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A 3/2 STABILITY RESULT FOR A REGULATED LOGISTIC GROWTH MODEL

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Abstract. A sufficient condition is established for globally asymptotic stability of the positive equilibrium of a regulated logistic growth model with a delay in the state feedback. The result improves some existing criteria for this model. It is in a form that is related to the number 3/2 and the coupling strength, and thus, is comparable to the well-known 3/2 condition for the uncontrolled delayed logistic equation. The comparison seems to suggest that the mechanism of the control in this model might be inappropriate and new mechanism should be introduced.

1. **Introduction.** In a biologically meaningful model a globally stable positive equilibrium plays a crucial rule. It is well known [2,4,7,8] that the delay logistic equation

$$N'(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right] \tag{1.1}$$

has a positive equilibrium K which is globally asymptotically stable if $r\tau \leq 3/2$. In some situations, one may need to adjust the size of the positive equilibrium (see, e.g., [1]). For this purpose for (1.1), Gopalsamy et al [2,3] first put forward a mechanism of " feedback regulation" to (1.1) by considering the following control system

$$\begin{cases} n'(t) = rn(t)\left[1 - \frac{n(t-\tau)}{K} - cu(t)\right] \\ \\ u'(t) = -au(t) + bn(t-\tau) \end{cases} \tag{1.2}$$

where u functions as a "feedback" control variable in system (1.2), $K, r, a, b, \tau, c \in (0, \infty)$ are constants. Due to biological reasons, (1.2) is assigned initial conditions of the form

$$\begin{cases} n(s) = \phi(s) \geq 0, \quad u(0) = u_0 > 0 \\ \\ \phi(0) > 0, \quad \phi \in C([-\tau, 0], [0, \infty)). \end{cases} \tag{1.3}$$

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One can easily prove, by using the method of steps, that the solution of (1.2)-(1.3) are defined for all $t \geq 0$ and remain positive for $t \geq 0$.

It is easy to verify that (1.2) has a unique nontrivial steady state (n^*, u^*), where

$$n^* = \frac{aK}{a + Kbc}, \quad u^* = \frac{bK}{a + Kbc}. \tag{1.4}$$

The population component n^* of the equilibrium given by (1.4) can be adjusted by varying the constants a, b and c. But one needs to make sure that (n^*, u^*) is globally asymptotically stable. Gopalsamy and Weng [3] proved that if

$$e^{r\tau}\left[r\tau\left(1 + \frac{bcK}{a}\right) + \frac{bcK}{a}\right] < \frac{1}{2}, \tag{1.5}$$

then every solution $(n(t), u(t))$ of (1.2)-(1.3) satisfies

$$\lim_{t\to\infty}(n(t), u(t)) = (n^*, u^*). \tag{1.6}$$

Furthermore, they also *conjectured* that the conclusion (1.6) remains true if condition (1.5) is replaced by

$$e^{r\tau}\left[r\tau\left(1 + \frac{bcK}{a}\right) + \frac{bcK}{a}\right] < 1, \tag{1.7}$$

or by

$$r\tau\left(1 + \frac{bcK}{a}\right) + \frac{bcK}{a} < 1. \tag{1.8}$$

Using a completely different method, Kuang [5] confirmed the first conjecture by giving an even better (than (1.7)) condition

$$r\tau e^{r\tau}\left(1 + \frac{bcK}{a}\right) < 1 - \frac{bcK}{a}. \tag{1.9}$$

Concerning the second conjecture, Lalli et al [6], obtained the following set of conditions:

$$\left(r\tau + \frac{bcK}{a}\right)\left[r\tau\left(1 + \frac{bcK}{a}\right) + \frac{bcK}{a}\right] \leq 1. \tag{1.10}$$

$$\frac{bcK}{a}e^{r\tau} < 1. \tag{1.11}$$

We remark that (1.10) implies (1.11). In fact, if (1.10) holds, then

$$\left(r\tau + \frac{bcK}{a}\right)^2 < \left(r\tau + \frac{bcK}{a}\right)\left[r\tau\left(1 + \frac{bcK}{a}\right) + \frac{bcK}{a}\right] \leq 1,$$

and so

$$\frac{bcK}{a}e^{r\tau} < \frac{bcK}{a}e^{1-bcK/a} < 1.$$

Therefore, (1.11) can be derived from (1.10), and thus it is unnecessary. Obviously, condition (1.10) is better than (1.8). Thus the second conjecture is also confirmed under an even weaker condition (1.10).

As pointed out in the discussion section in Kuang [5], both (1.9) and (1.10) have much room for various improvements, and the 3/2 stability result for (1.1) should provide at least one motivation for further improvement. This paper makes an attempt toward this direction. More precisely, in this paper we will establish a 3/2 type criterion which further improve (1.9) and (1.10) to

$$r\tau \leq \frac{3}{2}\left(1 - \frac{bcK}{a}\right). \tag{1.12}$$

In other words, we will prove the following main theorem.

Theorem 1.1. *If (1.12) holds, then every solution of (1.2)-(1.3) satisfies (1.6).*

It is worth noting that when $c = 0$, (1.12) reduces to the well-known criterion $r\tau \le \frac{3}{2}$ for globally asymptotic stability of the positive equilibrium of system (1.1). But as mentioned in Kuang [5], none of the existing criteria for (1.2) can be reduced to this condition. In other words, our result is comparable to the well-known 3/2 result for the uncontrolled delay logistic equation (1.1). A comparison seems to suggest that the mechanism of the control in this model might be inappropriate and new control mechanism should be introduced. For details of such a discussion, see Section 4.

2. **Preliminaries.** For convenience, we define $x(t), y(t), \mu, A, B$ as follows:

$$x(t) = \frac{n(t)}{n^*} - 1, \quad y(t) = \frac{u(t)}{u^*} - 1, \tag{2.1}$$

$$\mu = \frac{bcK}{a}, \ \ A = \frac{r}{1+\mu}, \ \ B = a. \tag{2.2}$$

Then (1.2) and (1.3) are transformed, respectively, into

$$\begin{cases} x'(t) = -A[1 + x(t)][x(t-\tau) + \mu y(t)] \\ \\ y'(t) = B\,[-y(t) + x(t-\tau)] \end{cases} \tag{2.3}$$

and

$$\begin{cases} x(s) = \phi(s) \ge -1, \quad y(0) = y_0 > -1, \\ \\ \phi(0) > -1, \quad \phi \in C([-\tau, 0], [-1, \infty)). \end{cases} \tag{2.4}$$

Lemma 2.1. Let $0 < \mu < 1$. The system of inequalities

$$\begin{cases} y \le (1+\mu x)\exp\left[(1-\mu)x - \frac{(1-\mu)^2}{6(1+\mu)}x^2\right] - 1 \\ \\ x \le 1 - (1-\mu y)\exp\left[-(1-\mu)y - \frac{(1-\mu)^2}{6(1+\mu)}y^2\right] \end{cases} \tag{2.5}$$

has a unique solution: $(x, y) = (0, 0)$ in the region $D = \{(x, y) : 0 \le x < 1,\ 0 \le y < 1/\mu\}$.

Proof. Let

$$\varphi(x) = (1-\mu)x - \frac{(1-\mu)^2}{6(1+\mu)}x^2, \quad \psi(y) = (1-\mu)y + \frac{(1-\mu)^2}{6(1+\mu)}y^2.$$

Then (2.5) can be written as

$$\begin{cases} y \le (1+\mu x)e^{\varphi(x)} - 1, \\ \\ x \le 1 - (1-\mu y)e^{-\psi(y)}. \end{cases} \tag{2.6}$$

Assume that (2.6) has another solution in the region D other than $(0, 0)$, say (x_0, y_0). Then $0 < x_0 < 1$ and $0 < y_0 < 1/\mu$. Define two curves Γ_1 and Γ_2 as follows:

$$\Gamma_1 : \ \ y = (1+\mu x)e^{\varphi(x)} - 1, \quad \Gamma_2 : \ \ x = 1 - (1-\mu y)e^{-\psi(y)}. \tag{2.7}$$

By direct calculation, we have for curve Γ_1:

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{(0,0)} &= 1,\\
\left.\frac{d^2y}{dx^2}\right|_{(0,0)} &= \frac{(1-\mu)(\mu+2)(3\mu+1)}{3(1+\mu)},\\
\left.\frac{d^3y}{dx^3}\right|_{(0,0)} &= \frac{\mu(1-\mu)^2(2\mu+3)}{1+\mu},
\end{aligned}$$

and for curve Γ_2:

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 1, \quad \left.\frac{d^2y}{dx^2}\right|_{(0,0)} = \frac{(1-\mu)(\mu+2)(3\mu+1)}{3(1+\mu)},$$

$$\left.\frac{d^3y}{dx^3}\right|_{(0,0)} = -\frac{\mu(1-\mu)^2(2\mu+3)}{1+\mu} + \frac{(1-\mu)^2(\mu+2)^2(3\mu+1)^2}{3(1+\mu)^2}.$$

Hence Γ_2 lies above Γ_1 near (0,0). The existence of (x_0, y_0) implies that the curves Γ_1 and Γ_2 must intersect at a point in the region D besides $(0,0)$. Let (x_1, y_1) be the first such point, i.e. x_1 is smallest. Then the slope of Γ_1 at (x_1, y_1) is no less than the slope of Γ_2 at (x_1, y_1), i.e.

$$[\mu + (1+\mu x_1)\varphi'(x_1)]e^{\varphi(x_1)} \geq \frac{1}{\mu + (1-\mu y_1)\psi'(y_1)} e^{\psi(y_1)}$$

or

$$[\mu + (1+\mu x_1)\varphi'(x_1)][\mu + (1-\mu y_1)\psi'(y_1)] \geq e^{\psi(y_1)-\varphi(x_1)}. \tag{2.8}$$

From (2.7), we have

$$\begin{aligned}
-\ln(1-x_1) &= -\ln(1-\mu y_1) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)}y_1^2\\
&= \left(\mu y_1 + \frac{\mu^2}{2}y_1^2 + \frac{\mu^3}{3}y_1^3 + \cdots\right) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)}y_1^2\\
&< y_1 + \frac{1}{2}y_1^2 + \frac{1}{3}y_1^3 + \cdots\\
&= -\ln(1-y_1).
\end{aligned}$$

This implies that

$$x_1 < y_1. \tag{2.9}$$

Using (2.9), we derive that

$$\begin{aligned}
& [\mu+(1+\mu x_1)\varphi'(x_1)][\mu+(1-\mu y_1)\psi'(y_1)] \\
= \;& 1+\left[\frac{(1-\mu)^2}{3(1+\mu)}-\mu(1-\mu)\right](y_1-x_1)-\left[\frac{(1-\mu)^2}{3(1+\mu)}-\mu(1-\mu)\right]^2 x_1y_1 \\
& -\frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2+y_1^2)+\frac{\mu(1-\mu)^3}{3(1+\mu)}\left[\frac{1-\mu}{3(1+\mu)}-\mu\right]x_1y_1(y_1-x_1) \\
& +\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2y_1^2 \\
< \;& 1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)-\frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2+y_1^2) \\
& +\frac{\mu(1-\mu)^4}{9(1+\mu)^2}x_1y_1(y_1-x_1)+\frac{\mu^2(1-\mu)^4}{9(1+\mu)^2}x_1^2y_1^2 \\
< \;& 1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)
\end{aligned}$$

and

$$\begin{aligned}
e^{\psi(y_1)-\varphi(x_1)} &= \exp\left[(1-\mu)(y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2)\right] \\
&> 1+(1-\mu)(y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
& e^{\psi(y_1)-\varphi(x_1)}-[\mu+(1+\mu x_1)\varphi'(x_1)][\mu+(1-\mu y_1)\psi'(y_1)] \\
> \;& \left[1+(1-\mu)(y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2)\right] \\
& -\left[1+(1-\mu)\left(\frac{1-\mu}{3(1+\mu)}-\mu\right)(y_1-x_1)\right] \\
= \;& (1-\mu)\left[1+\mu-\frac{1-\mu}{3(1+\mu)}\right](y_1-x_1)+\frac{(1-\mu)^2}{6(1+\mu)}(x_1^2+y_1^2) \\
> \;& 0,
\end{aligned}$$

which contradicts (2.8). The proof is complete.

Lemma 2.2. *Let $(x(t), y(t))$ be the solution of (2.3) and (2.4). If*

$$A\tau \le \frac{3(1-\mu)}{2(1+\mu)}, \tag{2.10}$$

then

$$-1 < -1+(1-\mu M)e^{-3(1-\mu)M/2} \le \liminf_{t\to\infty} x(t) \le 0 \le \limsup_{t\to\infty} x(t) \le M. \tag{2.11}$$

where $M = (1+\mu)e^{1-\mu}-1$.

Proof. By the method of steps, it is easy to prove that $(x(t), y(t))$ is defined for all $t \ge 0$ and satisfies

$$x(t) > -1, \quad y(t) > -1, \quad t \ge 0. \tag{2.12}$$

Substituting (2.12) into the first equation in (2.3) we get

$$x'(t) \le A[1+x(t)][-x(t-\tau)+\mu] \le A(1+\mu)[1+x(t)], \quad t \ge t_1. \tag{2.13}$$

There are two possibilities: $x(t)$ is oscillatory about μ or $x(t)$ is nonoscillatory about μ.

If $x(t)$ is not oscillatory about μ, then there exists a $t_2 > t_1$ such that

$$\text{either } \; x(t) \ge \mu \; \text{ for } \; t \ge t_2 \; \text{ or } \; x(t) \le \mu \; \text{ for } \; t \ge t_2. \tag{2.14}$$

Similar to the proof of Lemma 3.1 in [3], using (2.13) and (2.14) we can derive that

$$x(t) \le (1+\mu)e^{1-\mu} - 1 = M \quad \text{for large } \; t. \tag{2.15}$$

Indeed, if the second alternative in (2.14) holds, then (2.15) is obviously true (noting that (1.12) implies $\mu < 1$ and hence $\mu < M$). Suppose $x(t) \ge \mu$ for $t \ge t_2$. Then, by (2.13), $x'(t) \le 0$ for $t \ge t_2 + \tau$. It follows that the limit $\lim_{t\to\infty} x(t) = x_0$ exists and $x_0 \ge \mu$. From (2.12)-(2.13), it is easy to show that $x_0 = \mu < M$ and so (2.15) also holds.

If $x(t)$ is oscillatory about μ, then we can choose an arbitrary local maximum point t^* of $x(t)$ such that $x'(t^*) = 0$ and $x(t^*) > \mu$. By (2.13), $x(t^* - \tau) \le \mu$. Thus, there exists $\xi \in [t^* - \tau, t^*)$ such that $x(\xi) = \mu$. For $t \in [\xi, t^*]$, integrating (2.13) from $t - \tau$ to ξ we get

$$-\ln \frac{1 + x(t-\tau)}{1 + x(\xi)} \le A(1+\mu)(\xi + \tau - t),$$

or

$$x(t-\tau) \ge -1 + (1+\mu)\exp[-A(1+\mu)(\xi+\tau-t)], \quad \xi \le t \le t^*.$$

Substituting this into the first inequality in (2.13), we obtain

$$x'(t) \le A[1+x(t)](1+\mu)\{1 - \exp[-A(1+\mu)(\xi+\tau-t)]\}, \quad \xi \le t \le t^*. \tag{2.16}$$

Integrating (2.16) and using (2.10), we have

$$\begin{aligned}
&\ln \frac{1+x(t^*)}{1+\mu} \\
\le\ & A(1+\mu)(t^*-\xi) - A(1+\mu)\int_{\xi}^{t^*} \exp[-A(1+\mu)(\xi+\tau-t)]dt \\
=\ & (1+\mu)\left\{A(t^*-\xi) - \frac{1-\exp(-A(1+\mu)(t^*-\xi))}{1+\mu}\exp[-A(1+\mu)(\xi+\tau-t^*)]\right\} \\
\le\ & (1+\mu)\left\{A(t^*-\xi) - \frac{1-\exp(-A(1+\mu)(t^*-\xi))}{1+\mu}\right. \\
& \qquad \left.\times \exp\left[-(1+\mu)\left(\frac{3(1-\mu)}{2(1+\mu)} - A(t^*-\xi)\right)\right]\right\} \\
\le\ & \frac{3(1-\mu)}{2} - 1 + e^{-3(1-\mu)/2} \\
<\ & (1-\mu).
\end{aligned}$$

Consequently, we obtain

$$x(t^*) \le (1+\mu)e^{1-\mu} - 1.$$

It follows from arbitrariness of t^* that (2.15) holds and that $u \equiv \limsup_{t\to\infty} x(t) \le M$. Since $\mu e^{1-\mu} < 1$, it follows that $\mu M < 1$. Therefore, there exists a $\epsilon > 0$ such that $\mu(M+\epsilon)<1$. Let $t_3>t_2$ such that (2.15) holds for $t\geq t_3$. Then, from (2.3) and (2.15), we have again

$$\begin{aligned} y(t) &= e^{-Bt}\left[y(t_3)+B\int_{t_3}^{t}x(s-\tau)e^{Bs}ds\right] \\ &\leq e^{-Bt}\left[y(t_3)e^{Bt_3}+M\left(e^{Bt}-e^{Bt_3}\right)\right] \\ &\to M \text{ as } t\to\infty. \end{aligned}$$

It follows that there exists a $t_4>t_3$ such that

$$y(t)<M+\epsilon, \quad t\geq t_4. \tag{2.17}$$

Substituting (2.15) and (2.17) into the first equation in (2.3) we get

$$\begin{aligned} x'(t) &\geq -A[1+x(t)][x(t-\tau)+\mu(M+\epsilon)] \\ &\geq -A(1+\mu)(M+\epsilon)[1+x(t)], \quad t\geq t_4. \end{aligned} \tag{2.18}$$

Using (2.18), one can , in a similar way of Lemma 4.1 in [4], derive that

$$x(t)\geq(1-\mu(M+\epsilon))e^{-3(1-\mu)(M+\epsilon)/2}-1>-1, \quad \text{for large } t. \tag{2.19}$$

This implies that

$$v\equiv\liminf_{t\to\infty}x(t)\geq(1-\mu M)e^{-3(1-\mu)M/2}-1>-1.$$

Next, we prove that $v\leq 0\leq u$. If $v>0$, then, similar to (2.18), we can derive that

$$x'(t)\leq -A[1+x(t)]x(t-\tau), \quad \text{for large } t.$$

This implies that $v=\lim_{t\to\infty}x(t)=0$, leading to a contradiction. Therefore, $v\leq 0$. In a similar way, one can derive that $u\geq 0$. The proof is complete.

3. **Main Results.** In this section, we prove our main result (Theorem 1.1)), which is a direct consequence of the following theorem.

Theorem 3.1. Assume that (2.10) holds. Then every solution (x(t),y(t)) of (2.3) and (2.4) satisfies

$$\lim_{t\to\infty}(x(t),y(t))=(0,0). \tag{3.1}$$

Proof. Set

$$\liminf_{t\to\infty}x(t)=-v \quad \text{and} \quad \limsup_{t\to\infty}x(t)=u. \tag{3.2}$$

In view of Lemma 2.2,

$$0\leq v<1, \quad 0\leq u<\infty. \tag{3.3}$$

In what follows, we show that v and u satisfy the inequalities

$$1+u\leq(1+\mu v)\exp\left[(1-\mu)v-\frac{(1-\mu)^2}{6(1+\mu)}v^2\right] \tag{3.4}$$

and

$$1-v\geq(1-\mu u)\exp\left[-(1-\mu)u-\frac{(1-\mu)^2}{6(1+\mu)}u^2\right]. \tag{3.5}$$

First, we prove that (3.4) holds. If $u\leq\mu v$, then (3.4) obviously holds. Therefore, we will prove (3.4) only in the case when $u>\mu v$. For any $\epsilon\in(0,\min\{(1-v),(u-\mu v)\}/4)$, it follows from (3.2) and (3.3) that there exist a $t_1>t_0$ such that

$$-(v_1-\epsilon)\equiv -v-\epsilon<x(t-\tau)<u+\epsilon\equiv u_1-\epsilon, \quad t\geq t_1. \tag{3.6}$$

Then, from (2.3), we have

$$\begin{aligned} y(t) &= e^{-Bt}\left[y(t_1)e^{Bt_1} + B\int_{t_1}^{t} x(s-\tau)e^{Bs}ds\right] \\ &\le e^{-Bt}\left[y(t_1)e^{Bt_1} + (u_1-\epsilon)(e^{Bt}-e^{Bt_1})\right] \\ &\to u_1-\epsilon \ \text{ as } \ t\to\infty \end{aligned}$$

and

$$\begin{aligned} y(t) &\ge e^{-Bt}\left[y(t_1)e^{Bt_1} - (v_1-\epsilon)(e^{Bt}-e^{Bt_1})\right] \\ &\to -(v_1-\epsilon) \ \text{ as } \ t\to\infty. \end{aligned}$$

Hence, we can choose a $t_2 > t_1$ such that

$$-v_1 < y(t) < u_1, \quad t \ge t_2. \tag{3.7}$$

Substituting (3.6) and (3.7) into the first equation in (2.3) we obtain

$$x'(t) \le A[1+x(t)][-x(t-\tau)+\mu v_1] \le Av_2[1+x(t)], \quad t\ge t_2 \tag{3.8}$$

and

$$-x'(t) \le A[1+x(t)][x(t-\tau)+\mu u_1] \le Au_2[1+x(t)], \quad t\ge t_2. \tag{3.9}$$

where $v_2 = (1+\mu)v_1$ and $u_2 = (1+\mu)u_1$. Since $u > \mu v_1$, we cannot have $x(t) \le \mu v_1$ eventually. On the other hand, if $x(t) \ge \mu v_1$ eventually, then it follows from the first inequality in (3.8) that $x(t)$ is nonincreasing and $u = \lim_{t\to\infty} x(t) = \mu v_1$. This is also impossible. Therefore, we assume that $x(t)$ oscillates about μv_1.

Let $\{p_n\}$ be an increasing sequence such that $p_n \ge t_2+\tau, x'(p_n) = 0, x(p_n) \ge \mu v_1, \lim_{n\to\infty} p_n = \infty$ and $\lim_{n\to\infty} x(p_n) = u$. By (3.8), $x(p_n-\tau) \le \mu v_1$. Thus, there exists $\xi_n \in [p_n-\tau, p_n]$ such that $x(\xi_n) = \mu v_1$. For $t \in [\xi_n, p_n]$, integrating (3.8) from $t-\tau$ to ξ_n we get

$$-\ln\frac{1+x(t-\tau)}{1+x(\xi_n)} \le Av_2(\xi_n+\tau-t),$$

or

$$x(t-\tau) \ge -1 + (1+\mu v_1)\exp[-Av_2(\xi_n+\tau-t)], \quad \xi_n \le t \le p_n.$$

Substituting this into the first inequality in (3.8), we obtain

$$x'(t) \le A[1+x(t)](1+\mu v_1)\{1-\exp[-Av_2(\xi_n+\tau-t)]\}, \quad \xi_n\le t\le p_n.$$

Combining this with (3.8), we have

$$\frac{x'(t)}{1+x(t)} \le \min\{Av_2, A(1+\mu v_1)\{1-\exp[-Av_2(\xi_n+\tau-t)]\}\}, \quad \xi_n \le t\le p_n. \tag{3.10}$$

To prove (3.4), we consider the following two possible cases.

Case 1. $A(p_n-\xi_n)\le -\frac{1}{v_2}\ln[1-(1-\mu)v_1]$. Then by (2.10) and (3.10)

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
\le\ & A(1+\mu v_1)(p_n-\xi_n)-A(1+\mu v_1)\int_{\xi_n}^{p_n}\exp[-Av_2(\xi_n+\tau-t)]dt\\
=\ & (1+\mu v_1)\{A(p_n-\xi_n)\\
&\quad -\frac{1}{v_2}\exp[-Av_2(\xi_n+\tau-p_n)][1-\exp(-Av_2(p_n-\xi_n))]\Big\}\\
\le\ & (1+\mu v_1)\left\{A(p_n-\xi_n)-\frac{1-\exp(-Av_2(p_n-\xi_n))}{v_2}\right.\\
&\qquad\left.\times\exp\left[-v_2\left(\frac{3(1-\mu)}{2(1+\mu)}-A(p_n-\xi_n)\right)\right]\right\}.
\end{aligned}$$

If $A(p_n-\xi_n)\le -\frac{1}{v_2}\ln[1-(1-\mu)v_1]\le 3(1-\mu)/2(1+\mu)$, then

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
\le\ & (1+\mu v_1)\left\{-\frac{1}{v_2}\ln[1-(1-\mu)v_1]\right.\\
&\qquad\left.-\frac{1-\mu}{1+\mu}\exp\left[-v_2\left(\frac{3(1-\mu)}{2(1+\mu)}+\frac{\ln[1-(1-\mu)v_1]}{v_2}\right)\right]\right\}\\
\le\ & (1+\mu v_1)\left\{-\frac{1}{v_2}\ln[1-(1-\mu)v_1]\right.\\
&\qquad\left.-\frac{1-\mu}{1+\mu}\left[1-v_2\left(\frac{3(1-\mu)}{2(1+\mu)}+\frac{\ln[1-(1-\mu)v_1]}{v_2}\right)\right]\right\}\\
=\ & \frac{1+\mu v_1}{1+\mu}\left\{-\frac{1}{v_1}\ln[1-(1-\mu)v_1]\right.\\
&\qquad\left.-(1-\mu)\left[1-\frac{3(1-\mu)}{2}v_1-\ln[1-(1-\mu)v_1]\right]\right\}\\
=\ & \frac{1+\mu v_1}{1+\mu}\left\{-\frac{1}{v_1}[1-(1-\mu)v_1]\ln[1-(1-\mu)v_1]-(1-\mu)+\frac{3(1-\mu)^2}{2}v_1\right\}\\
\le\ & \frac{1+\mu v_1}{1+\mu}\left[(1-\mu)^2v_1-\frac{(1-\mu)^3}{6}v_1^2\right]\\
<\ & (1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}$$

In the above third inequality, we have used the following inequality

$$\begin{aligned}
&[1-(1-\mu)v_1]\ln[1-(1-\mu)v_1]\\
&\ge -(1-\mu)v_1+\tfrac{(1-\mu)^2}{2}v_1^2+\tfrac{(1-\mu)^3}{6}v_1^3.
\end{aligned}\tag{3.11}$$

If $A(p_n-\xi_n)\le 3(1-\mu)/2(1+\mu)\le -\frac{1}{v_2}\ln[1-(1-\mu)v_1]$, then

$$\frac{3}{2}(1-\mu)\le -\frac{1}{v_1}\ln[1-(1-\mu)v_1]\le \frac{1-\mu}{1-(1-\mu)v_1}\left[1-\frac{1-\mu}{2}v_1-\frac{(1-\mu)^2}{6}v_1^2\right]$$

which implies that $(1-\mu)v_1 > 1/2$. Hence,

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
\le\ & (1+\mu v_1)\left\{\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{v_2}\left[1-\exp\left(-\frac{3}{2}(1-\mu)v_1\right)\right]\right\}\\
=\ & \frac{1+\mu v_1}{1+\mu}\left[\frac{3}{2}(1-\mu)-\frac{1}{v_1}\left(1-e^{-3(1-\mu)v_1/2}\right)\right]\\
\le\ & \frac{1+\mu v_1}{1+\mu}\left\{\frac{3}{2}(1-\mu)-\left[\frac{3}{2}(1-\mu)-\frac{9}{8}(1-\mu)^2v_1+\frac{9}{16}(1-\mu)^3v_1^2\right.\right.\\
&\qquad\left.\left.-\frac{27}{128}(1-\mu)^4v_1^3\right]\right\}\\
=\ & \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[\frac{9}{8}(1-\mu)v_1-\frac{9}{16}(1-\mu)^2v_1^2+\frac{27}{128}(1-\mu)^3v_1^3\right]\\
\le\ & \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[(1-\mu)v_1-\frac{1}{6}(1-\mu)^2v_1^2\right]\\
<\ & (1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}$$

Case 2. $-\frac{1}{v_2}\ln[1-(1-\mu)v_1] < A(p_n-\xi_n) \le 3(1-\mu)/2(1+\mu)$. Choose $l_n \in (\xi_n, p_n)$ such that $A(p_n-l_n) = -\frac{1}{v_2}\ln[1-(1-\mu)v_1]$. Then by (2.10) and (3.10),

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
\le\ & Av_2(l_n-\xi_n)+(1+\mu v_1)\left\{A(p_n-l_n)-A\int_{l_n}^{p_n}\exp[-Av_2(\xi_n+\tau-t)]dt\right\}\\
=\ & Av_2(l_n-\xi_n)+(1+\mu v_1)\\
&\times\left\{A(p_n-l_n)-\frac{1}{v_2}\exp[-Av_2(\xi_n+\tau-p_n)][1-\exp(-Av_2(p_n-l_n))]\right\}\\
=\ & Av_2(l_n-\xi_n)+(1+\mu v_1)\left\{A(p_n-l_n)-\frac{1-\mu}{1+\mu}\exp[-Av_2(\xi_n+\tau-p_n)]\right\}\\
\le\ & Av_2(l_n-\xi_n)+(1+\mu v_1)\left\{A(p_n-l_n)-\frac{1-\mu}{1+\mu}+\frac{1-\mu}{1+\mu}Av_2(\xi_n+\tau-p_n)\right\}\\
\le\ & Av_2\tau+(1-v_1)A(p_n-l_n)-\frac{1-\mu}{1+\mu}\\
=\ & A\tau v_2-\frac{1}{v_2}(1-v_1)\ln[1-(1-\mu)v_1]-\frac{1-\mu}{1+\mu}\\
\le\ & \frac{3}{2}(1-\mu)v_1-\frac{1}{1+\mu}\left[-(1-\mu)+\frac{(1-\mu)(1+\mu)}{2}v_1+\frac{(1-\mu)^2(1+2\mu)}{6}v_1^2\right]\\
&\qquad-\frac{1-\mu}{1+\mu}\\
=\ & (1-\mu)v_1-\frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}v_1^2\\
<\ & (1-\mu)v_1-\frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}$$

In the above fourth inequality, we have used the following inequality

$$(1-v_1)\ln[1-(1-\mu)v_1] \geq -(1-\mu)v_1 + \frac{(1-\mu)(1+\mu)}{2}v_1^2 + \frac{(1-\mu)^2(1+2\mu)}{6}v_1^3.$$

Combining Case 1 with Case 2, we have proved that

$$\ln\frac{1+x(p_n)}{1+\mu v_1} \leq (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)}v_1^2, \quad n = 1,\ 2,\ \ldots .$$

Letting $n \to \infty$ and $\epsilon \to 0$, we have

$$\ln\frac{1+u}{1+\mu v} \leq (1-\mu)v - \frac{(1-\mu)^2}{6(1+\mu)}v^2.$$

This shows that (3.4) holds. Next, we will prove that (3.5) holds as well. From (3.4), we have

$$u < (1+\mu)e^{1-\mu} - 1 < 2, \quad \mu u \leq \mu[(1+\mu v)e^{(1-\mu)v} - 1] \leq v < 1. \tag{3.12}$$

If $v \leq \mu u$, then (3.5) holds naturally. Thus we may assume, without loss of generality, that $v > \mu u_1$. In view of this and (3.9), we can show that neither $x(t) \geq -\mu u_1$ eventually nor $x(t) \leq -\mu u_1$ eventually. Therefore, $x(t)$ oscillates about $-\mu u_1$.

Let $\{q_n\}$ be an increasing sequence such that $q_n \geq t_2 + \tau, x'(q_n) = 0, x(q_n) \leq -\mu u_1, \lim_{n\to\infty} q_n = \infty$ and $\lim_{n\to\infty} x(q_n) = -v$. By (3.9), $x(q_n - \tau) \geq -\mu u_1$. Thus, there exists $\eta_n \in [q_n - \tau, q_n]$ such that $x(\eta_n) = -\mu u_1$. For $t \in [\eta_n, q_n]$, integrating (3.9) from $t-\tau$ to η_n, we have

$$x(t-\tau) \leq (1-\mu u_1)\exp[Au_2(\eta_n + \tau - t)] - 1, \quad \eta_n \leq t \leq q_n.$$

Substituting this into the first inequality in (3.9), we obtain

$$-x'(t) \leq A[1+x(t)](1-\mu u_1)\{\exp[Au_2(\eta_n+\tau-t)] - 1\}, \quad \eta_n \leq t \leq q_n.$$

Combining this with (3.9), we have

$$-\frac{x'(t)}{1+x(t)} \leq \min\{Au_2, A(1-\mu u_1)\{\exp[Au_2(\eta_n+\tau-t)]-1\}\}, \quad \eta_n \leq t \leq q_n. \tag{3.13}$$

There are two possibilities:

Case 1. $A(q_n - \eta_n) \leq \frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Integrating (3.13) from η_n to q_n and using the inequality

$$\ln[1+(1-\mu)u_1] \geq \frac{1}{2}(1-\mu)u_1 - \frac{(1-\mu)^2}{6(1+\mu)}u_1^2,$$

we have

$$\begin{aligned} -\ln\frac{1+x(q_n)}{1-\mu u_1} &\leq Au_2(q_n - \eta_n) \\ &\leq u_2\left\{\frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{u_2}\ln[1+(1-\mu)u_1]\right\} \\ &= \frac{3}{2}(1-\mu)u_1 - \ln[1+(1-\mu)u_1] \\ &\leq (1-\mu)u_1 + \frac{(1-\mu)^2}{6(1+\mu)}u_1^2. \end{aligned}$$

Case 2. $A(q_n-\eta_n) > \frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Choose $h_n \in (\eta_n, q_n)$ such that

$$A(h_n-\eta_n) = \frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{u_2}\ln[1+(1-\mu)u_1].$$

Then by (2.10),(3.13) we have

$$\begin{aligned}
&-\ln\frac{1+x(q_n)}{1-\mu u_1}\\
\le\ & Au_2(h_n-\eta_n) + (1-\mu u_1)\left\{A\int_{h_n}^{q_n}\exp[Au_2(\eta_n+\tau-t)]dt - A(q_n-h_n)\right\}\\
=\ & Au_2(h_n-\eta_n) + (1-\mu u_1)\\
&\times\left\{\frac{1}{u_2}[\exp(Au_2(\eta_n+\tau-h_n)) - \exp(Au_2(\eta_n+\tau-q_n))] - A(q_n-h_n)\right\}\\
=\ & Au_2(h_n-\eta_n) - A(1-\mu u_1)(q_n-h_n)\\
&+\frac{1-\mu u_1}{u_2}e^{A\tau u_2}\left\{[1+(1-\mu)u_1]\exp\left(-\frac{3(1-\mu)}{2(1+\mu)}u_2\right) - e^{-Au_2(q_n-\eta_n)}\right\}\\
\le\ & Au_2(h_n-\eta_n) - A(1-\mu u_1)(q_n-h_n)\\
&+\frac{1-\mu u_1}{u_2}\left\{1+(1-\mu)u_1 - \exp\left[u_2\left(\frac{3(1-\mu)}{2(1+\mu)} - A(q_n-\eta_n)\right)\right]\right\}\\
\le\ & Au_2(h_n-\eta_n) - A(1-\mu u_1)(q_n-h_n)\\
&+\frac{1-\mu u_1}{u_2}\left\{(1-\mu)u_1 - (1+\mu)u_1\left[\frac{3(1-\mu)}{2(1+\mu)} - A(q_n-\eta_n)\right]\right\}\\
=\ & (1+u_1)A(h_n-\eta_n) - \frac{1-\mu}{2(1+\mu)}(1-\mu u_1)\\
=\ & \frac{3(1-\mu)}{2(1+\mu)}(1+u_1) - \frac{(1+u_1)\ln[1+(1-\mu)u_1]}{(1+\mu)u_1} - \frac{1-\mu}{2(1+\mu)}(1-\mu u_1)\\
\le\ & \frac{1-\mu}{1+\mu}u_1 + \frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}u_1^2\\
\le\ & (1-\mu)u_1 + \frac{(1-\mu)^2}{6(1+\mu)}u_1^2.
\end{aligned}$$

In the above fourth inequality, we have used the following inequality

$$(1+u_1)\ln[1+(1-\mu)u_1] \ge (1-\mu)u_1 + \frac{(1-\mu)(1+\mu)}{2}u_1^2 - \frac{(1-\mu)^2(1+2\mu)}{6}u_1^3.$$

Combining Case 1 with Case 2, we have shown that

$$-\ln\frac{1+x(q_n)}{1-\mu u_1} \le (1-\mu)u_1 + \frac{(1-\mu)^2}{6(1+\mu)}u_1^2, \quad n=1,\ 2,\ \dots.$$

Letting $n\to\infty$ and $\epsilon\to 0$, we have

$$-\ln\frac{1-v}{1-\mu u} \le (1-\mu)u + \frac{(1-\mu)^2}{6(1+\mu)}u^2,$$

which implies that (3.5) holds. In view of Lemma 2.1, it follows from (3.4) and (3.5) that $u=v=0$. Thus, $\lim_{t\to\infty}(x(t),y(t)) = (0,0)$. The proof is complete.

As a corollary of Theorem 3.1, we immediately obtain Theorem 1.1.

4. **Discussion.** We establish a criterion (1.12) for the global stability of the the positive equilibrium of system (1.2), which improves the existing ones. Another merit of our main result is that it is related to the well-known 3/2 stability criterion for the uncontrolled system (1.1). Although it remains open if 3/2 serves as the benchmark for $r\tau$ for the global asymptotic stability of (1.1), it is acknowledged to be the best bound for $r\tau$ so far obtained. When it comes to (1.2), to the best of our knowledge, (1.12) gives the best estimate for $r\tau$ in the same context. Note that the bound in (1.2) reduces to 3/2 when $bc = 0$ (uncontrolled), but is actually lower than 3/2 when $bc > 0$ which means more restrictive for $r\tau$. This seems to suggest that the introduction of the control mechanism as in (1.2) is a failure. A further look at this model, we see that the control term is introduced into the model in such a way that it serves as a term in the per capita growth function, and in reality this is almost impossible to implement, and thus, makes little sense. More reasonable way might be to think the control as an inhibiting term, and replace the first equation of (1.2) by

$$n'(t) = rn(t)\left[1 - \frac{n(t-\tau)}{K}\right] - u(t). \tag{4.1}$$

This would enable us to interpret the second equation of (1.2) as: the change rate of control force = the need of control that is proportional to the density of $n(t-\tau)$ - the cost of the control (assuming it costs a for each unit of control force). This is, of course, a completely different model and may require very different methods of analysis. We have to leave it as a future project.

On the other hand, the population component n^* of the equilibrium of (1.2) (see (1.4)) is also lower than the equilibrium K of the un-controlled system (1.1), and this is not desirable if the population under consideration is a favorable species. This also suggests consideration of other new control mechanism.

Also, it has been suggested that a proper single species growth model should identify the birth and death processes, that is $N'(t)$ = birth rate -death rate. Typically, the birth rate is a delayed term $b(N(t-\tau))$, due to the maturation of the species, such as $N(t-\tau)e^{-\alpha N(t-\tau)}$ as is used in the Nicholson's blowflies model, where r is the specific birth rate of an adult, and α is the so-called through-stage survival rate, and τ is the maturation delay; or other one hump shape functions for $b(\cdot)$.

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