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EXISTENCE OF TRAVELING WAVEFRONTS OF DELAYED REACTION DIFFUSION SYSTEMS WITHOUT MONOTONICITY

Jianhua Huang
Dept. Math., Central China Normal University,Wuhan, HuBei 430079, China

Xingfu Zou
Department of Mathematics and Statistics
Memorial University of Newfoundland, St. John's, NF, A1C5S7, Canada

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Abstract. In this paper, we establish the existence of traveling wavefronts for delayed reaction diffusion systems without quasimonotonicity in the reaction term, by using Schauder's fixed point theorem. We show the merit of our result by applying it to the Belousov-Zhabotinskii reaction model with two delays.

1. **Introduction.** Traveling wave solutions for reaction diffusion systems without time delay have been extensively and intensively studied, see, for example, the books Britton [1], Fife [2], Murray [6], Volpert et al [11], the book review Gardner[3], the recent model study on this topic by Satnoianu et al [7], and the references therein. But little work has been done on traveling wave solutions for reaction diffusion systems with time delay. Schaaf [8] is the pioneer work, where systematically studied is a *scalar* reaction diffusion equation with a *single* discrete delay by using the phase-plane technique, the maximum principle for parabolic functional differential equations and general theory for ordinary functional differential equations.

Zou and Wu [15] consider *systems* with quasimonotonicity and a single delay, and establish existence of traveling wavefronts by first truncating the unbounded domain and then passing to a limit. Recently, Wu and Zou [13] further study more general reaction diffusion systems with general finite delays, where both quasimonotone and non-quasimonotone reaction terms are explored. The approach in Wu and Zou [13] is a monotone iteration scheme combined with upper-lower solutions technique. In addition to the applications in Wu and Zou [13], the main theorems in [13] have been applied to various delayed systems in S. Gourley [4], So, Wu and Zou [9] and So and Zou [10].

Ma [5] goes along the direction of Wu and Zou [13], but gives up the monotonicity of the iteration. Instead, he employs the Schauder's fixed point theorem to the operator used in Wu and Zou [13] in a properly chosen subset in the Banach space $C(R, R^n)$ equipped with the so called exponentially decay norm. The subset is constructed in terms of a pair of upper-lower solutions, which is less restrictive than the upper-lower solutions required in Wu and Zou [13]. This makes the searching for the pair of upper-lower solutions less harder. For example, an upper solutions

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does not have to be in the profile set. But Ma [5] only considers systems with quasimonotone reaction terms, that is, systems of the form

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + f(u_t(x)), \tag{1.1}$$

with $f : C([-\tau,0]), R^n) \to R^n$ satisfying the following quasimonotonicity:

(QM) There exists a matrix $\beta = diag(\beta_1, \cdots, \beta_n)$ with $\beta_i \geq 0, i = 1, \cdots, n$, such that

$$f(\phi(x)) - f(\psi(x)) + \beta[\phi(x)(0) - \psi(x)(0)] \geq 0$$

for $\phi(x), \psi(x) \in X = C([-\tau,0]; R^n)$ with $0 \leq \psi(x)(s) \leq \phi(x)(s) \leq K$ for $s \in [-\tau, 0]$.

Here, $t \in R^+,\ x \in R, u \in R^n, D = diag(d_1, \ldots, d_n)$ with $d_i > 0, i = 1, \cdots, n;\ f : C([-\tau,0]), R^n) \to R^n$ is continuous, $f(0) = 0 = f(K)$ and $f(u) \neq 0$ for $u \in (0, K)$. For any fixed $x \in R, u_t(x) \in C([-\tau,0], R^n)$ is defined by $u_t(x)(\theta) = u(t+\theta, x)$ for $\theta \in [-\tau, 0]$. In the sequel, when there is no confusion, we will drop the x and write $u_t(x) = u_t$.

On the other hand, it is quite common that the reaction term in a model system arising from a practical problem may not satisfy (QM). A simple but typical and important example is the so called Hutchinson equation

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1 - u(x,t-\tau)]. \tag{1.2}$$

Thus, it is worthwhile to further explore this topic for systems without quasimonotonicity, and this constitutes the purpose of this paper.

In this paper we will consider the existence of traveling wavefronts of (1.1). Motivated by Wu and Zou [13], we will propose a less restrictive condition on the reaction term as follows:

(QM*): There exists a matrix $\beta = diag(\beta_1, \cdots, \beta_n)$ with $\beta_i \geq 0, i = 1, \cdots, n$, such that

$$f(\phi(x)) - f(\psi(x)) + \beta[\phi(x)(0) - \psi(x)(0)] \geq 0$$

for $\phi(x), \psi(x) \in X = C([-\tau,0]; R^n)$ with (i) $0 \leq \psi(x)(s) \leq \phi(x)(s) \leq K$ for $s \in [-\tau, 0]$; and (ii) $e^{\beta s}[\phi(x)(s) - \psi(x)(s)]$ non-decreasing in $s \in [-\tau, 0]$.

As in Ma [5], we will construct a subset in the Banach space $C(R, R^n)$ equipped with the exponential decay norm, and apply the Schauder's fixed point theorem to the operator used in Wu and Zou [13] and Ma [5] to establish the existence of a traveling wavefront. The subset is obtained from a pair of upper-lower solutions, but unlike in Ma [5], we also take into account the fact that the reaction term only satisfies (QM*). A merit of our main theorem is that we do not require that the upper solution $\bar{\rho}(t)$ be monotone and satisfy $\lim_{t\to-\infty} \bar{\rho}(t) = 0$. This brings certain convenience in searching for an upper solution. For example, in the system case, some components of an upper solution can be chosen to be constants, and thereby, the corresponding wave system can be *decoupled* to some extent. Such a merit will be demonstrated by applying the main theorem to the Belousov-Zhabotinskii reaction model with two delays, to which, the original theory in Wu and Zou [13] is not easy (if not impossible) to apply and the main result in Ma [5] does not apply.

2. **Preliminaries.** In this paper, we use the usual notations for the standard ordering in R^n. That is, for $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T$. we denote $u \leq v$ if $u_i \leq v_i,\ i = 1, \ldots, n$,and $u < v$ if $u \leq v$ but $u \neq v$. In particular, we denote $u \ll v$ if $u \leq v$ but $u_i \neq v_i, i = 1, \ldots, n$. If $u \leq v$, we also denote $(u, v] = \{w \in R^n : u < w \leq v\}$, $[u, v) = \{w \in R^n : u \leq w < v\}$,and $[u, v] = \{w \in R^n : u \leq w \leq v\}$. Let $|\cdot|$ denotes the Euclidean norm in R^n and $||\cdot||$ denote the supremum norm in $C([-\tau, 0], R^n)$.

A traveling wave solution of (1.1) is a solution of the special from $u(x, t) = \phi(x + ct)$ where $\phi \in C^2(R, R^n)$, and $c > 0$ is a positive constant accounting for the wave speed. Substituting $u(x, t) = \phi(x + ct)$ and denoting $x + ct$ by t, we obtain the corresponding wave equation

$$D\phi''(t) - c\phi'(t) + f_c(\phi_t) = 0 \tag{2.1}$$

where $f_c(\phi_s) : X_c = C([-c\tau, 0], R^n) \to R^n$ is defined by

$$f_c(\psi) = f(\psi^c), \quad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0]. \tag{2.2}$$

If for some $c > 0$, (1.1) has a monotone solution ϕ defined on R, subject to the following asymptotic boundary condition

$$\lim_{t\to-\infty} \phi(t) = \phi_- \quad \text{and} \quad \lim_{t\to+\infty} \phi(t) = \phi_+, \tag{2.3}$$

then the corresponding solution $u(x, t) = \phi(x + ct)$ is called a traveling wave front with wave speed c. Therefore, (1.1) has a traveling wavefront if and only if (2.1) has a solution on R satisfying the asymptotic boundary condition (2.3). Without loss of generality, we assume $\phi_- = 0$ and $\phi_+ = K$, thus (2.3) can be replaced by

$$\lim_{t\to-\infty} \phi(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \phi(t) = K. \tag{2.4}$$

Correspondingly, we make the following hypotheses:

(A1) $f(\tilde{0}) = f(\tilde{K}) = 0$ with $0 < K$, where by $\tilde{u} : [-\tau, 0] \to R^n$ is the constant function with value u for all $t \in R$.

(A2) There exists a positive constants $L > 0$ such that

$$|f(\phi_t) - f(\psi_t)| \leq L||\phi - \psi||.$$

for $\phi, \psi \in C([-\tau, 0], R^n)$ with $0 \leq \phi(s), \psi(s) \leq K,\ s \in [-\tau, 0]$.

As mentioned in the introduction, we will use a pair of upper-lower solutions of (2.1) to construct a subset of $C(R, R^n)$ in which the Schauder's fixed point theorem can be applied to the related operator. To this end, we need to make it clear what upper and lower solutions mean.

Definition 2.1. A continuous function $\rho : R \to R^n$ is called an upper solution of (2.1) if ρ' and ρ'' exist almost everywhere (a.e.) in R and they are essentially bounded on R, and if ρ satisfies

$$D\rho''(t) - c\rho'(t) + f_c(\rho_t) \leq 0, \quad a.e. \quad in \quad R. \tag{2.5}$$

A lower solution of (2.1) is defined in a similar way by reversing the inequality in (2.5).

In what follows, we assume that an upper solution $\bar{\rho}$ and a lower solution $\underline{\rho}$ of (2.1) are given so that $(H1) \quad 0 \leq \underline{\rho} \leq \bar{\rho} \leq K$, $t \in R$,

(H2) $\quad \lim_{t\to-\infty} \underline{\rho}(t) = 0, \quad \lim_{t\to\infty} \bar{\rho}(t) = K$, (H3) The set

$$\Gamma(\underline{\rho},\bar{\rho}) = \left\{ \phi \in C(R,R^n): \begin{array}{l} (i) \quad \phi \text{ is nondecreasing in } R, \text{ and } \underline{\rho}(t) \le \phi(t) \le \bar{\rho}(t); \\ (ii) \quad e^{\beta t}[\bar{\rho}(t)-\phi(t)] \text{ and } e^{\beta t}[\phi(t)-\underline{\rho}(t)] \text{ are} \\ \qquad \text{nondecreasing in } t \in R; \\ (iii) \quad e^{\beta t}[\phi(t+s)-\phi(t)] \text{ is nondecreasing} \\ \qquad \text{in } t \in R \text{ for every } s>0. \end{array} \right\}$$

is non-empty.

We will further explore the properties of $\Gamma(\underline{\rho},\bar{\rho})$ in Section 3.

Assume that (QM*) holds. Define $H: C(R,R^n) \to C(R,R^n)$ by

$$H(\phi)(t) = f_c(\phi_t) + \beta\phi(t) \quad \phi \in C(R,R^n) \tag{2.6}$$

where β is as in (QM*). The operator H enjoys the following nice properties:

Lemma 2.1 (Wu and Zou [13]) *Assume that (QM*) and (A1) hold. Then for any $\phi \in \Gamma(\underline{\rho},\bar{\rho})$ we have (i) $H(\phi)(t) \ge 0, \quad t \in R$,*

(ii) *$H(\phi)(t)$ is nondecreasing in $t \in R$*

(iii) *$H(\psi)(t) \le H(\phi)(t)$ for $t \in R$ and $\psi \in C(R,R^n)$, with $0 \le \psi(t) \le \phi(t) \le K$ and $e^{\beta t}[\phi(t)-\psi(t)]$ non-decreasing in $t \in R$.*

In terms of H, (2.1) can be rewritten as

$$D\phi''(t) - c\phi'(t) - \beta\phi(t) + H(\phi)(t) = 0, \quad t \in R. \tag{2.7}$$

Define

$$\lambda_{1i} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i}, \quad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i}.$$

Let $C_{[0,K]}(R,R^n) = \{\phi \in C(R,R^n): \quad 0 \le \phi(s) \le K, \ s \in R\}$. By (H1)-(H3), $\Gamma(\underline{\rho},\bar{\rho}) \subset C_{[0,K]}(R,R^n)$. Define $F: C_{[0,K]}(R,R^n) \to C(R,R^n)$ by

$$(F\phi)_i(t) = \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)} H_i(\phi)(s)ds + \int_t^{\infty} e^{\lambda_{2i}(t-s)} H_i(\phi)(s)ds\Big] \tag{2.8}$$

for $i = 1,\ldots,n$ and $\phi \in C_{[0,K]}(R,R^n)$. It is easy to show that F is well-defined on $C_{[0,K]}(R,R^n)$, and for any $\phi \in C_{[0,K]}(R,R^n)$, $F(\phi)$ satisfies

$$D(F\phi)'' - c(F\phi)' - \beta(F\phi) + H(\phi) = 0 \tag{2.9}$$

Thus, if $F(\phi) = \phi$, i.e., ϕ is a fixed point of F, then (2.7) has a solution ϕ. If this solution is monotone and satisfies the boundary condition (2.4), then we obtain a traveling wave front for (1.1).

Corresponding to Lemma 2.1, we have the following lemma for F, which is a direct consequence of Lemma 2.1.

Lemma 2.2 *Assume that (QM*) and (A1) hold, then for any $\phi \in \Gamma(\underline{\rho},\bar{\rho})$, we have*

(i) *$F(\phi)(t)$ is nondecreasing in $t \in R$;*

(ii) *$F(\psi)(t) \le F(\phi)(t)$ for $t \in R$ and $\psi \in C_{[0,K]}(R,R^n)$ with $0 \le \psi(t) \le \phi(t) \le K$ and $e^{\beta t}[\phi(t)-\psi(t)]$ non-decreasing in $t \in R$.*

In the next section, we will apply Schauder's fixed point theorem to F in the subset $\Gamma(\underline{\rho},\bar{\rho})$. For this purpose, we need to introduce a topology in $C(R,R^n)$. Let $\mu > 0$ be such that $\mu < \min\{-\lambda_{1i}, \lambda_{2i}, i = 1,\ldots,n.\}$. Equip $C(R,R^n)$ with the exponential decay norm defined by $|\phi|_\mu = \sup_{t\in R}|\phi(t)|_{R^n} e^{-\mu|t|}$. Denote correspondingly $B_\mu(R,R^n) = \{\phi \in C(R,R^n): \ \sup_{t\in R}|\phi(t)|_{R^n} e^{-\mu|t|} < \infty\}$. Then it is easy to show that $(B_\mu(R,R^n), |\cdot|_\mu)$ is a Banach space.

3. **Main result.** Now we state our main theorem.

Theorem 3.1 *Assume that (QM*), (A1) and (A2) hold. In addition to (H1)-(H3) , we assume the upper solution* $\bar{\rho}(t)$ *and the lower solution* $\underline{\rho}(t)$ *further satisfy*

(H4) $f(\tilde{u}) \neq 0$, *for* $u \in (0, \inf_{t\in R}\bar{\rho}(t)] \bigcup [\sup_{t\in R}\underline{\rho}(t), K)$.

Then, (1.1) has a traveling wavefront solution.

In the remainder of this section, we will assume $c > 1 - \min\{\beta_i d_i;\ i = 1, \cdots, n\}$, which can always be realized by choosing β_i sufficiently large, without violating (QM*). In order to prove this theorem, we first establish the following lemmas.

Lemma 3.1 *Under the assumptions of Theorem 3.1,* $\Gamma(\underline{\rho}, \bar{\rho})$ *is a closed, bounded and convex subset of* $B_\mu(R, R^n)$.

Proof Boundedness is obvious. Let $\phi_n \in \Gamma(\underline{\rho}, \bar{\rho})$ with $\phi_n \to \phi$ in $B_\mu(R, R^n)$, i.e.,

$$\lim_{n\to+\infty} \sup_{t\in R} |\phi_n(t) - \phi(t)| e^{-\mu|t|} = 0.$$

Thus, $\phi_n(t)$ converges to $\phi(t)$ point wise for every $t \in R$ as $n \to +\infty$. Obviously, $\phi(t)$ satisfies (i) of $\Gamma(\underline{\rho}, \bar{\rho})$. For any $t_1, t_2 \in R$ with $t_1 \geq t_2$ as $s > 0$, by condition (ii) and (iii) for $\Gamma(\underline{\rho}, \bar{\rho})$, it follows that

$$\begin{aligned} &e^{\beta t_1}[\phi_n(t_1 + s) - \phi_n(t_1)] \geq e^{\beta t_2}[\phi_n(t_2 + s) - \phi_n(t_2)], \\ &e^{\beta t_1}[\bar{\rho}(t_1) - \phi_n(t_1)] \geq e^{\beta t_1}[\bar{\rho}(t_1) - \phi_n(t_1)], \\ &e^{\beta t_1}[\phi_n(t_1) - \underline{\rho}(t_1)] \geq e^{\beta t_1}[\phi_n(t_1) - \underline{\rho}(t_1)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} &e^{\beta t_1}[\phi(t_1 + s) - \phi(t_1)] \geq e^{\beta t_2}[\phi(t_2 + s) - \phi(t_2)], \\ &e^{\beta t_1}[\bar{\rho}(t_1) - \phi(t_1)] \geq e^{\beta t_1}[\bar{\rho}(t_1) - \phi(t_1)], \\ &e^{\beta t_1}[\phi(t_1) - \underline{\rho}(t_1)] \geq e^{\beta t_1}[\phi(t_1) - \underline{\rho}(t_1)]. \end{aligned}$$

This implies that $\phi(t)$ satisfies (ii) and (iii) of $\Gamma(\underline{\rho}, \bar{\rho})$. Hence $\phi \in \Gamma(\underline{\rho}, \bar{\rho})$ and therefore, $\Gamma(\underline{\rho}, \bar{\rho})$ is closed. The convex property is a direct verification by definition. This completes the proof.

Lemma 3.2 *Under the assumptions of Theorem 3.1,* $F(\Gamma(\underline{\rho}, \bar{\rho})) \subset \Gamma(\underline{\rho}, \bar{\rho})$.

Proof. For any $\phi \in \Gamma(\underline{\rho}, \bar{\rho})$, $\underline{\rho}(t) \leq \phi(t) \leq \bar{\rho}(t)$ and by Lemma 2.2 (i), $F(\phi)(t)$ is nondecreasing in $t \in R$. Lemma 2.2 (ii) implies

$$F(\underline{\rho})(t) \leq F(\phi)(t) \leq F(\bar{\rho})(t).$$

Repeating the proof of Wu and Zou ([13], Lemma 3.3-(iii)) gives

$$F(\bar{\rho})(t) \leq \bar{\rho}(t), \quad F(\underline{\rho})(t) \geq \underline{\rho}(t),$$

and hence $\underline{\rho}(t) \leq F(\phi)(t) \leq \bar{\rho}(t)$.

Note that $c > 1 - \min\{\beta_i d_i;\ i = 1, \cdots, n\}$ implies $\beta_i + \lambda_{1i} > 0$ and $\beta_i + \lambda_{2i} > 0$ (see Wu and Zou ([13] Lemma 4.2-(ii)), and thus,

$$\begin{aligned} &\frac{d}{dt} e^{\beta_i t}[(F\phi)_i(t + s) - (F\phi)_i(t)] \\ &= (\beta_i + \lambda_{1i}) e^{\beta_i + \lambda_{1i})t} \int_{-\infty}^{t} \frac{e^{-\lambda_{1i}\theta}}{d_i(\lambda_{2i} - \lambda_{1i})} [H_i(\phi)(\theta + s) - H_i(\phi)(\theta)] d\theta \\ &\quad + (\beta_i + \lambda_{2i}) e^{(\beta_i + \lambda_{2i})t} \int_{t}^{\infty} \frac{e^{-\lambda_{2i}\theta}}{d_i(\lambda_{2i} - \lambda_{1i})} [H_i(\phi)(\theta + s) - H_i(\phi)(\theta)] d\theta \\ &\geq 0, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

Let $w(t)=\bar{\rho}(t)-(F\phi)(t)$. Since $\bar{\rho}(t)$ is a upper solution of (2.1), by(2.9), we get

$$cw'(t)\geq Dw''(t)-\beta w(t)+[H(\bar{\rho})(t)-H(\phi)(t)]\geq Dw''(t)-\beta w(t).$$

Repeating the proof of Wu and Zou ([13] Lemma 4.3), we can compute

$$\frac{d}{dt}\left\{e^{\beta t}[\bar{\rho}(t)-F(\phi)(t)]\right\}\geq 0,\quad t\in R.$$

Similarly, $e^{\beta t}[F(\phi)(t)-\underline{\rho}(t)]$ is nondecreasing in $t\in R$. Thus, $F(\phi)\in\Gamma(\underline{\rho},\bar{\rho})$, and $F(\Gamma(\underline{\rho},\bar{\rho}))\subset\Gamma(\underline{\rho},\bar{\rho})$. This completes the proof.

Lemma 3.3 *Under the assumptions of Theorem 3.1, F is continuous with respective to the norm $|\cdot|_\mu$ in $B_\mu(R,R^n)$.*

Proof We prove Lemma 3.3 by two steps. The first step is to prove that $H: B_\mu(R,R^n)\to B_\mu(R,R^n)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^n)$. For any fixed $\varepsilon>0$, let $\delta<\frac{\varepsilon}{Le^{\mu c\tau}+||\beta||}$. Then for $\phi,\psi\in B_\mu(R,R^n)$ with

$$|\phi-\psi|_\mu=\sup_{t\in R}|\phi(t)-\psi(t)|e^{-\mu|t|}<\delta,$$

we have

$$\begin{aligned}
|(H\phi)(t)-(H\psi)(t)|e^{-\mu|t|} &\leq |f(\phi_t)-f(\psi_t)|e^{-\mu|t|}+||\beta|||\phi-\psi|_\mu\\
&\leq L||\phi-\psi||_{X_c}e^{-\mu|t|}+||\beta|||\phi-\psi|_\mu\\
&= Lsup_{s\in[-c\tau,0]}|\phi(s+t)-\psi(s+t)|e^{-\mu|t|}+||\beta|||\phi-\psi|_\mu\\
&\leq Lsup_{(s+t)\in R}|\phi(s+t)-\psi(s+t)|e^{-\mu|t|}+||\beta|||\phi-\psi|_\mu\\
&\leq Lsup_{\theta\in R}|\phi(\theta)-\psi(\theta)||e^{-\mu|\theta|}e^{\mu|s|}+||\beta|||\phi-\psi|_\mu\\
&\leq Le^{\mu c\tau}|\phi-\psi|_\mu+||\beta|||\phi-\psi|_\mu\\
&\leq (Le^{\mu c\tau}+||\beta||)|\phi-\psi|_\mu\leq\varepsilon.
\end{aligned}$$

Therefore $H: B_\mu(R,R^n)\to B_\mu(R,R^n)$ is continuous.

We next prove the continuity of F. If $t\geq 0$, it follows that

$$\begin{aligned}
&|(F\phi)_i(t)-(F\psi)_i(t)|e^{-\mu|t|}\\
&\leq\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\frac{\lambda_{2i}-\lambda_{1i}}{(\mu-\lambda_{1i})(\lambda_{2i}-\mu)}+\frac{2\mu}{\lambda_{1i}^2-\mu^2}e^{(\lambda_{1i}-\mu)t}\Big]|H(\phi)-H(\psi)|_\mu\\
&\leq\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\frac{\lambda_{2i}-\lambda_{1i}}{(\mu-\lambda_{1i})(\lambda_{2i}-\mu)}+\frac{2\mu}{\lambda_{1i}^2-\mu^2}\Big]|H(\phi)-H(\psi)|_\mu.
\end{aligned}$$

If $t<0$,we have

$$\begin{aligned}
&|(F\phi)_i(t)-(F\psi)_i(t)|e^{-\mu|t|}\\
&\leq\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\frac{\lambda_{2i}-\lambda_{1i}}{-(\mu+\lambda_{1i})(\lambda_{2i}+\mu)}+\frac{2\mu}{\lambda_{2i}^2-\mu^2}e^{(\lambda_{2i}+\mu)t}\Big]|H(\phi)-H(\psi)|_\mu\\
&\leq\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\frac{\lambda_{2i}-\lambda_{1i}}{-(\mu+\lambda_{1i})(\lambda_{2i}+\mu)}+\frac{2\mu}{\lambda_{2i}^2-\mu^2}\Big]|H(\phi)-H(\psi)|_\mu.
\end{aligned}$$

Now the continuity of F follows from that of H, and this completes the proof.

Lemma 3.4 *Under the assumptions of Theorem 3.1, $F:\Gamma(\underline{\rho},\bar{\rho})\to\Gamma(\underline{\rho},\bar{\rho})$ is compact.*

Proof We first established an estimate for F. For any $\phi \in \Gamma[\underline{\rho}, \bar{\rho}]$, direct calculation shows

$$\begin{aligned} 0 \le (F\phi)'_i(t) &= \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\lambda_{1i}e^{\lambda_{1i}t}\int_{-\infty}^{t} e^{-\lambda_{1i}\theta}(H\phi)_i(\theta)d\theta + (H\phi)_i(t)\Big] \\ &+\frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\lambda_{2i}e^{\lambda_{2i}t}\int_{t}^{+\infty} e^{-\lambda_{2i}\theta}(H\phi)_i(\theta)d\theta + (H\phi)_i(t)\Big], \end{aligned}$$

Since $(H\phi)_i(\theta) \ge 0,\ \lambda_{1i} < 0,\ \lambda_{2i} > 0$, we then have

$$\begin{aligned} 0 \le (F\phi)'_i(t) &\le \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\lambda_{2i}e^{\lambda_{2i}t}\int_{t}^{+\infty} e^{-\lambda_{2i}\theta}(H\phi)_i(\theta)d\theta + 2(H\phi)_i(t)\Big] \\ &\le \frac{1}{d_i(\lambda_{2i}-\lambda_{1i})}\Big[\lambda_{2i}e^{\lambda_{2i}t}(H\bar{\rho})_i(t))\int_{t}^{+\infty} e^{-\lambda_{2i}\theta}d\theta + 2(H\phi)_i(t)\Big] \\ &\le \frac{3}{d_i(\lambda_{2i}-\lambda_{1i})}(H\bar{\rho})_i(t). \end{aligned}$$

(H1)-(H2) and Lemma 3.3 imply that $H(\bar{\rho})(t)$ is uniformly bounded, hence, it follows that $F(\Gamma(\underline{\rho}, \bar{\rho}))$ is equicontinuous on $\Gamma(\underline{\rho}, \bar{\rho})$. It is also easily seen that $F(\Gamma(\underline{\rho}, \bar{\rho}))$ is uniformly bounded.

Define

$$(F_n\phi)(t) = \begin{cases} (F\phi)(t), & t \in [-n, n]; \\ (F\phi)(n), & t \in (n, +\infty); \\ (F\phi)(-n), & t \in (-\infty, -n). \end{cases}$$

Then for each $n \ge 1$, $F_n(\Gamma(\underline{\rho}, \bar{\rho}))$ is also equicontinuous and uniformly bounded on $\Gamma(\underline{\rho}, \bar{\rho})$. Now, as $F_n(\phi)$ has a uniformly compact support for every $\phi \in \Gamma(\underline{\rho}, \bar{\rho})$, Ascoli-Arzela lemma can be applied to F_n, implying that F_n is compact.

Next we prove that $F_n \to F$ in $B_\mu(R, R^n)$ as $n \to \infty$. This is obtained by the following estimate.

$$\begin{aligned} \sup_{t\in R}|(F_n\phi)(t) - (F\phi)(t)|e^{-\mu|t|} &= \sup_{t\in(-\infty,-n)\bigcup(n,\infty)}|(F_n\phi)(t) - (F\phi)(t)|e^{-\mu|t|} \\ &\le 2Ke^{-\mu n} \to 0, \quad n \to \infty. \end{aligned}$$

Now, by Proposition 2.12 in [14], it follows that $F : \Gamma(\underline{\rho}, \bar{\rho}) \to \Gamma(\underline{\rho}, \bar{\rho})$ is also compact. The proof is completed.

Proof of the Theorem 3.1. Combining Lemmas 3.1-3.4 with Schauder's fixed point theorem, we know that there exits a fixed point ϕ of F in $\Gamma(\underline{\rho}, \bar{\rho})$. In order to show this fixed point is traveling wave front solution, we need to verify the boundary condition (2.4).

First of all, $\phi \in \Gamma(\underline{\rho}, \bar{\rho})$ implies that ϕ is monotone. Secondly, since $0 \le \underline{\rho} \le \phi(t) \le \bar{\rho}(t) \le K$, we have

$$0 \le \phi_- = \lim_{t\to-\infty} \phi(t) \le \inf_{t\in R}\bar{\rho}(t) \le K$$

and

$$0 \le \sup_{t\in R}\underline{\rho}(t) \le \phi_+ = \lim_{t\to\infty} \phi(t) \le K.$$

Applying Proposition 2.1 in [13], we have $f_c(\bar{\phi}_-) = f_c(\bar{\phi}_+)) = 0$. Now (H4) implies that $\phi_- = 0$ and $\phi_+ = K$, that is,

$$\lim_{t\to-\infty} \phi(t) = 0, \quad \lim_{t\to\infty} \phi(t) = K.$$

Therefore, the fixed point does give a traveling wave front solution of (2.4).

Remark 3.1. Although the upper and lower solutions $\bar{\rho}(t)$ and $\underline{\rho}(t)$ are required to satisfy (H2), we do not require $\bar{\rho}(t)$ to be monotone and to satisfy $\lim_{t\to-\infty}\bar{\rho}(t) = 0$. This is a significant improvement of the corresponding result in Wu and Zou [13]. As will be seen in the next section, such an improvement does make the searching of an upper solution easier.

4. **Applications.** We consider Belousov-Zhabotinskii system

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1 - u(x,t) - rv(x,t)] \\ \frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - bu(x,t)v(x,t), \end{cases} \tag{4.1}$$

where $r > 0$ and $b > 0$ are constants, u and v correspond respectively to the Bromic acid and bromide ion concentrations. This system has been studied by many authors. (see e.g. Murray [6]) and can be regarded as a model for many other biochemical and biological processes.

By incorporating two discrete time delay $\tau_1 \geq 0, \tau_2 \geq 0$ into (4.1), we obtain a delayed system

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1 - u(x,t-\tau_1) - rv(x,t-\tau_2)], \\ \frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t) - bu(x,t)v(x,t). \end{cases} \tag{4.2}$$

Wu and Zou [13] and Ma [5] consider the case of $\tau_1 = 0$. In such a case, $f = (f_1,\ f_2)$ satisfies the quasimontonicity (QM). But if $\tau_1 \neq 0$, then $f = (f_1,\ f_2)$ is not quasimontone, and the existence of traveling wavefront solution of (4.2) has not been considered elsewhere.

Substituting $u(x,t) = \phi_1(s),\ v(x,t) = \phi_2(s), s = x + ct$ into (4.2), and still denoting traveling wave coordinate s by t, the corresponding wave equation can be written as

$$\begin{cases} \phi_1''(t) - c\phi_1'(t) + \phi_1(t)[1 - \phi_1(t - c\tau_1) - r\phi_2(t - c\tau_2)] = 0 \\ \phi_2''(t) - c\phi_2'(t) - b\phi_1(t)\phi_2(t) = 0 \end{cases} \tag{4.3}$$

with asymptotic boundary condition

$$\begin{cases} \lim_{t\to-\infty}\phi_1(t) = 0, \quad \lim_{t\to\infty}\phi_1(t) = 1 \\ \lim_{t\to-\infty}\phi_2(t) = 1, \quad \lim_{t\to\infty}\phi_2(t) = 0. \end{cases} \tag{4.4}$$

Now, by making change of variable $\phi_2^* = 1 - \phi_2,\ s = 1 - r$, and still denoting it by ϕ_2 for the convenience of notations, (4.3) and (4.4) become respectively

$$\begin{cases} \phi_1''(t) - c\phi_1'(t) + \phi_1(t)[s - \phi_1(t - c\tau_1) + r\phi_2(t - c\tau_2)] = 0 \\ \phi_2''(t) - c\phi_2'(t) + b\phi_1(t)[1 - \phi_2(t)] = 0 \end{cases} \tag{4.5}$$

with asymptotic boundary condition

$$\begin{cases} \lim_{t\to-\infty}(\phi_1(t), \phi_2(t)) = (0,0) \\ \lim_{t\to\infty}(\phi_1(t), \phi_2(t)) = (1,1). \end{cases} \tag{4.6}$$

Denote $\phi = (\phi_1(t), \phi_2(t))$ and

$$f_1(\phi)(x) = \phi_1(x)(0)[s - \phi_1(x)(-\tau_1) + r\phi_2(x)(-\tau_2)]$$
$$f_2(\phi)(x) = b\phi_1(x)(0)[1 - \phi_2(x)(0)]$$

For convenience, we identify $\phi_i(x)(s) = \phi_i(s),\ i = 1, 2$, omitting x in the notation.

Lemma 4.1 *If τ_1 is sufficiently small, then $f_1(\phi)$ and $f_2(\phi)$ satisfy the non-quasimonotonicity condition (QM^*).*

Proof Let $\tau = max\{\tau_1, \tau_2\}$. For any $\phi = (\phi_1, \phi_2),\ \psi = (\psi_1, \psi_2) \in X_\tau = C([-\tau, 0]; R^2)$ with (i) $0 \leq \psi_i(s) \leq \phi_i(s) \leq K$ for $s \in [-\tau, 0]$; (ii) $e^{\beta_i s}[\phi_i(s) - \psi_i(s)]$ non-decreasing in $s \in [-\tau, 0], i = 1, 2.$ we have

$$\begin{aligned}
&f_1(\phi) - f_1(\psi)\\
&= s[\phi_1(0) - \psi_1(0)] - [\phi_1(0)\phi_1(-\tau_1) - \psi_1(0)\psi_(-\tau_1)]\\
&\quad + r[\phi_1(0)\phi_2(-\tau_2) - \psi_1(0)\psi_2(-\tau_2)]\\
&= s[\phi_1(0) - \psi_1(0)] - \phi_1(-\tau_1)[\phi_1(0) - \psi_1(0)] - \psi_1(0)[\phi_1(-\tau_1) - \psi_1(-\tau_1)]\\
&\quad + r\phi_2(-\tau_2)[\phi_1(0) - \psi_1(0)] + r\psi_1(0)[\phi_2(-\tau_2) - \psi_2(-\tau_2)]\\
&\geq [s - \phi_1(-\tau_1) - \psi_1(0)e^{\beta_1\tau_1} + r\psi_2(-\tau_2)][\phi_1(0) - \psi_1(0)]\\
&\geq -(r + e^{\beta_1\tau_1})[\phi_1(0) - \psi_1(0)].
\end{aligned}$$

Similarly, we have $f_2(\phi) - f_2(\psi) \geq -b[\phi_2(0) - \psi_2(0)]$ Thus

$$f_1(\phi) - f_1(\psi) + \beta_1[\phi_1(0) - \psi_1(0)] \geq (\beta_1 - r - e^{\beta_1\tau_1})[\phi_1(0) - \psi_1(0)]$$
$$f_2(\phi) - f_2(\psi) + \beta_2[\phi_2(0) - \psi_2(0)] \geq (\beta_2 - b)[\phi_2(0) - \psi_2(0)]$$

If we choose

$$\beta_1 > r + 1, \quad \beta_2 \geq b. \tag{4.7}$$

then, for sufficiently small τ_1, we have

$$\beta_1 - r - e^{\beta_1\tau_1} \geq 0, \quad \beta_2 \geq b. \tag{4.8}$$

Hence (QM^*) holds for $f = (f_1,\ f_2)$. This completes the proof.

Assume $0 < r < 1,\ b > 0$. For $c > 2$, we can define

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4}}{2}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4r}}{2}. \tag{4.9}$$

Fix $\varepsilon > 0$ such that

$$0 < \varepsilon < 1 - r. \qquad 0 < \varepsilon \leq \frac{\lambda_1 + \beta_1}{2\beta_1}, \qquad 0 < \varepsilon \leq \frac{\beta_1}{\beta_1 + \lambda_2}. \tag{4.10}$$

Then, we can choose sufficiently small $\alpha > 0$ such that

$$0 < \varepsilon < \frac{1}{1 + \alpha}, \tag{4.11}$$

$$0 < \varepsilon < \frac{\lambda_1 + \beta_1}{(2 + \alpha)\beta_1}, \tag{4.12}$$

$$0 < \varepsilon < \frac{\lambda_1 + \beta_1}{\alpha(\beta_1 + \lambda_2)}, \tag{4.13}$$

$$0 < \varepsilon < \frac{\beta_1}{(1 + 2\alpha)(\beta_1 + \lambda_2)}. \tag{4.14}$$

Define $\bar{\rho}(t) = (\bar{\rho}_1(t),\ \bar{\rho}_2(t))$ and $\underline{\rho}(t) = (\underline{\rho}_1(t), \underline{\rho}_2(t))$ by

$$\bar{\rho}_1(t) = \tfrac{1}{1+\alpha e^{-\lambda_1 t}}, \quad t \in R; \qquad \bar{\rho}_2(t) = 1. \quad t \in R,$$

$$\underline{\rho}_1(t) = \begin{cases} \varepsilon e^{\lambda_2 t} & t \le 0, \\ \varepsilon, & t > 0. \end{cases} ; \qquad \underline{\rho}_2(t) = 0, \quad t \in R.$$

Then, by (4.11), $0 \le \underline{\rho}(t) \le \bar{\rho}(t) \le 1$. In fact, if $t > 0$, then $\bar{\rho}_1(t) \ge \frac{1}{1+\alpha}$, $\underline{\rho}_1(t) = \varepsilon$. By (4.11), we have $\bar{\rho}_1(t) \ge \underline{\rho}_1(t)$. If $t \le 0$, simple calculation shows that

$$\frac{1}{1+\alpha e^{-\lambda_1 t}} - \varepsilon e^{\lambda_2 t} = \frac{1 - \varepsilon e^{\lambda_3 t} - \alpha \varepsilon e^{(\lambda_2 - \lambda_1)t}}{1+\alpha e^{-\lambda_1 t}} \ge \frac{1-\varepsilon(1+\alpha)}{1+\alpha e^{-\lambda_1 t}} > 0.$$

Lemma 4.2 *(i) If τ_1 is sufficiently small , then $\bar{\rho}(t)$ is an upper solution of (4.5); (ii) $\underline{\rho}(t)$ is a lower solution of (4.5).*

Proof For $\bar{\rho}_1(t)$, it is easy to know that

$$\begin{aligned} &\bar{\rho}_1''(t) - c\bar{\rho}_1'(t) + \bar{\rho}_1(t)[s - \bar{\rho}_1(t - c\tau_1) + r\bar{\rho}_2(t - c\tau_2)] \\ &\le \bar{\rho}_1''(t) - c\bar{\rho}_1'(t) + \bar{\rho}_1(t)[s - \bar{\rho}_1(t - c\tau_1) + r] \\ &= \bar{\rho}_1''(t) - c\bar{\rho}_1'(t) + \bar{\rho}_1(t)[1 - \bar{\rho}_1(t - c\tau_1)]. \end{aligned}$$

We know from Proposition 5.1.2 in Wu and Zou [13] that $\bar{\rho}_1(t)$ satisfies

$$\bar{\rho}_1''(t) - c\bar{\rho}_1'(t) + \bar{\rho}_1(t)[1 - \bar{\rho}_1(t - c\tau_1)] \le 0,$$

provided that τ_1 is sufficiently small.

For $\bar{\rho}_2(t) = 1$, it is easy to verify that

$$\bar{\rho}_2''(t) - c\bar{\rho}_2'(t) + b\bar{\rho}_1(t)[1 - \bar{\rho}_2(t)] = 0$$

Thus, $\bar{\rho}(t) = (\bar{\rho}_1(t), \bar{\rho}_2(t)) = (\bar{\rho}_1(t), 1)$ is an upper solution of (4.5).

For $\underline{\rho}(t) = (\underline{\rho}_1, \underline{\rho}_2) = (\underline{\rho}_1, 0)$, it is easy to know that

$$\begin{aligned} &\underline{\rho}_1''(t) - c\underline{\rho}_1'(t) + \underline{\rho}_1(t)[s - \underline{\rho}_1(t - c\tau_1) + r\underline{\rho}_2(t - c\tau_2)] \\ &= \underline{\rho}_1''(t) - c\underline{\rho}_1'(t) + \underline{\rho}_1(t)[s - \underline{\rho}_1(t - c\tau_1)] \\ &\ge \underline{\rho}_1''(t) - c\underline{\rho}_1'(t) + \underline{\rho}_1(t)[s - \varepsilon]. \end{aligned}$$

If $t > 0$, then $\underline{\rho}_1(t) = \varepsilon$. Since $\varepsilon < 1 - r = s$, it follows that

$$\underline{\rho}_1''(t) - c\underline{\rho}_1'(t) + \underline{\rho}_1(t)[s - \varepsilon] = \varepsilon[s - \varepsilon] > 0.$$

If $t \le 0$, then $\underline{\rho}_1(t) = \varepsilon e^{\lambda_2 t}$. Direct calculation shows that

$$\begin{aligned} \underline{\rho}_1''(t) - c\underline{\rho}_1'(t) + \underline{\rho}_1(t)[s - \varepsilon] \qquad &= \varepsilon e^{\lambda_2 t}[\lambda_2^2 - c\lambda_2 + (s - \varepsilon)] \\ &\ge \varepsilon e^{\lambda_2 t}[\lambda_2^2 - c\lambda_2 + s - 1] = 0. \end{aligned}$$

For the second component, we have

$$\underline{\rho}_2''(t) - c\underline{\rho}_2'(t) + b\underline{\rho}_1(t)[1 - \underline{\rho}_2(t)] = b\underline{\rho}_1(t) \ge 0.$$

Therefore, $\underline{\rho}(t) = (\underline{\rho}_1(t), \underline{\rho}_2(t))$ is a lower solution of (4.5). This completes the proof.

Lemma 4.3 *Let $\beta_1 > \lambda_1$, and $\alpha > 0$ and $0 < \varepsilon < 1$ be such that (4.10)-(4.14) hold. Then $\Gamma(\underline{\rho}, \bar{\rho})$ is non-empty.*

Proof Let $\phi(t) = (\phi_1(t), \phi_2(t)) = (\bar{\rho}_1(t), \frac{1}{2})$. We claim that $\phi(t) \in \Gamma(\underline{\rho}, \bar{\rho})$. In fact, $\bar{\rho}_1(t)$ is nondecreasing since

$$\bar{\rho}_1'(t) = \frac{\alpha\lambda_2 e^{-\lambda_2 t}}{[1+\alpha e^{-\lambda_2 t}]^2} > 0.$$

Obviously, $\phi_2(t) = \frac{1}{2}$ is also nondecreasing in R, and $\underline{\rho}(t) \le \phi(t) \le \bar{\rho}(t)$.

We need to prove $e^{\beta t}[\phi(t) - \underline{\rho}(t)]$ and $e^{\beta t}[\bar{\rho}(t) - \phi(t)]$ are nondecreasing in $t \in R$. For $e^{\beta_t}[\phi(t) - \underline{\rho}(t)]$: If $t > 0$, then $\underline{\rho}_1(t) = \varepsilon$. Direct calculation shows that

$$\begin{aligned}
&\frac{d}{dt}\big\{e^{\beta_1 t}[\phi_1(t) - \underline{\rho}_1(t)]\big\} = \frac{d}{dt}\big\{e^{\beta_1 t}[\frac{1}{1+\alpha e^{-\lambda_1 t}} - \varepsilon]\big\} \\
&= \frac{e^{\beta_1 t}}{[1+\alpha e^{-\lambda_1 t}]^2}\big[\beta_1(1-\varepsilon) + (\alpha\beta_1 + \alpha\lambda_1 - 2\alpha\beta_1\varepsilon)e^{-\lambda_1 t} - \alpha^2\varepsilon\beta_1 e^{-2\lambda_1 t}\big] \\
&\ge \frac{e^{\beta_1 t}}{[1+\alpha e^{-\lambda_1 t}]^2}\big[\beta_1(1-\varepsilon) + (\alpha\beta_1 + \alpha\lambda_1 - 2\alpha\beta_1\varepsilon - \alpha^2\varepsilon\beta_1)e^{-\lambda_1 t}\big].
\end{aligned}$$

By (4.12), we have $\beta_1(1-\varepsilon) > 0, \quad \alpha\beta_1 + \alpha\lambda_1 - 2\alpha\beta_1\varepsilon - \alpha^2\varepsilon\beta_1 \ge 0$, which imply

$$\frac{d}{dt}\big\{e^{\beta_1 t}[\phi_1(t) - \underline{\rho}_1(t)]\big\} > 0.$$

If $t \le 0$, then $\underline{\rho}_1(t) = \varepsilon e^{\lambda_2 t}$. Since $\lambda_1 < \lambda_2$, $e^{\lambda_1 t} \ge e^{\lambda_2 t}$. After some calculation, we obtain,

$$\begin{aligned}
&\frac{d}{dt}\{e^{\beta_1 t}[\phi_1(t) - \underline{\rho}_1(t)]\} = \frac{d}{dt}\big\{\frac{e^{\beta_1 t}}{1+\alpha e^{-\lambda_1 t}} - \varepsilon e^{(\beta_1+\lambda_2)t}\big\} \\
&= \frac{e^{\beta_1 t}}{[1+\alpha e^{-\lambda_1 t}]^2}\Big[\beta_1 + \alpha\beta_1 e^{-\lambda_1 t} + \alpha\lambda_1 e^{-\lambda_1 t} - \varepsilon(\beta_1+\lambda_2)e^{\lambda_2 t} \\
&\quad -2\alpha\varepsilon(\beta_1+\lambda_2)e^{(\lambda_2-\lambda_1)t} - \alpha^2\varepsilon(\beta_1+\lambda_2)e^{(\lambda_2-2\lambda_1)t}\Big] \\
&\ge \frac{e^{\beta_1 t}}{[1+\alpha e^{-\lambda_1 t}]^2}\Big[\alpha\beta_1 + \alpha\lambda_1 - \alpha^2\varepsilon(\beta_1+\lambda_2)]e^{-\lambda_1 t} + \beta_1 \\
&\quad -2\alpha\varepsilon(\beta_1+\lambda_2) - \varepsilon(\beta_1+\lambda_2)\Big].
\end{aligned}$$

By (4.13) and (4.14), it follows that

$$\begin{cases} \alpha\beta_1 + \alpha\lambda_1 - \alpha^2\varepsilon(\beta_1+\lambda_2) > 0, \\ \beta_1 - 2\alpha\varepsilon(\beta_1+\lambda_2) - \varepsilon(\beta_1+\lambda_2) > 0. \end{cases}$$

which imply

$$\frac{d}{dt}\big\{\frac{e^{\beta_1 t}}{1+\alpha e^{-\lambda_1 t}} - \varepsilon e^{(\beta_1+\lambda_2)t}\big\} \ge 0.$$

Therefore, $e^{\beta_1 t}[\phi_1(t) - \underline{\rho}_1(t)]$ is nondecreasing in $t \in R$. Clearly, $e^{\beta_2 t}[\phi_2(t) - \underline{\rho}_2(t)] = \frac{1}{2}e^{\beta_2 t}$ is also nondecreasing. For $e^{\beta t}[\bar{\rho}(t) - \phi(t)]$, we can compute as follows. First, $e^{\beta_1 t}[\bar{\rho}_1(t) - \phi_1(t)] = e^{\beta_1 t} \cdot 0 = 0$ is nondecreasing. Secondly, $e^{\beta_2 t}[\bar{\rho}_2(t) - \phi_2(t)] = e^{\beta_2 t}[1 - \frac{1}{2}] = \frac{1}{2}e^{\beta_2 t}$ is also nondecreasing.

Finally, $e^{\beta t}[\phi(t+s) - \phi(t)]$ is nondecreasing in $t \in R$ for every $s > 0$. In fact,

$$\begin{aligned}
&\frac{d}{dt}\{e^{\beta_1 t}[\phi_1(t+s) - \phi_1(t)]\} = \frac{d}{dt}\Big\{\frac{\alpha e^{(\beta_1-\lambda_1)t}[1-e^{-\lambda_1 s}]}{(1+\alpha e^{-\lambda_1(s+t)})(1+\alpha e^{-\lambda_2 t})}\Big\} \\
&= \frac{\alpha[1-e^{-\lambda_1 s}]e^{(\beta_1-\lambda_1)t}\big\{(\beta_1-\lambda_1) + \beta_1\alpha e^{-\lambda_1 t} + \beta_1\alpha e^{-\lambda_1(s+t)}}{[(1+\alpha e^{-\lambda_1(s+t)})(1+\alpha e^{-\lambda_1 t})]^2} \\
&\quad + \frac{(\beta_1+\lambda_1)\alpha^2 e^{-\lambda_1(s+2t))}\big\}}{[(1+\alpha e^{-\lambda_1(s+t)})(1+\alpha e^{-\lambda_1 t})]^2} \ge 0.
\end{aligned}$$

which implies that $e^{\beta_1 t}[\phi_1(t+s) - \phi_1(t)]$ is nondecreasing in $t \in R$ for every $s > 0$. $e^{\beta_2 t}[\phi_2(t+s) - \phi_2(t)] = e^{\beta_2 t}[\frac{1}{2} - \frac{1}{2}] = 0$ is also nondecreasing. Hence,

$\phi(t) = (\phi_1(t), \phi_2(t)) = (\phi_1(t),\ \frac{1}{2}) \in \Gamma(\underline{\rho}, \bar{\rho})$, and this completes the proof.

Since $\inf_{t\in R}\phi_1(t) = 0,\ \inf_{t\in R}\phi_2(t) = 1,\quad \sup_{t\in R}\psi_1(t) = \varepsilon$ and $\sup_{t\in R}\psi_2(t) = 0$, we see that $f(\tilde{u}) \neq 0$ for $u \in (0, \inf_{t\in R}\Phi(t)] \bigcup [\sup_{t\in R}\Psi(t), K) = [\varepsilon, 1) \times [0, 1)$. Applying Lemma 4.1-4.3 and Theorem 3.1 , we obtain

Theorem 4.1 *Assume that $b > 0,\ 0 < r < 1$. Then for every $c > 2$, (4.5) has a traveling wave front solution with wave speed c which connecting $(0,0)$ and $(1,1)$, provided that τ_1 is sufficiently small.*

Remark 4.1: From the proof of Theorem 4.1, we can see that by setting the second component of the upper and lower solutions to be constants, we can take advantage of upper-lower solutions of the *scalar* equations. This shows the novelty of our main theorem .

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E-mail address: `jhhuang@ccnu.edu.cn`
E-mail address: `xzou@math.mun.ca`