge 0,\ \text{for } y\le\xi_0,$$

which implies

$$-d_2 I(y)+c\mathscr{G}(y)\ge\eta\int_{-\infty}^{y}\mathscr{G}(\xi)d\xi\ge 0,\ \text{for } y\le\xi_0.$$

Therefore, we obtain that, for $y\le\xi_0$,

$$d_2 I''(y)\le cI'(y),\ d_2 I'(y)\le cI(y),\ d_2 I(y)\le c\mathscr{G}(y), \tag{44}$$

and

$$\int_{-\infty}^{y}\mathscr{G}(\xi)d\xi\le\frac{c}{\eta}\mathscr{G}(y).$$

Note that $\mathscr{G}(y)$ is non-decreasing, so for any $\delta>0$,

$$\mathscr{G}(y-\delta)\delta\le\frac{c}{\eta}\mathscr{G}(y),\ \text{for } y\le\xi_0.$$

Take $\delta>\frac{c}{\eta}$. Then, there is $\lambda_0\in\left(0,\frac{1}{\delta}\ln\frac{\eta\delta}{c}\right]$ such that

$$\mathscr{G}(y-\delta)e^{-\lambda_0(y-\delta)}\le\mathscr{G}(y)e^{-\lambda_0 y},\ \text{for } y\le\xi_0,$$

which implies that

$$\mathscr{G}(y)e^{-\lambda_0 y}\le\max_{y\in[\xi_0-\delta,\xi_0]}\mathscr{G}(y)e^{-\lambda_0 y},\ \text{for } y\le\xi_0.$$

Thus, by (43), we conclude that $\mathscr{G}(y)e^{-\lambda_0 y}$ is bounded on $\mathbb{R}$. Therefore, $I(y)e^{-\lambda_0 y}$, $I'(y)e^{-\lambda_0 y}$, and $I''(y)e^{-\lambda_0 y}$ are all bounded on $\mathbb{R}$ from (44). Hence, for $\varepsilon>0$ sufficiently small, there hold

$$\mathscr{G}(y)=o\left(e^{(\lambda_0-\epsilon)y}\right),\ I(y)=o\left(e^{(\lambda_0-\epsilon)y}\right),$$

$$I'(y) = o\left(e^{(\lambda_0-\epsilon)y}\right),\ I''(y) = o\left(e^{(\lambda_0-\epsilon)y}\right) \tag{45}$$

as $y \to -\infty$.

On the other hand, $I(y)$ is bound on $\mathbb{R}$, so $\mathcal{B}(\lambda, I) := \int_{-\infty}^{\infty} e^{-\lambda y} I(y)dy = \int_{-\infty}^{\infty} e^{-\lambda y} d\mathscr{G}(y)$ converges on the strip $0 < Re\lambda < \lambda_0$. Let $\lambda_a, \lambda_b \in \mathbb{R}$ ($\lambda_a \le \lambda_b$) be the abscissas of convergence of $\mathcal{B}(\lambda, I)$. Then, $(0, \lambda_0) \subset (\lambda_a, \lambda_b)$. Now we verify that $\lambda_a and \lambda_b$ are finite. Differentiating (41) gives that

$$\delta_2^- I(y) \le I'(y) \le \delta_2^+ I(y),$$

implying that $(I(y)e^{-\delta_2^- y})' \ge 0$ and $(I(y)e^{-\delta_2^+ y})' \le 0$ for $y \in \mathbb{R}$. So, we have $I(y) \ge I(0)e^{\delta_2^+ y}$ for all $y \le 0$, and $I(y) \ge I(0)e^{\delta_2^- y}$ for $y \ge 0$, which implies that $\mathcal{B}(\lambda, I)$ will diverge for $Re\lambda < \delta_2^-$ and $Re\lambda > \delta_2^+$. Clearly, we have $\delta_2^- \le \lambda_a \le \lambda_b \le \delta_2^+$. Therefore, by applying Theorem 2.5a, 2.5b in [21], we see that $\mathcal{B}(\lambda, I)$ is analytic on the strip $\lambda_a < Re\lambda < \lambda_b$, and $\lambda = \lambda_b$ is a singularity of $\mathcal{B}(\lambda, I)$ because $\mathscr{G}(y)$ is non-decreasing.

Next, we show that $\lambda = \lambda_b$ is an analytic point of $\mathcal{B}(\lambda, I)$ to achieve a contradiction. For any sufficiently small $\epsilon > 0$, $\mathcal{B}(\lambda, I)$ converges for $\lambda = \lambda_b - \epsilon$. Then utilizing Theorem 8.3a in [21], we conclude that

$$\mathscr{G}(y) = o\left(e^{(\lambda_b-\epsilon)y}\right),\ y \to -\infty,$$

which, from (44), implies that

$$I(y) = o\left(e^{(\lambda_b-\epsilon)y}\right),\ I'(y) = o\left(e^{(\lambda_b-\epsilon)y}\right),\ I''(y) = o\left(e^{(\lambda_b-\epsilon)y}\right),\ y \to -\infty.$$

Now, considering the fact that $I(y) and I'(y)$ are both bounded on $\mathbb{R}$, we then obtain

$$\int_{-\infty}^{\infty} e^{-\lambda y} I'(y)dy = \lambda\mathcal{B}(\lambda, I),\ \int_{-\infty}^{\infty} e^{-\lambda y} I''(y)dy = \lambda^2\mathcal{B}(\lambda, I)$$

for $0 < Re\lambda < \lambda_b$ by integration by parts. Observe that (9b) can be rewritten as

$$\begin{aligned} &- d_2 I''(y) + cI'(y) + \left(\gamma - f'(0^+)P(0)S_0\right) I(y) \\ &= f(I(y))P\left(L(y)\right)S(y) - f'(0^+)P(0)S_0 I(y). \end{aligned}$$

Conducting the bilateral Laplace transform to this equation, we then obtain

$$\begin{aligned} &K(c,\lambda)\int_{-\infty}^{\infty} e^{-\lambda y} I(y)dy \\ &= \int_{-\infty}^{\infty} e^{-\lambda y} I(y)\left(\frac{f(I(y))}{I(y)}P\left(L(y)\right)S(y) - f'(0^+)P(0)S_0\right)dy. \end{aligned} \tag{46}$$

Since $\mathscr{H}(y) := \frac{f(I(y))}{I(y)}P\left(L(y)\right)S(y) - f'(0^+)P(0)S_0$ is continuous and

$$\lim_{y\to-\infty} \mathscr{H}(y) = 0,\ \lim_{y\to\infty} \mathscr{H}(y) = f'(0^+)P(0)\left(S_\infty - S_0\right),$$

$\mathscr{H}(y)$ is bounded on $\mathbb{R}$. Hence, the integrals on both sides of (46) are well-defined on the strip $0 < Re\lambda < \lambda_b$.

From the fact that $I(-\infty) = 0$ and Assumption 1.1, there exists $y_0 < 0$ such that

$$0 \le f'(0^+) - \frac{f(I(y))}{I(y)} \le kI(y)^{a-1},\ \text{for } y \le y_0.$$

Therefore, for $y \le y_0$, there holds

$$\begin{aligned} |\mathscr{H}(y)| \le & \left|\frac{f(I(y))}{I(y)} S(y)\left(P\left(L(y)\right)-P(0)\right)\right| + \left|P(0)S(y)\left(\frac{f(I(y))}{I(y)} - f'(0^+)\right)\right| \\ & + f'(0^+)P(0)\left|S(y)-S_0\right| \\ \le & f'(0^+)S_0P'_M\left|L(y)\right| + P(0)S_0k\left|I(y)^{a-1}\right| + f'(0^+)P(0)\left|S(y)-S_0\right|. \end{aligned} \tag{47}$$

Now, integrating (9) from $-\infty$ to y, we obtain

$$d_1S'(y) - c(S(y)-S_0) = \int_{-\infty}^{y} f(I(\xi))P(L(\xi))S(\xi)d\xi,$$

which implies that

$$\left(e^{-\frac{c}{d_1}y}(S(y)-S_0)\right)' = \frac{1}{d_1}e^{-\frac{c}{d_1}y}\int_{-\infty}^{y} f(I(\xi))P(L(\xi))S(\xi)d\xi.$$

It then follows that

$$S(y)-S_0 = (S(0)-S_0)e^{\frac{c}{d_1}y} + \frac{1}{d_1}\int_0^y e^{\frac{c}{d_1}(y-x)}\int_{-\infty}^{x} f(I(\xi))P(L(\xi))S(\xi)d\xi dx. \tag{48}$$

Now, integrating (9b) from $-\infty$ to y yields that

$$\int_{-\infty}^{y} f(I(\xi))P(L(\xi))S(\xi)d\xi = -d_1I'(y) + cI(y) + \gamma\mathscr{G}(y),$$

which together with (45) implies that for $\epsilon > 0$ very small, there holds

$$\int_{-\infty}^{y} f(I(\xi))P(L(\xi))S(\xi)d\xi = o\left(e^{(\lambda_0-\epsilon)y}\right)$$

as y tends to $-\infty$. Then, applying L'Hôpital's rule, we infer that, for $0 < \delta < \min\{\lambda_0, \frac{c}{d_1}\}$, there holds

$$\int_0^y e^{\frac{c}{d_1}(y-x)}\int_{-\infty}^{x} f(I(\xi))P(L(\xi))S(\xi)d\xi dx = o\left(e^{\delta y}\right)$$

as y tends to $-\infty$. So, $S(y)-S_0 = o\left(e^{\delta y}\right)$ as $y \to -\infty$ by (48), and $\left(S(y)-S_0\right)e^{-\delta y}$ is bounded on $\mathbb{R}$.

On the other hand, we have known that $I(y)e^{-\lambda_0 y}$ is bounded on $\mathbb{R}$, which implies that $L(y)e^{-\lambda_0 y} = \int_0^\tau w(\theta)I(y-c\theta)d\theta e^{-\lambda_0 y}$ and $I(y)^{a-1}e^{-(a-1)\lambda_0 y}$ are both bounded on $\mathbb{R}$. Letting

$$\delta_0 = \min\left\{\lambda_0, (a-1)\lambda_0, \frac{\min\{\lambda_0, \frac{c}{d_1}\}}{2}\right\},$$

then $\mathscr{H}(y)e^{-\delta_0 y}$ is bounded on $\mathbb{R}$ by (47). It follows that $\int_{-\infty}^{\infty} e^{-\lambda y}I(y)\mathscr{H}(y)dy$ converges for any $0 < Re\lambda < \lambda_b + \delta_0$, so it is analytic for $0 < Re\lambda < \lambda_b + \delta_0$ by Theorem 2.5a [21]. Observe that $K(c,\lambda) < 0$ for all $\lambda > 0$ when $0 < c < c_*$ and

$$\int_{-\infty}^{\infty} e^{-\lambda y}I(y)dy = \frac{\int_{-\infty}^{\infty} e^{-\lambda y}I(y)\mathscr{H}(y)dy}{K(c,\lambda)}, \text{ for any } 0 < \lambda < \lambda_b + \delta_0.$$

So, $\lambda = \lambda_b$ is an analytic point of $\mathcal{B}(\lambda, I)$, a contradiction. Consequently, system (8) also has no traveling wave solutions that satisfies (10) in this case. The proof of Theorem 2.4 is complete. □

7. **Demonstration with numerical simulations.** In this section, we present some examples and numerical simulations to demonstrate the obtained theoretical results, hoping to help us better understand spatial-temporal dynamics of an infectious disease.

To begin with, we choose $f(I) = \beta I$ and $L(x,t) = I(x,t)$. Then, system (8) becomes

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - \beta I(x,t) P\left(I(x,t)\right) S(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + \beta I(x,t) P\left(I(x,t)\right) S(x,t) - \gamma I(x,t). \end{cases} \tag{49}$$

It is not difficult to verify that $f(I) = \beta I$ satisfies Assumption 1.1. For these choices, in addition to the results stated in Theorems 2.1, we can further prove that the profile of the I component $I(y)$ of the traveling wave for (49) is actually <u>*unimodal*</u>, as claimed in the following theorem.

Theorem 7.1. *Assume that Assumption 1.3 holds and $P(L) \not\equiv 1$. If $R_0 = \beta P(0) S_0/\gamma > 1$ and $c \geq c_* = 2\sqrt{d_2(\beta P(0) S_0 - \gamma)}$, system (49) has a non-trivial and non-negative traveling wave solution $(S(y), I(y))$ satisfying the boundary conditions (10). Moreover,*

(i) $S(y)$ is non-increasing on $\mathbb{R}$;

(ii) $0 < I(y) \leq \dfrac{2c\,(S_0 - S_\infty)}{c + \sqrt{c^2 + 4d_2\gamma}}$ for all $y \in \mathbb{R}$ and $I(y)$ is <u>unimodal</u> on $\mathbb{R}$;

(iii) there holds the following integral equality:

$$\gamma \int_{-\infty}^{+\infty} I(y)dy = \int_{-\infty}^{+\infty} \beta I(y) P\left(I(y)\right) S(y)dy = c\left(S_0 - S_\infty\right). \tag{50}$$

Proof. We just need to prove that $I(y)$ is unimodal on $\mathbb{R}$ because the rest of the results follow from Theorem 2.1. First, $(S(y), I(y))$ must satisfy

$$\begin{cases} cS'(y) = d_1 S''(y) - \beta I(y) P(I(y)) S(y), & \text{(51a)} \\ cI'(y) = d_2 I''(y) + \beta I(y) P(I(y)) S(y) - \gamma I(y), & \text{(51b)} \end{cases}$$

Since $I(y)$ is positive on $\mathbb{R}$ and $I(\pm\infty) = 0$, $I(y)$ must have at least one peak (maximum). We just need to show the maximum is unique. Otherwise, assume there are two maximum points $y_1 < y_2$ with $I(y_2) < I(y_1)$. Then, there will be a minimum point $y_3 \in (y_1, y_2)$ at which $I'(y_3) = 0$ and $I''(y_3) \geq 0$. So, (51b) implies that

$$S(y_3) \leq \frac{\gamma}{\beta P(I(y_3))}.$$

On the other hand, at the maximum point y_2, $I(y_2) > I(y_3)$, $I'(y_2) = 0$ and $I''(y_2) \leq 0$. It follows from (51b) that

$$S(y_2) \geq \frac{\gamma}{\beta P\left(I(y_2)\right)} > \frac{\gamma}{\beta P\left(I(y_3)\right)} \geq S(y_3),$$

which is a contradiction because $S(y)$ is non-increasing. The proof is completed. □

When choosing $f(I) = \beta I$ and $L(x,t) = I(x,t)$, Theorem 7.1 clearly indicates that $I(y)$ is *unimodal*, meaning that there is only a single outbreak. However, if the disease severity level $L(x,t)$ is measured involving disease surveillance of the past, $I(y)$ is not necessarily unimodal, and it may have multiple peaks, as is illustrated in Figure 6. To show this, we adopt $P(L) = 1/(1+\alpha L^2)$ and still choose $f(I) = \beta I$, but use $L(x,t) = \int_0^\tau w(\theta)I(x,t-\theta)d\theta$ to measure the severity. Assuming $\int_0^\tau w(\theta)d\theta = 1$, this $L(x,t)$ is a weighted total of infectious individuals in a past period of time (τ time units ago). Then, the model is reduced to the following specific one:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - \dfrac{\beta I(x,t)S(x,t)}{1+\alpha\left(\int_0^\tau w(\theta)I(x,t-\theta)d\theta\right)^2}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + \dfrac{\beta I(x,t)S(x,t)}{1+\alpha\left(\int_0^\tau w(\theta)I(x,t-\theta)d\theta\right)^2} - \gamma I(x,t). \end{cases} \tag{52}$$

For this specific system, by Theorem 2.1, we still conclude that when $R_0 = \beta S_0/\gamma > 1$ and $c \geq c_* = 2\sqrt{d_2(\beta S_0 - \gamma)}$, (52) has a non-trivial and non-negative traveling wave solution satisfying the boundary condition (10). It is obvious that

(a) c_* is increasing with β;
(b) c_* is proportional to $\sqrt{d_2}$;
(c) c_* is independent of d_1, α, and τ.

Although τ, α, and k_1 do not affect R_0 and c_*, they can have an impact on the profile of the traveling wave that accounts for the outbreak pattern, including the number and magnitudes of outbreaks, as is illustrated numerically in Figures 5-9 in what follows.

Below, for convenience of comparison, we perform some numerical simulations for (52) for the cases of $w(\theta) = \delta(\theta)$ (Dirac delta function) and some non-Dirac functions for $w(\theta)$. In all these cases, we will employ the numeric methods developed in [12].

Case 1. $w(\theta) = \delta(\theta)$.

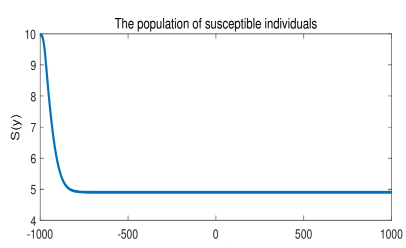
In this case, a traveling wave solution $(S(y), I(y))$ ($y = x + ct$) for (52) satisfies

$$\begin{cases} cS'(y) = d_1 S''(y) - \dfrac{\beta S(y)I(y)}{1+\alpha I^2(y)}, y \in \mathbb{R}, \\ cI'(y) = d_2 I''(y) + \dfrac{\beta S(y)I(y)}{1+\alpha I^2(y)} - \gamma I(y). \end{cases} \tag{53}$$

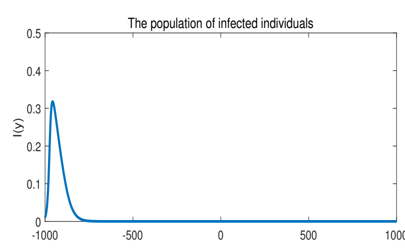
and the boundary conditions (10). We fix the following parameter values:

$$d_1 = 1,\ d_2 = 1,\ \beta = 0.05,\ \alpha = 4,\ \gamma = 0.3, S_0 = 10, \tag{54}$$

which lead to $R_0 \approx 1.6667 > 1$ and $c_* \approx 0.8944$.



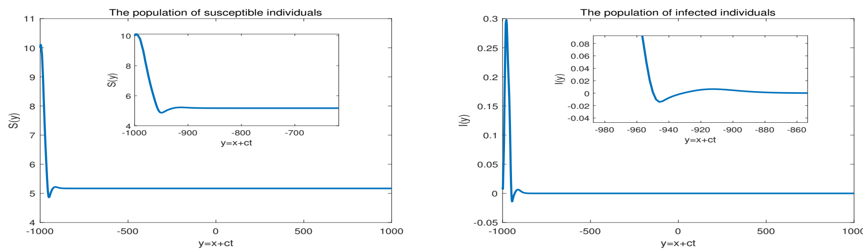
(a)



(b)

FIGURE 1. The solution of (53) with parameters (54) and speed $c = 3 > c_* \approx 0.8944$.

Next, as in [12], we first truncate the domain of y to $[-1000, 1000]$ and use the algorithm in [12] to (53) in the truncated interval to generate the plots in Figure

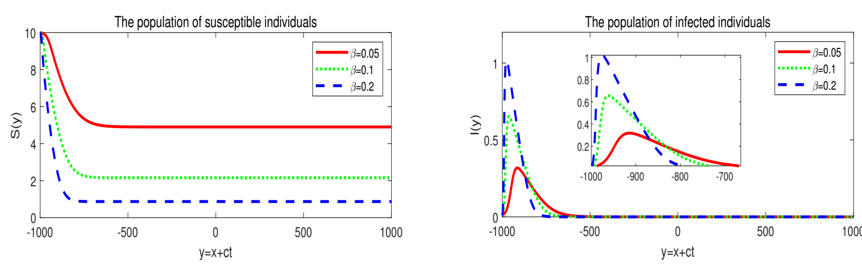


(a) (b)

FIGURE 2. The solution of (53) with parameters (54) and $c = 0.5 < c_* \approx 0.8944$; there is no traveling wave solution with speed $c = 0.5$: The infected I may take negative values.

1, which clearly shows the existence of a positive solution to (53) with (10) for $c = 3 > c_* \approx 0.8944$. Further, we observe that $S(y)$ decreases from the initial size S_0 to the final S_∞, and $I(y)$ is unimodal, which is consistent with Theorem 7.1. When setting $c = 0.5 < c_* = 0.8944$, we can see from Figure 2 that there is no positive traveling wave solution since the infected component I may take negative values.

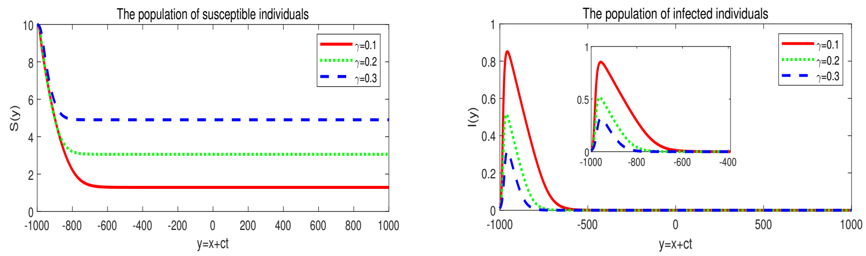
Note that the final size S_∞ and the magnitude I_{max} of the peak of $I(y)$ reflect how severe the epidemic is. Thus, it is meaningful to see how they are affected by model parameters. When $d_1 = 0$, for the simple incidence rate function βSI, a simple and clear formula of the final size S_∞ was deduced in [10,19]. When $d_1 > 0$, however, although we have obtained an equation (11) that implicitly determines S_∞, it seems difficult to obtain a simple formula for S_∞. Below, we numerically demonstrate how the parameters β, γ, and α affect the final size and the peak size for the more specific model (53).



(a) (b)

FIGURE 3. Impact of β on the final size and peak size for (53): (a) S_∞ decreases as β increases; (b) the peak size of $I(y)$ increases as β increases. Other parameters are as in (54) and $c = 3$, which is larger than c^* for all three values of β.

First, we fix the parameters in (54), except the transmission rate β, and choose β to be 0.05, 0.1, and 0.2, respectively. The corresponding values of R_0 are then calculated to be $1.6667, 3.3333,$ and 6.6667, and the minimal wave speed c_* is calculated to be $0.8944, 1.6733,$ and 2.6077, respectively. Choosing the same speed



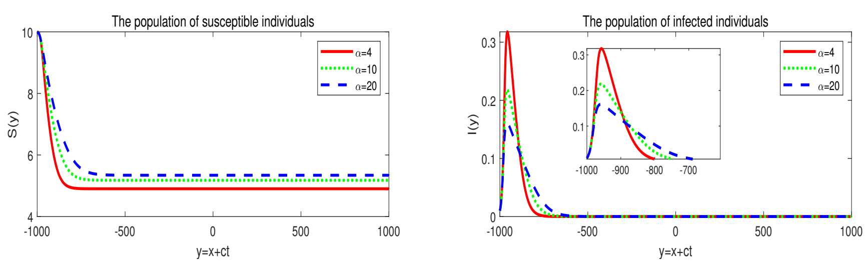
(a) (b)

FIGURE 4. Impact of γ on the final size and peak size in (53): (a) S_∞ increases as γ increases; (b) the peak size of $I(y)$ decreases as γ increases. Other parameters are as in (54) and $c = 1.5$ which is larger than c_* for all three values of γ.

$c = 3$ that is larger than c_* for all three values of β, the numerical results in Figure 3 show that as β increases from 0.05 to 0.2, S_∞ decreases and the peak size of $I(y)$ increases.

Next, we fix the parameters in (54), except for the recovery rate γ, and take three different values of γ as $0.1, 0.2,$ and 0.3. In this case, the corresponding values of R_0 are $5, 2.5,$ and 1.6667, and c_* are $1.2649, 1.0954,$ and 0.8944, respectively. The numerical results are plotted in Figure 4, in which we find that S_∞ increases and the peak size of $I(y)$ decreases as γ increases from 0.1 to 0.3.

Finally, we look at the effect of α on S_∞. To this end, we fix the parameter values as (54) except for α, which we will alter as $\alpha = 4, 10, 20$. Since α does not affect the values of R_0 and c_*, we still have $R_0 \approx 1.6667 > 1$ and $c_* \approx 0.8944$. The numerical results are presented in Figure 5 from which we can see that S_∞ increases as α increases, while the peak size of $I(y)$ decreases as α increases. Noting that α in the adopted fraction function form $1/(1+\alpha L^2)$ accounts for the precaution level of susceptible population, the above observed effect of α is not surprising: The more precautious the susceptible individuals are, the less disastrous the epidemic will be.



(a) (b)

FIGURE 5. Impact of α on the final size and peak size for (53) with $c = 1.5 > c_* \approx 0.8944$: (a) S_∞ increases as α increases; (b) the peak size of $I(y)$ decreases as α increases. Other parameters are as in (54).

Case 2. $w(\theta) = k_0\delta(\theta) + k_1\delta(\theta - \tau)$ with $k_0 + k_1 = 1$.

For this weight function, the disease severity level $L(x,t) = k_0 I(x,t) + k_1 I(x, t-\tau)$ also depends on disease surveillance of the past time $t-\tau$, and the traveling wave solution $(S(y), I(y))$ of (52) satisfies

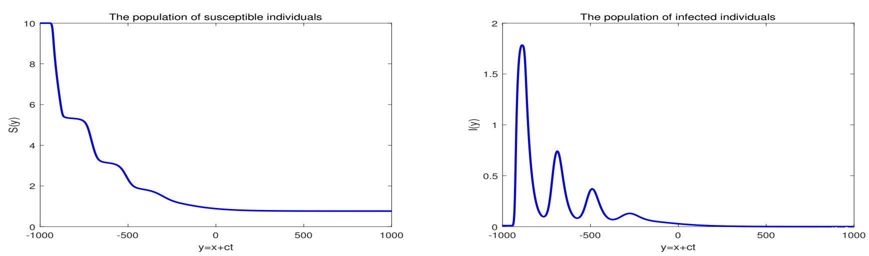
$$\begin{cases} cS'(y) = d_1 S''(y) - \dfrac{\beta S(y) I(y)}{1+\alpha(k_0 I(y) + k_1 I(y-c\tau))^2}, y \in \mathbb{R}, \\ cI'(y) = d_2 I''(y) + \dfrac{\beta S(y) I(y)}{1+\alpha(k_0 I(y) + k_1 I(y-c\tau))^2} - \gamma I(y), \end{cases} \tag{55}$$

and the boundary conditions (10).

We are interested in the effect of τ on $(S(y), I(y))$. Note that the influence of τ in (55) is related to the values of α and the weight k_1. So, we will numerically explore the impact of τ in conjunction with the different values of weight parameter k_1, and the precaution level parameter α. To this end, we fix other parameters in (56) as

$$d_1 = 1,\ d_2 = 1,\ \beta = 0.05,\ \gamma = 0.05,\ S_0 = 10. \tag{56}$$

With these values, we can calculate $R_0 = 10 > 1$ and $c_* \approx 1.3416$.

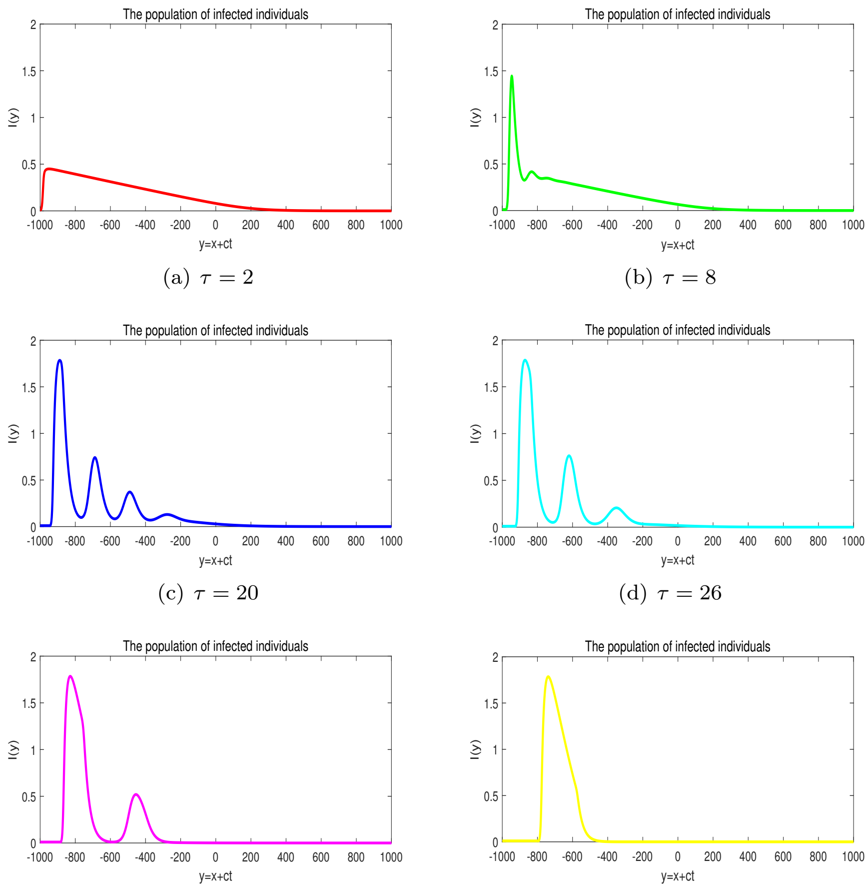


(a) (b)

FIGURE 6. The solution of (55) when speed $c = 1.5 > c_* \approx 1.3416$ – $I(y)$ is not unimodal, and multiple outbreaks occur.

Recall that τ, α, and k_1 do not affect R_0 and c_*. Now take $c = 1.5 > c_*$, $\alpha = 40$, $\tau = 20$, and $k_1 = 0.8$. Numerical simulation of (55) shows that the $I(t)$ component of the traveling wave is no longer unimodal, it has multiple peaks representing multiple outbreaks, see Figure 6. Interestingly, if we fix all other parameters as above but increase the value of τ (e.g., as $2, 8, 20, 26, 40,$ and 70, respectively), we find that $I(y)$ is unimodal when $0 < \tau \le 6$, but as τ increases, the number of peaks firstly increases and then decreases and eventually it returns to be unimodal; see Figure 7. Unfortunately, we are not able to explain this phenomenon analytically, at least at the present. Hence, τ may alter the pattern of the waveform of $I(y)$ and lead to multiple outbreaks of disease.

The discrete weights k_1 and $k_0 = 1 - k_1$ reflect how much weight is put in the past surveillance at previous time $t - \tau$ in measuring the severity $L(t)$. Next, by numerical exploration, we show that outbreak pattern can also be affected by these weights. To this end, we use the same parameter values as above and $\alpha = 40$ and $\tau = 20$. We saw earlier that when $k_1 = 0$, $I(y)$ is unimodal. Now, for four different values of k_1, we plot the $I(y)$ component of the numerical solutions of (55) in Figure 8, from which we find that as k_1 increases in $[0, 1]$, the number of peaks of $I(y)$ and their sizes increase.



(a) $\tau = 2$ (b) $\tau = 8$

(c) $\tau = 20$ (d) $\tau = 26$

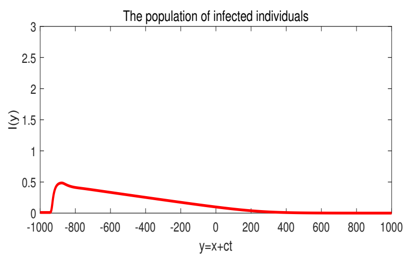
(e) $\tau = 40$ (f) $\tau = 70$

FIGURE 7. The solution of (55) for different τ when speed $c = 1.5 > c_* \approx 1.3416$. (a) $I(y)$ is unimodal when $\tau = 2$ is small; (b)-(e) $I(y)$ appears to have multiple peaks as τ increases and then converges to 0; (f) $I(y)$ will be unimodal again when τ increases to 70.

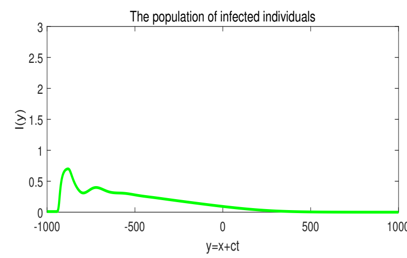
Finally, we explore the impact of α on the pattern of outbreaks for (55). We take the parameter values given in (56) and $k_1 = 0.8$ and $\tau = 20$, and we set α to four different values: 2, 20, 40, and 80. Then, for $c = 1.5 > c_*$, the $I(y)$ component of the numerical solution to (55) is plotted in Figure 9 for these four values of α, clearly showing that the number of outbreaks increases with α.

Case 3. Some other continuous forms of the weight function $w(\theta)$.

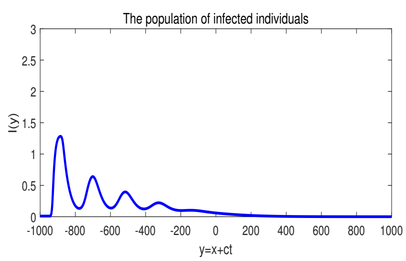
The above two forms of the weight function $w(\theta)$ are discontinuous, which lead to special cases of the model that either has no delay or has a single discrete delay. As far as numerical simulations go, we can also consider various forms of the continuous weight function $w(\theta)$ which account for true distributed delays with different emphases. For such a general continuous function $w(\theta)$, traveling wave solutions



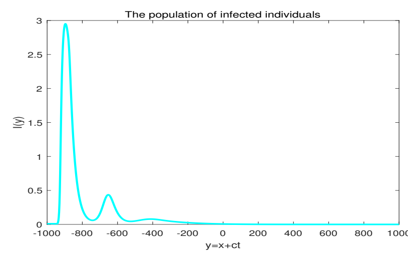
(a) $k_0 = 0.9, k_1 = 0.1$



(b) $k_0 = 0.6, k_1 = 0.4$

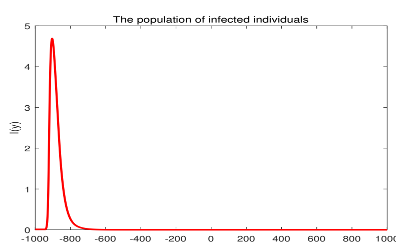


(c) $k_0 = 0.3, k_1 = 0.7$

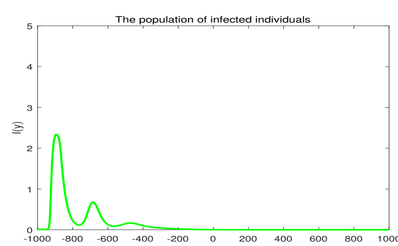


(d) $k_0 = 0.1, k_1 = 0.9$

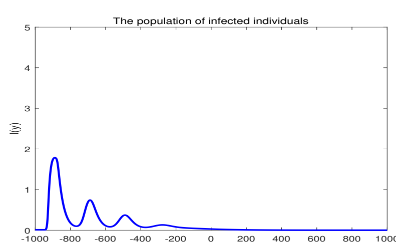
FIGURE 8. Impact of the weight k_1 on the profile of $I(y)$ component of the solution to (55) when speed $c = 1.5 > c_* \approx 1.3416$, $\alpha = 40,\ \tau = 20$.



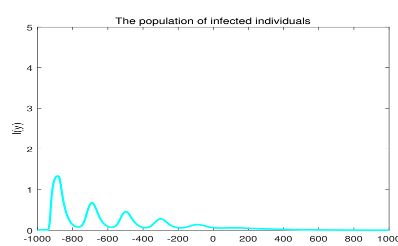
(a) $\alpha = 2$



(b) $\alpha = 20$



(c) $\alpha = 40$



(d) $\alpha = 80$

FIGURE 9. Impact of α on the profile of the $I(y)$ component of the solution to (55) for different α when speed $c = 1.5 > c_* \approx 1.3416$. The number of outbreaks increases as α increases.

$(S(y), I(y))$ of (52) are governed by

$$\begin{cases} cS'(y) = d_1 S''(y) - \dfrac{\beta S(y)I(y)}{1+\alpha\left[\int_0^\tau w(\theta)I(y-c\theta)\right]^2}, \\ cI'(y) = d_2 I''(y) + \dfrac{\beta S(y)I(y)}{1+\alpha\left[\int_0^\tau w(\theta)I(y-c\theta)\right]^2} - \gamma I(y), \end{cases} \quad y \in \mathbb{R}, \tag{57}$$

and the boundary conditions (10).

For the purpose of demonstration, we consider the following forms:

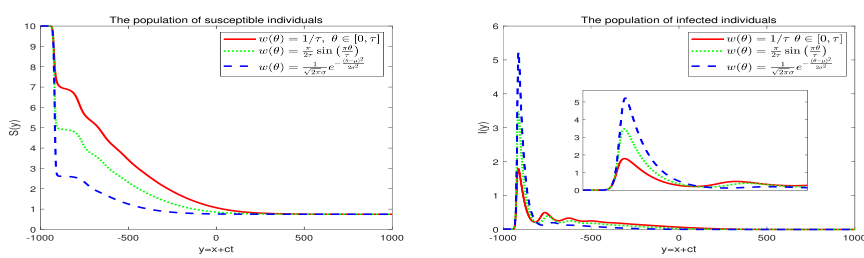
$$\text{(A) } w(\theta) = 1/\tau; \text{ (B) } w(\theta) = \frac{\pi}{2\tau}\sin\left(\frac{\pi\theta}{\tau}\right);$$

$$\text{and } (C) \ w(\theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \left(\mu = \frac{\tau}{2}, \sigma = \frac{\tau}{6}\right), \ \theta \in [0, \tau].$$

All these functions satisfy that $\int_0^\tau w(\theta)d\theta = 1$ for all $\tau > 0$. Now we choose the following parameter values:

$$d_1 = 1, \ d_2 = 1, \ \beta = 0.05, \ \gamma = 0.05, \ S_0 = 10, \ \alpha = 40, \ \tau = 20, \tag{58}$$

which lead to the same $R_0 = 10 > 1$ and $c_* \approx 1.3416$. Then, we can numerically solve (57) to generate the plots in Figure 10 for the above choices (A)-(C) of $w(\theta)$. These numerical results seem to suggest that, while different forms of the weight functions $w(\theta)$ do not affect the final size S_∞, they do influence the outbreak pattern of the epidemics reflected by the shape of $I(y)$ in Figure 10.



(a) (b)

FIGURE 10. The solutions of (57) for different distributed delay weight functions $w(\theta)$ when $c = 1.5 > c^* \approx 1.3416$ and other parameters are as in (58).

8. **Conclusions and discussion.** In this paper, we have proposed a new diffusive SIR infectious disease model with an general incidence rate function in which we incorporate behavior change of the host reflected by a fraction function $P(L(t))$ where $L(t)$ is a measurement of the severity of the epidemic. The main concern is the spatial spreading represented by non-trivial positive traveling wave solutions. We have identifed the basic reproduction number R_0 and derived the minimal wave speed c_* by proving that when $R_0 > 1$, then for every $c \geq c_*$, the model has a traveling wave solution with speed c, while when $R_0 < 1$ or $c \in (0, c_*)$, there is no such traveling wave solution. We have also obtained an equation that implicitly determines the final size S_∞, which serves as a base for numerically computing S_∞. Our theoretical results generalize some exiting results in literature.

The explicit formulas for R_0 and c_* clearly show how they are affected by some model parameters as presented in Sections 2-6. Moreover, by the numerical simulations presented in Section 7, we have explored the impacts of some model parameters on the final size, the number of outbreaks, and their magnitudes. Particularly, and more interestingly, we have found that those parameters involved in the behavior change term (α, k_1 and τ) in the model can cause multiple outbreaks. For example, the simulation results show that

(i) higher level of precaution (i.e., larger α) may increase the final size, reduce the infection peak size, and cause multiple outbreaks;
(ii) relatively larger weight k_1 on the past surveillance data in measuring the severity of epidemics may cause multiple outbreaks;
(iii) using too old of data (large τ) in measuring the severity of epidemics may cause multiple outbreaks.

Such novel results/observations have their practical significance because they can help us better understand the role and importance of some non-pharmaceutical interventions in control measures. Mathematically, it is interesting to analytically determine the thresholds for these parameters for the occurrence of multiple peaks of the $I(y)$ component of a traveling wave solution. But, this seems to be very challenging, if not impossible, and we leave it for a future work.

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