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Periodic Traveling Waves In Reaction Diffusion Systems With Delays

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Abstract. Existence of periodic traveling waves in systems of reaction diffusion equations with delays are studied. Instead of bifurcation approach which usually involves analyzing very complicated characteristic equations, a continuation theorem of coincidence degree theory is employed.

Key words. Reaction diffusion equations, traveling waves, functional differential equations, periodic solutions, coincidence degree.
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1 Introduction

Consider the reaction diffusion system

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + f(u(x,t)), \quad t \geq 0, \quad x \in \Omega \subset R^m. \tag{1.1}$$

where $u \in R^n$, $D = diag(d_1, d_2, \cdots, d_n)$ with $d_i > 0$, $i = 1, \cdots, n$, and Δ is the Laplacian operator with respect to the spatial variable x, that is,

$$\Delta u(x,t) = \left(\sum_{k=1}^{m} \frac{\partial^2 u_1(x,t)}{\partial x_k^2}, \cdots, \sum_{k=1}^{m} \frac{\partial^2 u_n(x,t)}{\partial x_k^2} \right)^T$$

Eq. (1.1) is an very important type of partial differential equations, for such equations have been used to model various problems arising from physics, chemistry and biology, etc. Among various aspects for (1.1), are the so-called traveling wave solutions, which are solutions of the form $u(x,t) = \phi(x \cdot \theta + ct)$, where $\theta = (\theta_1, \theta_2, \cdots, \theta_m)$ is a unit vector describing the direction of the wave and c is a constant giving the velocity of the wave. Fisher [3] and Kolmogonov et al. [7] first revealed that, like wave equations, a reaction diffusion equation can also allow traveling wave solutions. Since then, there have been many studies on traveling waves of reaction diffusion equations, and it has been found that traveling wave solutions play a crucial role in describing the spatial-temporal patterns.

Substituting $u(x,t) = \phi(x \cdot \theta + ct)$ into (1.1) and denoting $x \cdot \theta + ct$ still by t gives a system of ordinary differential equations for the *profile* ϕ:

$$D\phi''(t) - c\phi'(t) + f(\phi(t)) = 0. \tag{1.2}$$

If the profile ϕ is monotone and saturates at $\pm\infty$, that is, $\lim_{t\to-\infty}\phi(t)$ and $\lim_{t\to\infty}\phi(t)$ exist, then the traveling wave solution is called a traveling wave front. If $\phi(t)$ is periodic, then the traveling wave solution is called a periodic traveling wave.

In many situations, time delay should be and has been incorporated into the realistic models in applications. The recent monograph Wu [12] provides a systematic coverage of the fundamental theory and some related topics in respect to the reaction diffusion system with delay:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + f(u_t(x)), \quad t \geq 0, \quad x \in \Omega \subset R^m, \tag{1.3}$$

where $f : C([-\tau, 0]; R^n) \to R^n$ is a functional satisfying some conditions, and $u_t(x)$ is an element in $C([-\tau, 0]; R^n)$ parameterized by $x \in \Omega$ and given by $u_t(x)(s) = u(x, t+s),\ s \in [-\tau, 0]$.

Traveling wave *fronts* for reaction diffusion systems *without* delay have been extensively studied in the literature. The recent book review of Gardner [5], the monographs of Fife [2], Britton [1], Murray [8] and Volpert et al. [11] provide full discussion of the subject. But for *delayed* reaction diffusion systems, very few papers on this topic are available in the literature. Schaaf [9] is the pioneer work in this aspect, where *scalar* reaction diffusion equations with a *discrete delay* were systematically studied for existence of traveling *fronts*, using the phase-plane technique, the maximum principle for parabolic functional differential equations and the general theory for ordinary functional differential equations. Wu and Zou [13], and Zou and Wu [14] made attempts to tackle the existence of traveling wave *fronts* of delayed reaction diffusion *systems* with certain nonlinearities, by employing monotone iteration technique and upper-lower solutions.

As far as *periodic* traveling waves are concerned for delayed reaction systems, bifurcation technique becomes a natural choice. See, for examples, Kopell and Howard [6], and Stech and Xin [10]. But when time delay is incorporated into the *system* ($n \geq 2$), the characteristic equation will be a *transcendental equation* with degree of *at least four*, and analyzing such a characteristic equation for the bifurcation purpose becomes, if not impossible, extremely difficult. That's why so little has been done in the literature on periodic traveling waves for *delayed reaction diffusion systems.* Thus it seems quite natural and reasonable to pursue an alternative way, and this constitutes the aim of this paper. In order to focus on the mathematical ideas, we consider a reaction diffusion system of only two equations with only discrete delays, that is

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= d_1 \sum_{k=1}^{m} \frac{\partial^2 u(x,t)}{\partial x_k^2} + f(u(x,t), v(x,t), u(x,t-r_{11}), v(x,t-r_{12})),\\ \frac{\partial v(x,t)}{\partial t} &= d_2 \sum_{k=1}^{m} \frac{\partial^2 v(x,t)}{\partial x_k^2} + g(u(x,t), v(x,t), u(x,t-r_{21}), v(x,t-r_{22})),\end{aligned} \tag{1.4}$$

where $t \geq 0$, $x \in R^m$, and $d_1 > 0$ and $d_2 > 0$ are diffusion coefficients. Instead of bifurcation approach, we will apply a continuation theorem in coincidence degree theory to establish existence of periodic traveling waves for (1.4).

The rest of the paper is organized as follows. In Section 2, we set up the problem into a form for which the continuation theorem will be employed, and state an appropriate version of the continuation theorem. The main results will be given and proved in session 3.

2 Preliminaries and the Main Result

We are concerned with the existence of periodic traveling waves for (1.4). Let $\theta = (\theta_1, \theta_2, \cdots, \theta_m)$ be a unit vector, and $c > 0$ be a constant. Substituting $u(x,t) = \phi_1(x \cdot \theta + ct)$ and $v(x,t) = \phi_2(x \cdot \theta + ct)$ into (1.4) yields

$$\begin{aligned}d_1\phi_1''(t) - c\phi_2'(t) + f(\phi_1(t), \phi_2(t), \phi_1(t-cr_{11}), \phi_2(t-cr_{12})) = 0,\\ d_2\phi_2''(t) - c\phi_2'(t) + g(\phi_1(t), \phi_2(t), \phi_1(t-cr_{21}), \phi_2(t-cr_{22})) = 0.\end{aligned} \tag{2.1}$$

Letting $y = \phi_1$ and $z = \phi_2$, (2.1) can be rewritten as

$$\begin{aligned}y''(t) + a_1 y'(t) + f_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2)) = 0,\\ z''(t) + b_1 z'(t) + f_2(y(t), z(t), y(t-\tau_3), z(t-\tau_4)) = 0.\end{aligned} \tag{2.2}$$

where $a_1 = \frac{-c}{d_1}$, $b_1 = \frac{-c}{d_2}$, $\tau_1 = cr_{11}$, $\tau_2 = cr_{12}$, $\tau_3 = cr_{21}$, $\tau_4 = cr_{22}$, and $f_1 = \frac{1}{d_1} f$, $f_2 = \frac{1}{d_2} g$. As we have seen in the introduction, a periodic solution of (2.2) corresponds to a periodic traveling wave of (1.4). In what follows, we will establish some results on existence of periodic solutions for (2.2) by combining a continuation theorem of coincidence degree in Gaines and Mawhin [4] with some differential inequality techniques for *a priori* bounds of periodic solutions of system (2.2). To this end, we first state an appropriate version of the continuation theorem.

Let X, Z be two Banach spaces, $L : DomL \subset X \to Z$ a linear mapping and $N : X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = CodimImL < +\infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = KerL, ImL = KerQ = Im(I - Q)$. It follows that $L|_{DomL \cap KerP} : (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$.

Lemma 2.1 (see [4]). *Let X and Z be Banach spaces and L: $DomL \cap X \to Z$ a Fredholm mapping of index zero. Assume that Ω is open and bounded subset in X and $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1), x \in \partial\Omega \bigcap$ Dom L, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap$ Ker L, $QNx \neq 0$ and*

$$\deg\{QN, \Omega \cap KerL, 0\} \neq 0,$$

where deg is the Leray-Schauder degree. Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

3 Existence of Periodic Traveling Waves

We first state our main results on the existence of 2π- periodic solutions of system (2.2).

Theorem 3.1. Assume that

(i) There exist constants a_2 and b_2 such that

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4) &= a_2x_1 + g_1(x_1,x_2,x_3,x_4) \\ f_2(x_1,x_2,x_3,x_4) &= b_2x_2 + g_2(x_1,x_2,x_3,x_4), \end{aligned}$$

for all $(x_1,x_2,x_3,x_4) \in R^4$.

(ii) There exist some constants $\alpha_i \geq 0$, $\beta_i \geq 0$, $M_i > 0$ such that

$$|g_i(x_1,x_2,x_3,x_4)| \leq M_i + \alpha_i|x_3| + \beta_i|x_4|, \quad i = 1,2, \forall (x_1,x_2,x_3,x_4) \in R^4$$

and

(iii) $\min\{|a_2| - \alpha_1, |b_2| - \beta_2\} > \max\{\beta_1, \alpha_2\}$.

Assume also that one of the following conditions holds:

(iv) $B > 0$ and $B^2A > \beta_1(2\pi|b_2|\sqrt{\beta_2}C + \beta_2B) + 2\pi|a_2|\sqrt{\beta_1}BC$,

or

(v) $D > 0$ and $DE > F$,

where

$$A = |a_2| - \alpha_1 - 2\pi|a_2|\sqrt{|a_2| + \alpha_1}, \qquad B = |b_2| - \alpha_2 - 2\pi|b_2|\sqrt{|b_2| + \alpha_2},$$

$$C = 2\pi|b_2|\sqrt{\beta_2} + \sqrt{\beta_2 B}, \qquad D = |b_1b_2| - \beta_2(|b_1| + 2\pi|b_2|),$$

$$E = |a_1a_2| - \alpha_1(|a_1| + 2\pi|a_2|), \qquad F = \beta_1\alpha_2(|a_1| + 2\pi|a_2|)(|b_1| + 2\pi|b_2|).$$

Then system (2.2) has at least one 2π-periodic solution, that is, system (1.4) has at least one 2π-periodic traveling wave.

Theorem 3.2. Assume that the conditions in Theorem 3.1 hold, and that

(vi) for $x_3 \neq 0$, $x_4 \neq 0$, $b_2x_4g_1(x_1,x_2,x_3,x_4) \neq a_2x_3g_2(x_1,x_2,x_3,x_4)$.

(vii) $g_1(0,x_2,0,x_4) \neq 0$ for $x_2, x_4 \in R$; $g_2(x_1,0,x_3,0) \neq 0$ for $x_1, x_3 \in R$.

Then system (2.2) has at least one non-constant 2π-periodic solution. That is, system (1.4) has at least one non-constant 2π-periodic traveling wave.

Theorem 3.3. Suppose f_1 and f_2 are continuous. Assume that

(i) There exist some constants $m_i \geq 0$, $n_i \geq 0$, $p_i \geq 0$, $r_i \geq 0$ and $q_i > 0$ such that

$$|f_i(x_1,x_2,x_3,x_4)| \leq m_i|x_1| + n_i|x_2| + p_i|x_3| + r_i|x_4|,$$

for $i = 1, 2$, $(x_1,x_2,x_3,x_4) \in R^4$.

(ii) There exists a constant $h > 0$ such that when $\min\{x_1, x_3\} > h$,

$$f_1(x_1,x_2,x_3,x_4) > 0, \quad f_1(-x_1,x_2,-x_3,x_4) < 0$$

and that when $\min\{x_2, x_4\} > h$,

$$f_2(x_1,x_2,x_3,x_4) > 0, \quad f_2(x_1,-x_2,x_3,-x_4) < 0.$$

Assume also that one of the following two conditions holds:

(iii) $4\pi^2(n_2+r_2) < 1$, $4\pi^2(m_1+p_1) < 1$ and

$$\left[1 - 4\pi^2(n_2+r_2)\right]\left[1 - 4\pi^2(m_1+p_1)\right] > 16\pi^4(n_1+r_1)(m_2+p_2)$$

or

(iv) $|a_1| > 2\pi(m_1+p_1)$, $|b_1| > 2\pi(n_2+r_2)$ and

$$\left[|a_1| - 2\pi(m_1+p_1)\right]\left[|b_1| - 2\pi(n_2+r_2)\right] > 4\pi^2(n_1+r_1)(m_2+p_2).$$

Then system (2.2) has at least one 2π-periodic solution. That is, system (1.4) has at least one 2π-periodic traveling wave.

In the remainder of this section, we give the proofs of the above theorems. In order to apply Lemma 2.1, it is crucial to find the required open and bounded subset in the properly chosen space. This can be achieved by establishing some *a priori* estimates, which is, as will be seen, quite technical. In what follows, we will always let

$$||x||_2 = \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{for} \quad x \in C(R,R),$$

where $|\cdot|$ denotes the Euclidean norm.

Proof of Theorem 3.1. We first consider the case when (i), (ii), (iii) and (iv) are satisfied.

In order to apply Lemma 2.1 to system (2.2), we consider the spaces

$$X = \{(y(t), z(t))^T \in C^1(R,R^2) : y(t+2\pi) = y(t), z(t+2\pi) = z(t)\}$$

and

$$Z = \{(y(t), z(t))^T \in C(R, R^2) : y(t + 2\pi) = y(t), z(t + 2\pi) = z(t)\}$$

equipped with the norms respectively

$$||(y, z)^T||_X = \max_{t\in[0,2\pi]} |y(t)| + \max_{t\in[0,2\pi]} |z(t)| + \max_{t\in[0,2\pi]} |y'(t)| + \max_{t\in[0,2\pi]} |z'(t)|$$

and

$$||(y, z)^T||_Z = \max_{t\in[0,2\pi]} |y(t)| + \max_{t\in[0,2\pi]} |z(t)|.$$

With the above norms, X and Z are Banach spaces. Define the operators N and L by

$$N \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -a_1 y'(t) - a_2 y(t) - g_1(y(t), z(t), y(t - \tau_1), z(t - \tau_2)) \\ -b_1 z'(t) - b_2 z(t) - g_2(y(t), z(t), y(t - \tau_3), z(t - \tau_4)) \end{bmatrix}$$

and

$$L \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y'' \\ z'' \end{bmatrix}.$$

The two project operators P and Q of L to Ker$L$ and $Z/$ImL, respectively, are given by

$$P : X \to X, \ \ P \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} y(t)dt \\ \frac{1}{2\pi}\int_0^{2\pi} z(t)dt \end{bmatrix}, \qquad \begin{bmatrix} y \\ z \end{bmatrix} \in X.$$

$$Q : Z \to Z, \ \ Q \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} y(t)dt \\ \frac{1}{2\pi}\int_0^{2\pi} z(t)dt \end{bmatrix}, \qquad \begin{bmatrix} y \\ z \end{bmatrix} \in Z.$$

Since Ker$L = R^2$ and

$$\text{Im } L = \left\{ \begin{bmatrix} y \\ z \end{bmatrix} \in Z : \int_0^{2\pi} y(t)dt = 0, \int_0^{2\pi} z(t)dt = 0 \right\},$$

ImL is closed and dimKer$L$ =dim$Z/$ImL = 2. Therefore, L is a Fredholm mapping of index 0.

In view of condition (i) in Theorem 3.1, for the above L and N, the equation $L(x, y)^T = \lambda N(x, y)^T$ reads

$$\begin{cases} y''(t) + \lambda a_1 y'(t) + \lambda a_2 y(t) + \lambda g_1(y(t), z(t), y(t - \tau_1), z(t - \tau_2)) = 0, \\ z''(t) + \lambda b_1 z'(t) + \lambda b_2 z(t) + \lambda g_2(y(t), z(t), y(t - \tau_3), z(t - \tau_4)) = 0. \end{cases} \tag{3.1}$$

Suppose that $(y(t), z(t))^T \in X$ is a solution of system (3.1) for a certain $\lambda \in (0, 1)$. Multiplying the first equation of system (3.1) by y and integrating over $[0, 2\pi]$, we obtain

$$-||y'||_2^2 + \lambda a_2 ||y||_2^2 + \lambda \int_0^{2\pi} y(t) g_1(y(t), z(t), y(t - \tau_1), z(t - \tau_2)) = 0,$$

from which, it follows that

$$||y'||_2^2 \leq |a_2|\ ||y||_2^2 + \int_0^{2\pi} |y(t)|[M_1 + \alpha_1|y(t-\tau_1)| + \beta_1|z(t-\tau_2)|]dt. \quad (3.2)$$

Using inequality

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^2 dt\right)^{1/2} \cdot \left(\int_a^b |g(x)|^2 dt\right)^{1/2} \quad (3.3)$$

and Hölder's inequality

$$\int_a^b |f(x)|^r dt \leq (2\pi)^{(1-r/s)} \left(\int_a^b |f(x)|^s ds\right)^{r/s} \quad \text{for} \quad 0 < r \leq s \quad (3.4)$$

to (3.2) gives

$$||y'||_2^2 \leq |a_2|\ ||y||_2^2 + ||y||_2[\sqrt{2\pi}M_1 + \alpha_1||y||_2^2 + \beta_1||z||_2]. \quad (3.5)$$

Applying the inequality

$$(a+b)^r \leq a^r + b^r \quad \text{for} \quad a > 0,\ b > 0 \quad \text{and} \quad 0 < r \leq 1, \quad (3.6)$$

to (3.5) yields

$$||y'||_2 \leq \sqrt{|a_2| + \alpha_1}||y||_2 + ||y||_2^{1/2}\left[\sqrt[4]{2\pi}\sqrt{M_1} + \sqrt{\beta_1}||z||_2^{1/2}\right]. \quad (3.7)$$

Similarly, multiplying the second equation of system (3.1) by z and integrating over $[0, 2\pi]$, an argument parallel to the one used in abtaining (3.7)) leads to

$$||z'||_2 \leq \sqrt{|b_2| + \alpha_2}||z||_2 + ||z||_2^{1/2}\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right]. \quad (3.8)$$

On the other hand, integrating the first equation of system (3.1), we obtain

$$a_2 \int_0^{2\pi} y(t)dt = -\int_0^{2\pi} g_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2))dt.$$

Thus there exists a point $\xi \in (0, 2\pi)$ such that

$$2\pi a_2 y(\xi) = -\int_0^{2\pi} g_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2))dt.$$

Hence

$$2\pi|a_2||y(\xi)| \leq 2\pi M_1 + \int_0^{2\pi} [\alpha_1|y(t-\tau_1)| + \beta_1|z(t-\tau_2)|]dt. \quad (3.9)$$

Applying (3.3) and (3.4) to (3.9) yields

$$2\pi|a_2||y(\xi)| \le 2\pi M_1 + \sqrt{2\pi}[\alpha_1||y||_2 + \beta_1||z||_2]. \tag{3.10}$$

Since for $\forall t \in [0, 2\pi]$,

$$|y(t)| \le |y(\xi)| + \int_0^{2\pi} |y'(t)|dt \le |y(\xi)| + \sqrt{2\pi}||y'||_2,$$

$$\begin{aligned} \sqrt{2\pi}|a_2|\ ||y||_2 &\le 2\pi|a_2|[|y(\xi)| + \sqrt{2\pi}||y'||_2] \\ &\le 2\pi M_1 + \sqrt{2\pi}[\alpha_1||y||_2 + \beta_1||z||_2 + 2\pi|a_2|\ ||y'||_2]. \end{aligned}$$

That is

$$(|a_2| - \alpha_1)||y||_2 \le \sqrt{2\pi}M_1 + \beta_1||z||_2 + 2\pi|a_2|\ ||y'||_2. \tag{3.11}$$

Integrating the second equation of system (3.1), and by a similar argument, we can parallelly establish

$$(|b_2| - \alpha_2)||z||_2 \le \sqrt{2\pi}M_2 + \beta_2||y||_2 + 2\pi|b_2|\ ||z'||_2. \tag{3.12}$$

From (3.7), (3.8), (3.11) and (3.12), we obtain

$$A||y||_2 \le \sqrt{2\pi}M_1 + \beta_1||z||_2 + 2\pi|a_2|\ ||y||_2^{1/2}\left[\sqrt[4]{2\pi M_1} + \sqrt{\beta_1}||z||_2^{1/2}\right] \tag{3.13}$$

and

$$B||z||_2 \le \sqrt{2\pi}M_2 + \beta_2||y||_2 + 2\pi|b_2|\ ||z||_2^{1/2}\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right]. \tag{3.14}$$

Now, by (3.14) and (3.6), we obtain

$$\begin{aligned} 2B||z||_2^{1/2} & \\ \le\ & 2\pi|b_2|\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right] \\ & + \Big\{4\pi^2 b_2^2\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right]^2 \\ & + 4B\left[\sqrt{2\pi}M_2 + \beta_2||y||_2\right]\Big\}^{1/2} \\ \le\ & 4\pi|b_2|\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right] \\ & + 2\sqrt{B}\left[\sqrt{\sqrt{2\pi}M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right]. \end{aligned}$$

Thus

$$B||z||_2^{1/2} \le v_1 + (2\pi|b_2|\sqrt{\beta_2} + \sqrt{B\beta_2})||y||_2^{1/2}, \tag{3.15}$$

where $v_1 = (2\pi|b_2| + \sqrt{B})\sqrt[4]{2\pi}\sqrt{M_2} > 0$. Substituting (3.15) into (3.14), we obtain

$$\begin{aligned} & B^2||z||_2 \\ \leq\ & \sqrt{2\pi}BM_2 + B\beta_2||y||_2 + 2\pi|b_2|\left[\sqrt[4]{2\pi}\sqrt{M_2} + \sqrt{\beta_2}||y||_2^{1/2}\right] \\ & \times\left[v_1 + \left(2\pi|b_2|\sqrt{\beta_2} + \sqrt{B\beta_2}\right)||y||_2^{1/2}\right] \\ =\ & u_1 + u_2||y||_2^{1/2} + \left\{2\pi|b_2|\sqrt{\beta_2}\left(2\pi|b_2|\sqrt{\beta_2} + \sqrt{\beta_2 B}\right) + B\beta_2\right\}||y||_2, \end{aligned} \tag{3.16}$$

where $u_1 > 0$ and $u_2 \geq 0$ are constants. Substituting (3.15) and (3.16) into (3.13) gives

$$\begin{aligned} & \left\{B^2A - \beta_1\left[2\pi|b_2|\sqrt{\beta_2}C + B\beta_2\right] - 2\pi|a_2|B\sqrt{\beta_1}C\right\}||y||_2 \\ & \leq u_3 + u_4||y||_2^{1/2}, \end{aligned} \tag{3.17}$$

where $u_3 > 0$ and $u_4 \geq 0$ are some constants. Now, from (3.17) and condition (iv), it follows that there exists a positive constant R_1 such that

$$||y||_2 \leq R_1. \tag{3.18}$$

This in return, together with (3.16), implies that there exists a positive constant R_2 such that

$$||z||_2 \leq R_2. \tag{3.19}$$

Combining (3.18) and (3.19) with (3.7) and (3.8), we know that there exist two positive constants R_3 and R_4 such that

$$||y'||_2 \leq R_3, \quad ||z'||_2 \leq R_4. \tag{3.20}$$

Multiplying the first equation and the second equation of system (3.1) by y'' and z'', respectively, and integrating both over $[0, 2\pi]$, we obtain

$$||y''||_2^2 \leq |a_2|\ ||y'||_2^2 + ||y''||_2\left[\sqrt{2\pi}M_1 + \alpha_1||y||_2 + \beta_1||z||_2\right] \tag{3.21}$$

and

$$||z''||_2^2 \leq |b_2|\ ||z'||_2^2 + ||z''||_2\left[\sqrt{2\pi}M_2 + \alpha_2||y||_2 + \beta_2||z||_2\right]. \tag{3.22}$$

From (3.18)-(3.22), it follows that there exist two positive constants R_5 and R_6 such that

$$||y''||_2 \leq R_5, \quad ||z''||_2 \leq R_6. \tag{3.23}$$

Consequently there exist four positive constants R_1^*, R_2^*, R_3^* and R_4^* such that

$$|y(t)| < R_1^*, \quad |z(t)| < R_2^*, \quad |y'(t)| < R_3^*, \quad |z'(t)| < R_4^*.$$

Clearly, $R_i^*(i=1,2,3,4)$ are independent of λ. Denote

$$M = R_1^* + R_2^* + R_3^* + R_4^* + C,$$

where $C > 0$ is taken sufficiently large so that

$$[\min\{|a_2| - \alpha_1, |b_2| - \beta_2\} - \max\{\beta_1, \alpha_2\}]M > M_1 + M_2.$$

Now we take $\Omega = \{(y(t), z(t))^T \in X : ||(y,z)^T||_X < M\}$. Then condition (a) in Lemma 2.1 is satisfied. Through an easy computation, we can find that the inverse K_P of L has the form $K_P : ImL \to DomL \cap KerP$,

$$K_p \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \int_0^t ds \int_0^s y(u)du - \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t ds \int_0^s y(u)du \\ \int_0^t ds \int_0^s z(u)du - \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^t ds \int_0^s z(u)du \end{bmatrix} \quad \text{for } \forall \begin{bmatrix} y \\ z \end{bmatrix} \in \overline{\Omega}.$$

It is easy to show that QN and $K_p(I-Q)N$ are continuous by the Lebesgue theorem. Moreover, $QN(\overline{\Omega})$, $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for bounded set $\Omega \subset X$. Therefore, N is L-compact on $\overline{\Omega}$. When $(y,z)^T \in \partial\Omega \cap$ Ker$L = \partial\Omega \cap R^2$, $(y,z)^T$ is a constant vector in R^2 with $|y| + |z| = M$. Then

$$QN \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -a_2 y - g_1(y,z,y,z) \\ -b_2 z - g_2(y,z,y,z) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} & ||QN(y,z)^T||_Z \\ = \ & |-a_2 y - g_1(y,z,y,z)| + |-b_2 z - g_2(y,z,y,z)| \\ \geq \ & (|a_2| - \alpha_1)|y| - M_1 - \beta_1|z| + (|b_2| - \beta_2)|z| - \alpha_2|y| - M_2 \\ \geq \ & [\min\{|a_2| - \alpha_1, |b_2| - \beta_2\} - \max\{\beta_1, \alpha_2\}]M - (M_1 + M_2) > 0. \end{aligned}$$

Consequently,

$$QN \begin{bmatrix} y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{for} \quad \begin{bmatrix} y \\ z \end{bmatrix} \in \partial\Omega \cap KerL.$$

In order to compute the Leray-Schauder degree, we define the homotopy ϕ :Dom$L \times [0,1] \to X$ by

$$\phi(y,z,\mu) = \mu(a_2 y, b_2 z)^T + (1-\mu) \begin{bmatrix} a_2 y + g_1(y,z,y,z) \\ b_2 z + g_2(y,z,y,z) \end{bmatrix},$$

for $y, z \in R$, $\mu \in [0,1]$.

When $(y,z)^T \in \partial\Omega \cap \mathrm{Ker}L$, $(y,z)^T$ is a constant vector in R^2 with $|y| + |z| = M$, and thus for $\mu \in [0,1]$,

$$\begin{aligned} & ||\phi(y,z,\mu)|| \\ = \; & |\mu a_2 y + (1-\mu)(a_2 y + g_1(y,z,y,z))| \\ & + |\mu b_2 z + (1-\mu)(b_2 z + g_2(y,z,y,z))| \\ = \; & |a_2 y + (1-\mu) g_1(y,z,y,z)| + |b_2 z + (1-\mu) g_2(y,z,y,z)| \\ \geq \; & (|a_2| - \alpha_1)|y| - M_1 - \beta_1 |z| + (|b_2| - \beta_2)|z| - \alpha_2 |y| - M_2 \\ \geq \; & [\min(|a_2| - \alpha_1, |b_2| - \beta_2) - \max(\beta_1, \alpha_2)]M - (M_1 + M_2) > 0. \end{aligned}$$

Therefore,

$$\phi(y,z,\mu) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{for} \quad (y,z)^T \in \partial\Omega \cap KerL.$$

As a result, we have

$$deg\{QN(y,z)^T, \Omega \cap KerL, (0,0)^T\} = deg((-a_2 y, -b_2 z)^T, \Omega \cap KerL, (0,0)^T) \neq 0.$$

By now we know that all conditions of Lemma 2.1 are satisfied for the above L, N and Ω, and therefore, system (2.2) has at least one 2π-periodic solution in the case when (i), (ii), (iii) and (iv) are satisfied.

Next, we consider the case when (i), (ii), (iii) and (v) are satisfied. Multiplying the first equation and second equation by $y'(t)$ and $z'(t)$, and both integrating over $[0, 2\pi]$, we obtain

$$a_1 ||y'||_2^2 + \int_0^{2\pi} y'(t) g_1(y(t), z(t), \; y(t-\tau_1), z(t-\tau_2)) dt = 0,$$

$$b_1 ||z'||_2^2 + \int_0^{2\pi} z'(t) g_2(y(t), z(t), \; y(t-\tau_3), z(t-\tau_4)) dt = 0.$$

The rest of the proof is similar to that of the case when (i), (ii), (iii) and (iv) are satisfied and we omit it. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. Since the conditions in Theorem 3.1 are satisfied, system (2.2) has at least one 2π-periodic solution, say, $(y(t), z(t))^T$. Next we will prove that $(y(t), z(t))^T$ is not a constant vector solution. In fact, if $(y(t), z(t))^T$ is identically equal to a constant vector $(c_1, c_2)^T$, then substituting $(y(t), z(t))^T = (c_1, c_2)^T$ into system (2.2), we obtain

$$a_2 c_1 + g_1(c_1, c_2, c_1, c_2) = 0 \tag{3.24}$$

and

$$b_2 c_2 + g_2(c_1, c_2, c_1, c_2) = 0. \tag{3.25}$$

We consider two possible cases: (I). $c_1 = 0$ or $c_2 = 0$; (II). $c_1 \neq 0, c_2 \neq 0$.

(I) When $c_1 = 0$ or $c_2 = 0$, (3.24) and (3.25) imply that

$$g_1(0, c_2, 0, c_2) = 0 \quad \text{or} \quad g_1(c_1, 0, c_1, 0) = 0,$$

which contradicts condition (vii) in Theorem 3.2.

(II) When $c_1 \neq 0, c_2 \neq 0$, (3.24) and (3.25) give

$$b_2 c_2 g_1(c_1, c_2, c_1, c_2) = a_2 c_1 g_2(c_1, c_2, c_1, c_2),$$

contracdicting (vi) in Theorem 3.2. Therefore, $(y(t), z(t))^T$ can not be a constant vector, and the proof is completed.

Proof of Theorem 3.3. We only consider the case when (i), (ii) and (iii) are satisfied, and the proof for the case when (i), (ii) and (iv) hold can be similarly carried out.

Let X and Z be as in the proof of Theorem 3.1 and define $N : X \to Z$ by

$$N \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -a_1 y'(t) - f_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2)) \\ -b_1 z'(t) - f_2(y(t), z(t), y(t-\tau_3), z(t-\tau_4)) \end{bmatrix},$$

Let L, P, Q, X also be the same as those in the proof of Theorem 3.1. Corresponding to the operator equation $Lx = \lambda N x$, $\lambda \in (0,1)$, we have

$$\begin{cases} y''(t) + \lambda a_1 y'(t) + \lambda f_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2)) = 0, \\ z''(t) + \lambda b_1 z'(t) + \lambda f_2(y(t), z(t), y(t-\tau_3), z(t-\tau_4)) = 0. \end{cases} \tag{3.26}$$

Suppose that $(y(t), z(t))^T \in DomL$ is a solution of system (3.26) for a certain $\lambda \in (0,1)$. Integrating system (3.26) over $[0, 2\pi]$ gives

$$\int_0^{2\pi} f_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2)) dt = 0 \tag{3.27}$$

and

$$\int_0^{2\pi} f_2(y(t), z(t), y(t-\tau_3), z(t-\tau_4)) dt = 0. \tag{3.28}$$

From condition (ii) in Theorem 3.3 and (3.27), we know that there exists $\tau \in \{0, \tau_1\}$, a point $t^* \in [0, 2\pi]$ and a constant $\overline{M} > 0$ such that

$$y(t^* - \tau) < \overline{M}. \tag{3.29}$$

If it is not true, then for $\forall M > 0$ and $t \in [0, 2\pi]$, we have

$$y(t) \geq M, \quad y(t-\tau_1) \geq M,$$

which, in view of condition (ii), contradicts (3.27). Thus (3.29) holds. Denoting $t^* - \tau = \xi_1 + 2\pi k$, here $\xi_1 \in [0, 2\pi]$ and k is a integer, then

$$y(\xi_1) < \overline{M}. \tag{3.30}$$

Similarly, we can find a point $\xi_2 \in [0, 2\pi]$ and a constant $K > 0$ such that

$$y(\xi_2) > -K. \tag{3.31}$$

Then from (3.30) and (3.31), we can obtain for $\forall t \in [0, 2\pi]$,

$$y(t) \le y(\xi_1) + \int_0^{2\pi} |y'(t)|dt < \overline{M} + \int_0^{2\pi} |y'(t)|dt$$

and

$$y(t) \ge y(\xi_2) - \int_0^{2\pi} |y'(t)|dt > -K - \int_0^{2\pi} |y'(t)|dt.$$

Consequently,

$$\begin{aligned} |y(t)| &< \max\left\{\overline{M} + \int_0^{2\pi} |y'(t)|dt, K + \int_0^{2\pi} |y'(t)|dt\right\} \\ &< \overline{M} + K + \int_0^{2\pi} |y'(t)|dt \overset{def}{=} d_1 + \int_0^{2\pi} |y'(t)|dt. \end{aligned} \tag{3.32}$$

By condition (ii) in Theorem 3.3 and (3.28) and using the same argument as in obtaining (3.32), we have

$$\begin{aligned} |z(t)| &< \max\left\{M^* + \int_0^{2\pi} |z'(t)|dt, K^* + \int_0^{2\pi} |y'(t)|dt\right\} \\ &< M^* + K^* + \int_0^{2\pi} |z'(t)|dt \overset{def}{=} d_2 + \int_0^{2\pi} |z'(t)|dt, \end{aligned} \tag{3.33}$$

where M^* and K^* are two positive constants.

On the other hand, multiplying the first equation and second equation by y and z, respectively, and integrating over $[0, 2\pi]$, we have

$$||y'||_2^2 = \lambda \int_0^{2\pi} y(t) f_1(y(t), z(t), y(t-\tau_1), z(t-\tau_2))dt \tag{3.34}$$

and

$$||z'||_2^2 = \lambda \int_0^{2\pi} z(t) f_2(y(t), z(t), y(t-\tau_3), z(t-\tau_4))dt, \tag{3.35}$$

from which and condition (i), it follows that

$$||y'||_2^2 \le ||y||_2 \left[(m_1+p_1)||y||_2 + (n_1+r_1)||z||_2\right] \tag{3.36}$$

and

$$||z'||_2^2 \le ||z||_2 \left[(m_2+p_2)||y||_2 + (n_2+r_2)||z||_2\right]. \tag{3.37}$$

Substituting (3.32) and (3.33) into (3.36) and (3.37) gives

$$\begin{aligned} ||y'||_2^2 \le 2\pi \left[d_1 + \sqrt{2\pi}||y'||_2\right] &\Big[(m_1+p_1)d_1 + (n_1+r_1)d_2 + \sqrt{2\pi}(m_1+p_1)||y'||_2 \\ &+ \sqrt{2\pi}(n_1+r_1)||z'||_2\Big] \end{aligned} \tag{3.38}$$

and

$$||z'||_2^2 \le 2\pi \left[d_2 + \sqrt{2\pi}||z'||_2\right] \cdot \left[(m_2+p_2)d_1 + (n_2+r_2)d_2 + \sqrt{2\pi}(m_2+p_2)||y'||_2 + \sqrt{2\pi}(n_2+r_2)||z'||_2\right]. \tag{3.39}$$

Combining inequality (3.6) with (3.38) and (3.39), we obtain

$$\begin{aligned}
&[1-4\pi^2(m_1+p_1)]||y'||_2 \\
&\le 2\pi\sqrt{2\pi}\left[2(m_1+p_1)d_1 + (n_1+r_1)d_2 + \sqrt{2\pi}(n_1+r_1)||z'||_2\right] \\
&+G\sqrt{1-4\pi^2(m_1+p_1)} \\
&+\sqrt{(1-4\pi^2(m_1+p_1))2\pi d_1\sqrt{2\pi}(n_1+r_1)}||z'||_2^{\frac{1}{2}}
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&[1-4\pi^2(n_2+r_2)]||z'||_2 \\
&\le 2\pi\sqrt{2\pi}\left[(m_2+p_2)d_1 + 2(n_2+r_2)d_2 + \sqrt{2\pi}(m_2+p_2)||y'||_2\right] \\
&+H\sqrt{1-4\pi^2(n_2+r_2)} \\
&+\sqrt{(1-4\pi^2(n_2+r_2))\sqrt{2\pi d_2}\sqrt{2\pi}(m_2+p_2)}||y'||_2^{\frac{1}{2}}.
\end{aligned} \tag{3.41}$$

where G and H are two positive constants. Combining (3.40) and (3.41) gives

$$\left\{[1-4\pi^2(n_2+r_2)][1-4\pi^2(m_1+p_1)] - 16\pi^4(n_1+r_1)(m_2+p_2)\right\}||y'||_2$$

$$\le d_3 + d_4||y'||_2^{\frac{1}{2}} + d_5||y'||_2^{\frac{1}{4}}, \tag{3.42}$$

where $d_i (i = 3, 4, 5)$ are three positive constants, from which, it follows that there exists a positive constant d_6 such that

$$||y'||_2 < d_6.$$

By similar arguments to those of the corresponding parts in the proof of Theorem 3.1, we can find a positive constant R^* and take $\Omega = \left\{(y(t), z(t))^T \in X : ||(y,z)^T||_X < R^*\right\}$, where R^* is taken sufficiently large such that $R^* > 2h$.

When $(y,z)^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^2$, $(y,z)^T$ is a constant vector in R^2 with $|y|+|z| = R^*$, and thus,

$$QN\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -f_1(y,z,y,z) \\ -f_2(y,z,y,z) \end{bmatrix}.$$

Therefore, when $(y,z)^T \in \partial\Omega \cap KerL$,

$$QN\begin{bmatrix} y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

by condition (ii) in Theorem 3.3. Finally, the condition that $\deg\{QN, \Omega \cap KerL, 0\} \neq 0$ can verified by the homotopy $\psi : DomL \times [0,1] \to X$ defined by

$$\psi(y,z) = (k_1 y, k_2 z)^T + \mu^* \begin{bmatrix} f_1(y,z,y,z) \\ f_2(y,z,y,z) \end{bmatrix},$$

where $y, z \in R, \mu^* \in [0,1]$, and $k_1 > 0$, $k_2 > 0$ are sufficiently large constants such that

$$\min\{k_1 - m_1 - p_1, k_2 - n_2 - r_2\} > \max\{n_1 + r_1, m_2 + p_2\}.$$

Hence, condition (b) of Lemma 2.1 is also satisfied for the above L, N and Ω. Theorefore, the coclusion of Theorem 3.3 follows by Lemma 2.1. This completes the proof.

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