COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **11**, Number **5**, September **2012**

doi:10.3934/cpaa.2012.11.1825

pp. **1825–1838**

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTION TO A NON-LOCAL DIFFERENTIAL EQUATION WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITION —A NON-MONOTONE CASE

Zhiming Guo

1. School of Mathematics and Information Sciences,
2. Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes,
Guangzhou University, Guangzhou, China, 510006

Zhichun Yang

College of Mathematics Science,
Chongqing Normal University, Chongqing, China, 400047

Xingfu Zou

Department of Applied Mathematics,
University of Western Ontario, London, Ontario, Canada N6A 5B7

(Communicated by Wei Feng)

Abstract. This paper deals with a class of non-local second order differential equations subject to the homogeneous Dirichlet boundary condition. The main concern is positive steady state of the boundary value problem, especially when the equation does not enjoy the monotonicity. Nonexistence, existence and uniqueness of positive steady state for the problem are addressed. In particular, developed is a technique that combines the method of super-sub solutions and the estimation of integral kernels, which enables us to obtain some sufficient conditions for the existence and uniqueness of a positive steady state. Two examples are given to illustrate the obtained results.

1. **Introduction.**

In studying the population dynamics for a single species with age structure habitating in a spatially continuous and *unbounded* domain $R = (-\infty, \infty)$, So *et. al.* [10] derived a model containing a spatial non-local term resulting from the mobility of immature individuals and a temporal delay accounting for the average maturation time. The model is given by the following non-local reaction diffusion equation with delay

$$\frac{\partial w(t,x)}{\partial t} = D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + \int_{-\infty}^{\infty} f_\alpha(x,y) b(w(t-\tau,y))dy, \ t \ge 0, \ x \in R. \quad (1)$$

When the habitat is a bounded domain, e.g., $\Omega = [0, \pi]$, similar models were obtained and numerically explored in [6] and theoretically investigated in [11, 12] by

2000 *Mathematics Subject Classification.* Primary: 34B18.

Key words and phrases. Non-local second order differential equation, Dirichlet boundary value problems, positive solutions, existence, uniqueness.

Research partially supported by National Natural Science Foundation of China (No.10871053, No.10971240) and by Natural Sciences and Engineering Research Council of Canada.

using a dynamical system approach. The differential equations of these models take the same form as (1), except that the kernel function $f_\alpha(x,y)$ would take different forms depending on the conditions posed on the boundary. Also the kernel depends on the parameter $\alpha \geq 0$ that measures the mobility of immature population. For homogeneous Neumann boundary condition, the recent work [12] has shown that the corresponding model

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\dfrac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + \displaystyle\int_0^\pi f_\alpha(x,y)b(w(t-\tau,y))dy, t \geq 0, x \in (0,\pi) \\ \dfrac{\partial}{\partial n}w(t,0) = \dfrac{\partial}{\partial n}w(t,\pi) = 0 \end{cases} \tag{2}$$

with non-negative initial functions only supports dynamics of converging to constant steady state, and hence, the global dynamics can be fully determined in terms of the model parameters.

When the homogeneous Dirichlet boundary condition is posed, the corresponding model is given by

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\dfrac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + \displaystyle\int_0^\pi f_\alpha(x,y)b(w(t-\tau,y))dy, t \geq 0, x \in (0,\pi) \\ w(t,0) = w(t,\pi) = 0, \end{cases} \tag{3}$$

for which, Xu and Zhao [11] studied the global dynamics and obtained some results on the global attractivity of a positive steady state by using the theory of monotone dynamical systems in the *monotone* case. However, in the *non-monotone* case, as pointed out by Zhao [12], it still remains an open and challenging problem to study the uniqueness and global attractivity of a positive steady state for (3).

It is well known that positive steady states play an important role in the study of the global dynamics of a reaction diffusion equation arising from population biology, and global convergence of solutions to a positive steady state requires existence of an unique positive steady state. In this paper we are concerned with the existence and uniqueness of a positive steady state of (3), that is, existence and uniqueness of a positive solution to the following Dirichlet boundary value problem

$$\begin{cases} -\dfrac{d^2 w}{dx^2} + k^2 w = \displaystyle\int_0^\pi f_\alpha(x,y)b(w(y))dy, t \geq 0, \text{for} x \in (0,\pi) \\ w(0) = w(\pi) = 0, \end{cases} \tag{4}$$

Here, by the derivation of (3) in [6], α and $k := \sqrt{d/D}$ are positive constants and the kernel function is given by

$$f_\alpha(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2\alpha} \sin nx \sin ny. \tag{5}$$

Since $b(w)$ represents a birth function, as in most models for population dynamics, we assume, throughout this paper, that b satisfies the following biologically reasonable assumptions:

(H1) $b(w) = wg(w)$, $g(w) > 0$ and $g'(w) < 0$ for all $w \geq 0$;

(H2) both $b(w)$ and $b'(w)$ are bounded for $w \geq 0$.

A typical such function is $b(u) = pue^{-qu}$ which corresponds to the birth function used in the Nicholson's blowflies equation in Gurney *et al* [4]. Clearly, $b'(w) = g(w) + wg'(w) \leq g(w)$ for $w \geq 0$ and $b'(0) = g(0) > 0$.

The rest of this paper is organized as follows: We will obtain some preliminary results in Section 2. In Section 3, we will give some results on the nonexistence, existence and uniqueness of positive solutions to (4) in the *monotone case*. These results may be deduced from [11], but for the purpose of self-containing and gradual proceeding, we include them in this section. Our main results are presented and proved in Section 4, where we obtain sufficient conditions to ensure the existence and uniqueness of positive solutions to (4) in a *non-monotone case*. This main theorem can also be applied to the monotone case (as a special case) to reproduce the corresponding result in Section 3. Section 5 is devoted to the applications of our main results to two models arising from population dynamics.

2. **Preliminaries.** Firstly, we show that the kernel $f_\alpha(x,y)$ in (5) enjoys the following properties.

Lemma 2.1. *For $\alpha > 0$, we have,*
(i) $f_\alpha(0,y) = f_\alpha(\pi,y) = f_\alpha(x,0) = f_\alpha(x,\pi) = 0$, *and*
(ii) $0 < f_\alpha(x,y) < \frac{2}{\pi(e^\alpha - 1)}$, *for $0 < x, y < \pi$.*

Proof. (i) is easy to verify. The right inequality of (ii) is a direct consequence of the following estimate:

$$\sum_{n=1}^{\infty} \left| e^{-n^2\alpha} \sin nx \sin ny \right| \le \sum_{n=1}^{\infty} e^{-n^2\alpha} \le \sum_{n=1}^{\infty} e^{-n\alpha} = \frac{1}{e^\alpha - 1}.$$

Also by the above estimate, the infinite series defining $f_\alpha(x,y)$ is absolutely summable, provided $\alpha > 0$. Hence, we can integrate or differentiate the series using termwise operation. Next, we fix any continuous $\phi(y) \ge 0$ with $\phi(0) = 0$ and define $\psi(\alpha,x) = \int_0^\pi f_\alpha(x,y)\phi(y)\,dy$. Then ψ satisfies the heat equation $\psi_\alpha = \psi_{xx}$. Furthermore, $\psi(0,x) = \phi(x) \ge 0$ and $\psi(\alpha,0) = \psi(\alpha,\pi) = 0$. Thus, $\psi \ge 0$, by maximum principle (see Theorem 2 on p. 168 of [9]). This also shows that $f_\alpha(x,y) \ge 0$. To show $f_\alpha(x,y) > 0$ for $0 < x, y < \pi$, we fix any $y \in (0,\pi)$. Consider the function $\xi(\alpha,x) = f_\alpha(x,y)$ for $\alpha > 0$ and $x \in [0,y]$. Then $\xi(\alpha,x)$ satisfies the heat equation $\xi_\alpha = \xi_{xx}$ and $\xi \ge 0$. Since $\xi(\alpha,y) > 0$ and 0 is the minimum of ξ, therefore, by maximum principle, $\xi(\alpha,x) > 0$ as well, for all $\alpha > 0$ and $0 < x < y$. By the arbitrariness of $y \in (0,\pi)$, we conclude that $\xi(\alpha,x) > 0$ for all $\alpha > 0$ and $0 < x < \pi$. □

Let $X = C^2(0,\pi) \cap C[0,\pi]$, $Y = C[0,\pi]$. Then both X and Y are ordered Banach spaces with the natural ordering, that is, for any $w_1, w_2 \in X$ (or Y), $w_1 \le w_2$ if and only if $w_1(x) \le w_2(x)$ for any $x \in [0,\pi]$. Define $\mathcal{L} : X \to Y$ by $\mathcal{L}w = -\frac{d^2w}{dx^2} + k^2 w$ and $\mathcal{J} : Y \to X$ by $(\mathcal{J}w)(x) = g(0)\int_0^\pi f_\alpha(x,y)w(y)dy$. Let $G(x,y)$ be the Green function of the operator $\mathcal{L}$ associated with $w(0) = w(\pi) = 0$. It can be easily verified that

$$G(x,y) = \begin{cases} \dfrac{\sinh(kx)\sinh k(\pi - y)}{k\sinh(k\pi)}, & \text{as } 0 \le x \le y \le \pi, \\ \dfrac{\sinh k(\pi - x)\sinh(ky)}{k\sinh(k\pi)}, & \text{as } 0 \le y \le x \le \pi, \end{cases} \tag{6}$$

where $\sinh(x) = (e^x - e^{-x})/2$. The following theorem gives a necessary condition for the existence of a positive solution to (4).

Theorem 2.2. *If*

$$1+k^2 \geq g(0)e^{-\alpha}, \tag{7}$$

then there is no positive solution to (4).

Proof. Consider the linear eigenvalue problem

$$\begin{cases} -\dfrac{d^2w}{dx^2}+k^2w=\lambda g(0)\displaystyle\int_0^\pi w(y)f_\alpha(x,y)dy, \\ w(0)=w(\pi)=0. \end{cases} \tag{8}$$

The linear equation (8) can be rewritten as $Tw=\frac{1}{\lambda}w$ where $T=\mathcal{L}^{-1}\mathcal{J}:Y\to X$. By Lemma 2.1 and the property of the differential operator $\mathcal{L}$, it is known (see *e.g.*, [1]) that T is a strongly positive compact endomorphism in $C_{\mathbf{e}}[0,\pi]$, where $\mathbf{e}$ is the unique solution of

$$\begin{cases} -\dfrac{d^2w}{dx^2}+k^2w=1, x\in(0,\pi), \\ w(0)=w(\pi)=0, \end{cases} \tag{9}$$

and $C_{\mathbf{e}}[0,\pi]$ is the Banach space generated by the order unit $\mathbf{e}\in X$ with order unit norm $\|\cdot\|_{\mathbf{e}}$ (see [1]). By the famous Krein-Rutman theorem and its sharper version for strongly positive linear operators (see Lemma 3.2 in [1]), the spectral radius $r(T)$ is a simple positive eigenvalue of T having a positive eigenvector. Indeed, one can easily determine $r(T)$ as

$$r(T)=\frac{g(0)}{(1+k^2)e^{\alpha}}.$$

Now, assume for the sake of contradiction that $w=w^*(x)$ is a positive solution to (4). Then

$$-\frac{d^2w^*}{dx^2}+k^2w^*=\int_0^\pi w^*(y)g(w^*(y))f_\alpha(x,y)dy. \tag{10}$$

Define $\overline{\mathcal{J}}:Y\to X$ by

$$(\overline{\mathcal{J}}w)(x)=\int_0^\pi f_\alpha(x,y)g(w^*(y))w(y)dy,$$

and let $\overline{T}=\mathcal{L}^{-1}\overline{\mathcal{J}}:Y\to X$. Clearly, $\overline{T}$ is also a strongly positive compact endomorphism of $C_{\mathbf{e}}[0,\pi]$. By (H1), for any $w>0$, $g(w)<g(0)$. Thus, for any $w\in C_{\mathbf{e}}[0,\pi]$, $\overline{T}w<Tw$. Again by Lemma 3.2 in [1], $r(\overline{T})<r(T)$, where $r(\overline{T})$ is the spectral radius of $\overline{T}$. It follows that

$$r(\overline{T})<r(T)=\frac{g(0)}{(1+k^2)e^{\alpha}}\leq 1.$$

On the other hand, (10) implies that 1 is an eigenvalue of $\overline{T}$ corresponding to a positive eigenvector w^*, contradicting $r(\overline{T})<1$. The proof is completed. □

Remark 1. It is interesting to compare (4) with the corresponding Neumann BVP

$$\begin{cases} -\dfrac{d^2w}{dx^2}+k^2w=\displaystyle\int_0^\pi f_\alpha^{(N)}(x,y)b(w(y))dy \\ w'(0)=w'(\pi)=0, \end{cases} \tag{11}$$

where

$$f_\alpha^{(N)}(x,y)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}e^{-n^2\alpha}\cos nx\cos ny. \tag{12}$$

Recently it is shown in [12] that (11) has no positive solution if and only if $k^2 \geq g(0)$. Note that $k^2 \geq g(0)$ would lead to $1+k^2 > k^2 \geq g(0) > g(0)e^{-\alpha}$. Thus when the Neumann BVP (11)-(12) has no positive solution, then the corresponding Dirichlet BVP (4)-(5) has no positive solution too. However, the reverse may not be true, because the conditions $(1+k^2)e^{\alpha} > g(0) > k^2$ are feasible.

3. **Existence and uniqueness of positive solution to (4)—monotone case.**

When $b(w) = wg(w)$ satisfies certain monotonicity coupled with some other conditions, (4) has a unique positive solution as stated in the theorem below.

Theorem 3.1. *Assume that $1+k^2 < g(0)e^{-\alpha}$. Suppose there is a positive constant M_0, such that*

(i) $b'(w) \geq 0$ *for any* $w \in [0, M_0]$*;*

(ii) $g(M_0)\gamma \leq k^2$, *where*

$$\gamma = \max_{x\in[0,\pi]} \int_0^{\pi} f_\alpha(x,y)dy = \max_{x\in[0,\pi]} \frac{4}{\pi} \sum_{n=1}^{+\infty} e^{-(2n-1)^2\alpha} \frac{\sin(2n-1)x}{2n-1}. \tag{13}$$

Then (4) has a unique positive solution $w(x)$ satisfying $0 < w(x) \leq M_0$ for $x \in (0,\pi)$.

Proof. Since $1+k^2 < g(0)e^{-\alpha}$, for sufficiently small ε, we have $1+k^2 < g(\varepsilon)e^{-\alpha}$. Let $w^{(l)}(x) = \varepsilon \sin x, \varepsilon > 0$. Thus, when ε is sufficiently small we have

$$\begin{aligned}
&-\frac{d^2w}{dx^2}^{(l)}(x) + k^2 w^{(l)}(x) - \int_0^{\pi} f_\alpha(x,y) b(w^{(l)}(y))dy \\
&= \varepsilon[(1+k^2)\sin x] - \varepsilon \int_0^{\pi} f_\alpha(x,y)(\sin y) g(\varepsilon \sin y)dy \\
&\leq \varepsilon[(1+k^2)\sin x] - \varepsilon g(\varepsilon) \int_0^{\pi} f_\alpha(x,y)(\sin y)dy \\
&= \varepsilon[(1+k^2) - g(\varepsilon)e^{-\alpha}]\sin x \\
&\leq 0 \qquad \text{for} \quad x \in (0,\pi),
\end{aligned}$$

implying that $w^{(l)}$ is a sub-solution to (4).

Next, we show that $w^{(u)}(x) \equiv M_0$ is a super-solution to (4). Indeed,

$$\begin{aligned}
&-\frac{d^2w}{dx^2}^{(u)}(x) + k^2 w^{(u)}(x) - \int_0^{\pi} f_\alpha(x,y) b(w^{(u)}(y))dy \\
&\geq M_0[k^2 - g(M_0) \int_0^{\pi} f_\alpha(x,y)dy] \\
&= M_0[k^2 - g(M_0)\gamma] \geq 0.
\end{aligned}$$

Now we consider the nonlinear operator $S : Y \to X$ defined by

$$(Sw)(x) = \mathcal{L}^{-1}\left(\int_0^{\pi} f_\alpha(x,y) b(w(y))dy\right), \forall w \in Y.$$

For a constant K, denote by $\hat{K}$ the constant function on $[0,\pi]$ taking the value K. By the monotonicity of b in the interval $[0, M_0]$, we know that S is strongly increasing in the order interval $[\hat{0}, \hat{M}_0]$. By a standard argument of sub and super solutions (e.g., [2, 8]), we conclude that there is a positive solution w_0 to (4) satisfying $w^{(l)}(x) \leq w_0(x) \leq w^{(u)}(x)$ for $x \in [0,\pi]$. The uniqueness follows from an argument similar to that in Hess [5]. □

Theorem 3.1 confirms that there exists a unique positive solution within the order interval $[\hat{0}, \hat{M}_0]$, but it does not answer if (4) has any positive solution that does not belong to this order interval. The corollary below addresses this question.

Corollary 1. *Suppose that the assumptions of Theorem 3.1 hold. Further assume that $b(w) \le b(M_0)$ for $w > M_0$. Then (4) has a unique positive solution.*

Proof. We only need to show that there is no positive solution to (1) beyond the order interval $[\hat{0}, \hat{M}_0]$. Assume, for the sake of contradiction, that $\bar{w}$ is a positive solution to (4) satisfying $\max_{x\in[0,\pi]} w(x) > M_0$. Let $x_0 \in (0, \pi)$ be such that $\bar{w}(x_0) = \max_{x\in[0,\pi]} w(x)$. Then

$$\begin{aligned} &-\frac{d^2\bar{w}}{dx^2}(x_0) + k^2\bar{w}(x_0) - \int_0^\pi f_\alpha(x_0, y)b(\bar{w}(y))dy \\ &> k^2M_0 - b(M_0)\int_0^\pi f_\alpha(x_0, y)dy \ge M_0(k^2 - g(M_0)\gamma) > 0, \end{aligned}$$

a contradiction. The proof is completed. □

Note that

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{e^{-(2n-1)^2\alpha}}{2n-1}\sin(2n-1)x &\le \sum_{n=1}^{+\infty} \frac{1}{2n-1}e^{(-4n+3)\alpha} \\ &= e^\alpha\left[\sum_{n=1}^{+\infty}\frac{1}{n}(e^{(-2\alpha)})^n - \sum_{n=1}^{+\infty}\frac{1}{2n}(e^{(-2\alpha)})^{2n}\right] \\ &= e^\alpha\left[-\ln(1-e^{-2\alpha}) + \frac{1}{2}\ln(1-e^{-4\alpha})\right] \\ &= \frac{1}{2}e^\alpha\ln\frac{e^{2\alpha}+1}{e^{2\alpha}-1}. \end{aligned}$$

Replacing γ by γ_1 which has the explicit formula

$$\gamma_1 = \frac{2e^\alpha}{\pi}\ln\frac{e^{2\alpha}+1}{e^{2\alpha}-1}, \tag{14}$$

we immediately have the following more convenient result.

Corollary 2. *Assume that $1 + k^2 < g(0)e^{-\alpha}$. Suppose that $b'(w) \ge 0$ in $[0, M_0]$ where $M_0 = \max_{w\in[0,\infty)} b(w)$. If $g(M_0)\gamma_1 < k^2$, then (4) has a unique positive solution.*

4. **Existence and uniqueness of positive solution to (4) —Non-monotone case.** The results in Section 3 require a monotone condition: $b'(w) \ge 0$ for $w \in [0, M_0]$, where M_0 satisfies $g(M_0)\gamma \le k^2$. Noting that $g(w)$ is decreasing in w, there is a balancing issue: larger M_0 will make the second condition easier to be but in the mean time, will make the first condition (monotone condition) harder to be satisfied. In this section, we develop an approach that enables us to drop the monotone condition. As a cost, we need pose some other conditions to guarantee the existence of a unique positive solution to (4).

By (H1) and (H2), $b'(w)$ is bounded from below. Let $\eta \in R$ be such that $\eta \le \inf_{w\ge0} b'(w)$ and set $b_0(w) = b(w) - \eta w$. Then (4) can be rewritten as

$$\begin{cases} -\dfrac{d^2w}{dx^2} + k^2w - \eta\displaystyle\int_0^\pi f_\alpha(x, y)w(y)dy = \int_0^\pi b_0(w(y))f_\alpha(x, y)dy \\ w(0) = w(\pi) = 0, \end{cases} \tag{15}$$

Clearly, $b_0'(w) \geq 0$ for any $w \geq 0$.

Since $\eta \geq 0$ implies $b'(w) \geq 0$ for all $w \geq 0$, which is the case discussed in Section 3, in the sequel we only consider the case $\eta < 0$.

Let $\mathcal{L}$ be as in Section 2, and define $\mathcal{K} : Y \to X$ by

$$(\mathcal{K}w)(x) = -\eta \int_0^\pi f_\alpha(x, y)w(y)dy.$$

The following two lemmas are related to these two operators.

Lemma 4.1. *If*

$$-\frac{4\eta}{(1+k^2)\pi}\frac{1}{e^\alpha - 1} < 1, \tag{16}$$

then $||\mathcal{L}^{-1}\mathcal{K}|| < 1$.

Proof. For any $w \in Y$,

$$\begin{aligned}
(\mathcal{L}^{-1}\mathcal{K}w)(x) &= -\eta \int_0^\pi G(x,u)\left[\int_0^\pi f_\alpha(u,y)w(y)dy\right]du \\
&= -\eta \int_0^\pi \left[\int_0^\pi G(x,u)f_\alpha(u,y)du\right]w(y)dy \\
&= -\frac{2\eta}{\pi}\sum_{n=1}^{+\infty} e^{-n^2\alpha}\int_0^\pi \left[\int_0^\pi G(x,u)\sin nudu\right]\sin nyw(y)dy \\
&= -\frac{2\eta}{\pi}\sum_{n=1}^{+\infty} e^{-n^2\alpha}\frac{1}{n^2+k^2}\sin nx\int_0^\pi \sin nyw(y)dy \\
&\leq -\frac{4\eta}{\pi}\sum_{n=1}^{+\infty} e^{-n^2\alpha}\frac{1}{n^2+k^2}\|w\| \\
&\leq -\frac{4\eta}{(1+k^2)\pi}\frac{1}{e^\alpha-1}\|w\|.
\end{aligned}$$

Thus, by (16), we have

$$\|\mathcal{L}^{-1}\mathcal{K}\| \leq -\frac{4\eta}{(1+k^2)\pi}\frac{1}{e^\alpha-1} < 1.$$

□

Lemma 4.2. *Assume that* (16) *holds. If*

$$-\frac{8\eta}{\pi}\frac{\sinh^4\frac{k\pi}{2}}{k^3\sinh k\pi}\frac{1}{e^\alpha-1} < 1, \tag{17}$$

then $(\mathcal{L}+\mathcal{K})^{-1}$ *is positive.*

Proof. Noticing that if $\|\mathcal{L}^{-1}\mathcal{K}\| < 1$ by Lemma 4.1, we have

$$\begin{aligned}
(\mathcal{L}+\mathcal{K})^{-1} &= [\mathcal{L}(I+\mathcal{L}^{-1}\mathcal{K})]^{-1} = (I+\mathcal{L}^{-1}\mathcal{K})^{-1}\mathcal{L}^{-1} = \sum_{k=0}^{+\infty}(-\mathcal{L}^{-1}\mathcal{K})^k\mathcal{L}^{-1} \\
&= [I - \mathcal{L}^{-1}\mathcal{K} + (\mathcal{L}^{-1}\mathcal{K})^2 - (\mathcal{L}^{-1}\mathcal{K})^3 + \cdots]\mathcal{L}^{-1} \\
&= [I + (\mathcal{L}^{-1}\mathcal{K})^2 + (\mathcal{L}^{-1}\mathcal{K})^4 + (\mathcal{L}^{-1}\mathcal{K})^6 + \cdots][I - \mathcal{L}^{-1}\mathcal{K}]\mathcal{L}^{-1} \\
&= \sum_{k=0}^{+\infty}(\mathcal{L}^{-1}\mathcal{K})^{2k}[\mathcal{L}^{-1} - \mathcal{L}^{-1}\mathcal{K}\mathcal{L}^{-1}].
\end{aligned}$$

It is easy to see that $\sum_{k=0}^{+\infty}(\mathcal{L}^{-1}\mathcal{K})^{2k}$ is positive due to the positivity of $\mathcal{L}^{-1}$ and $\mathcal{K}$. Therefore, we only need to show that $\mathcal{L}^{-1}-\mathcal{L}^{-1}\mathcal{K}\mathcal{L}^{-1}$ is positive. To this end, we show that for any $u \geq 0$, $\mathcal{L}^{-1}u \geq \mathcal{L}^{-1}\mathcal{K}\mathcal{L}^{-1}u$.

For any $w \in C[0,\pi]$,

$$(\mathcal{L}^{-1}w)(x) = \int_0^\pi G(x,y)w(y)\,dy,$$

and by the Fubini theorem,

$$\begin{aligned}(\mathcal{L}^{-1}\mathcal{K}\mathcal{L}^{-1}w)(x) &= -\int_0^\pi G(x,v)\left\{\eta\int_0^\pi f_\alpha(v,u)\left[\int_0^\pi G(u,y)w(y)dy\right]du\right\}dv\\ &= -\int_0^\pi\left\{\int_0^\pi G(x,v)\left[\eta\int_0^\pi f_\alpha(v,u)G(u,y)du\right]dv\right\}w(y)dy.\end{aligned}$$

Thus, we only need to show that for any $x, y \in [0,\pi]$,

$$-\int_0^\pi G(x,v)\left[\eta\int_0^\pi f_\alpha(v,u)G(u,y)du\right]dv \leq G(x,y). \tag{18}$$

Note that

$$\int_0^\pi f_\alpha(v,u)G(u,y)du = \frac{2}{\pi}\sum_{n=1}^{+\infty}e^{-n^2\alpha}\sin nv\left[\int_0^\pi \sin nuG(u,y)du\right].$$

It is known that the boundary value problem

$$\begin{cases} -\dfrac{d^2w}{dy^2}+k^2w = \sin ny\\ w(0)=w(\pi)=0,\end{cases} \tag{19}$$

has a unique solution which is given by

$$w(y) = \int_0^\pi \sin nuG(u,y)\,du$$

On the other hand, one can easily verify that $w(y) = \frac{1}{n^2+k^2}\sin ny$ satisfies (19). Thus,

$$\int_0^\pi \sin nuG(u,y)du = \frac{1}{k^2+n^2}\sin ny,$$

and therefore,

$$\int_0^\pi f_\alpha(v,u)G(u,y)du = \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{e^{-n^2\alpha}}{k^2+n^2}\sin nv\sin ny. \tag{20}$$

This leads to

$$\begin{aligned}&-\int_0^\pi G(x,v)\left[\eta\int_0^\pi f_\alpha(v,u)G(u,y)du\right]dv\\ &= -\eta\int_0^\pi G(x,v)\left[\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{e^{-n^2\alpha}}{k^2+n^2}\sin nv\sin ny\right]dv\\ &= -\frac{2\eta}{\pi}\sum_{n=1}^{+\infty}\frac{e^{-n^2\alpha}}{k^2+n^2}\sin ny\int_0^\pi G(x,v)\sin nvdv\\ &= -\frac{2\eta}{\pi}\sum_{n=1}^{+\infty}\frac{e^{-n^2\alpha}}{(k^2+n^2)^2}\sin nx\sin ny.\end{aligned}$$

Since

$$\int_0^\pi G(u,y)du = \int_0^\pi G(y,u)du = \frac{1}{k^2 \sinh k\pi}[\sinh k\pi - \sinh ky - \sinh k(\pi - y)],$$

we have

$$\begin{aligned}\frac{1}{k^2+n^2}\sin nx &= \int_0^\pi \sin nu G(u,x)du \le \int_0^\pi G(u,x)du \\ &= \frac{1}{k^2 \sinh k\pi}[\sinh k\pi - \sinh kx - \sinh k(\pi - x)].\end{aligned}$$

Therefore,

$$\begin{aligned}&-\int_0^\pi G(x,v)\left[\eta \int_0^\pi f_\alpha(v,u)G(u,y)du\right]dv \\ &\le -\frac{2\eta}{\pi}\frac{1}{k^4 \sinh^2 k\pi}[\sinh k\pi - \sinh kx - \sinh k(\pi - x)] \\ &\cdot [\sinh k\pi - \sinh ky - \sinh k(\pi - y)]\sum_{n=1}^{+\infty} e^{-n^2\alpha}.\end{aligned}$$

Set

$$c = \sqrt{\frac{-2\eta}{\pi}\frac{1}{k^3 \sinh k\pi}\sum_{n=1}^{+\infty} e^{-n^2\alpha}}.$$

We now prove that for any $x, y \in [0,\pi]$,

$$c[\sinh k\pi - \sinh kx - \sinh k(\pi - x)] \le \sinh kx, \tag{21}$$

and

$$c[\sinh k\pi - \sinh kx - \sinh k(\pi - x)] \le \sinh k(\pi - x). \tag{22}$$

Let $h(c,x) = (c+1)\sinh kx + c\sinh k(\pi - x) - c\sinh k\pi$. Simple calculations give $h'_x(c,x) = k(c+1)\cosh kx - kc\cosh k(\pi - x)$ and $h''_x(c,x) = k^2(c+1)\sinh kx + k^2 c\sinh k(\pi - x) \ge 0$ for all $x \in [0,\pi]$. Note that by (17) and Lemma 2.1-(ii), we have

$$c = \sqrt{\frac{-2\eta}{\pi}\frac{1}{k^3 \sinh k\pi}\sum_{n=1}^{+\infty} e^{-n^2\alpha}} < \sqrt{-\frac{2\eta}{\pi}\frac{1}{k^3 \sinh k\pi}\frac{1}{e^\alpha - 1}} < \frac{1}{2\sinh^2\frac{k\pi}{2}},$$

which implies that $h'_x(c,0) \ge 0$. This together with $h''_x(c,x) \ge 0$ further leads to $h'_x(c,x) \ge 0$ for all $x \in [0,\pi]$. From this and the fact that $h(c,0) = 0$, it follows that $h(c,x) \ge 0$ for all $x \in [0,\pi]$, confirming (21).

By a similar argument, we can prove (22). Therefore, we have proved that (18) holds, and thus, the operator $\mathcal{L}^{-1} - \mathcal{L}^{-1}\mathcal{K}\mathcal{L}^{-1}$ (hence $(\mathcal{L}+\mathcal{K})^{-1}$) is positive, completing the proof. □

Remark 2. The conditions (16) and (17) can hold simultaneously. To see this, we note that

$$\lim_{k\to 0}\frac{\sinh k\pi}{k\pi} = 1.$$

Thus, for any $\eta < 0$, we may take α such that

$$-\frac{\eta\pi^2}{2} < e^\alpha - 1.$$

Then,

$$-\frac{8\eta}{\pi}\frac{\sinh^4\frac{k\pi}{2}}{k^3\sinh k\pi}\frac{1}{e^\alpha-1}\to-\frac{\eta\pi^2}{2}\frac{1}{e^\alpha-1}<1,\ \text{ as }\ k\to 0$$

and

$$-\frac{4\eta}{(1+k^2)\pi}\frac{1}{e^\alpha-1}\to-\frac{4\eta}{\pi}\frac{1}{e^\alpha-1}<-\frac{\eta\pi^2}{2}\frac{1}{e^\alpha-1}<1,\ \text{ as }\ k\to 0,$$

implying that (16) and (17) both hold when $k>0$ is sufficiently small.

Remark 3. The main idea of the proof of Lemma 4.2 is essentially due to Freitas and Sweers [3], where the authors gave a general result on the Dirichlet boundary value problem of nonlocal elliptic equations in a bounded domain with dimension great than or equal to 3 . By using estimates of integral kernels involved, they gave some conditions which ensure that the monotonicity is preserved.

Now we are in a position to state and prove the main theorem of this section.

Theorem 4.3. *Suppose that $1+k^2<g(0)e^{-\alpha}$ and that there is a positive constant M_1 such that $k^2>g(M_1)\gamma_1$, where γ_1 is given by (14). If both (16) and (17) hold, then (4) has a unique positive solution.*

Proof. By the proof of Theorem 3.1, $w^{(l)}=\varepsilon\sin x$ is a sub-solution to (4) for sufficiently small $\varepsilon>0$. Also one can easily verify that $w^{(u)}=\hat{M}_1$ is a super solution. Define the nonlinear operator $\mathcal{T}:Y\to X$ by

$$(\mathcal{T}w)(x)=(\mathcal{L}+\mathcal{K})^{-1}\int_0^\pi f_\alpha(x,y)b_0(w(y))dy, \tag{23}$$

Since $b_0'(w)\geq 0$ for any $w\geq 0$, by Lemma 2.1, the operator $\tilde{\mathcal{S}}:Y\to X$ is positive and strongly monotone, where $\tilde{\mathcal{S}}$ is defined by

$$(\tilde{\mathcal{S}}w)(x)=\int_0^\pi f_\alpha(x,y)b_0(w(y))dy, \forall w\in Y. \tag{24}$$

This together with Lemma 4.2 implies that $\mathcal{T}$ is positive and strongly monotone. Employing a standard super and sub-solution argument, we know that (4) has a maximal positive solution and a minimal positive solution in the ordered interval $[w^{(l)},\hat{M}_1]$, denoted by $\bar{w}(x)$ and $\underline{w}(x)$ respectively.

Next, we prove the uniqueness of positive solution to (4) in the order interval $[w^{(l)},\hat{M}_1]$. Let w_0 be any positive solution to (4) satisfying $w^{(l)}\leq w_0\leq\hat{M}_1$. Then $w_0(x)\leq\bar{w}(x)$ for $x\in[0,\pi]$. If $w_0\neq\bar{w}$, then $w_0<\bar{w}$ in the sense of ordering in Banach space X.

Consider the eigenvalue problem

$$\begin{cases} -\dfrac{d^2w}{dx^2}+k^2w=\lambda\displaystyle\int_0^\pi g(\bar{w}(y))f_\alpha(x,y)w(y)dy \\ w(0)=w(\pi)=0. \end{cases} \tag{25}$$

Let $\mathcal{S}_1:Y\to X$ be a linear operator defined by

$$(\mathcal{S}_1w)(x)=\int_0^\pi g(\bar{w}(y))f_\alpha(x,y)w(y)dy, \forall w\in Y, \tag{26}$$

and $\mathcal{T}_1:Y\to X$ defined by $\mathcal{T}_1=\mathcal{L}^{-1}\mathcal{S}_1$. Clearly, $\mathcal{T}_1$ is a strongly positive compact endomorphism of $C_{\mathbf{e}}[0,\pi]$ (see the proof of Theorem 2.2). Again by Theorem 3.2 in [1], the spectral radius $r(\mathcal{T}_1)$ is the only eigenvalue having positive eigenvector.

It follows that $r(\mathcal{T}_1) = 1$ since $\bar{w}$ is a positive eigenvector corresponding to the eigenvalue 1 of the eigenvalue problem (25).

Similarly, consider the eigenvalue problem

$$\begin{cases} -\dfrac{d^2w}{dx^2} + k^2 w = \lambda \displaystyle\int_0^\pi g(w_0(y)) f_\alpha(x,y) w(y) dy \\ w(0) = w(\pi) = 0. \end{cases} \tag{27}$$

Define the operator $\mathcal{S}_2$ by

$$(\mathcal{S}_2 w)(x) = \int_0^\pi g(w_0(y)) f_\alpha(x,y) w(y), \forall w \in Y, \tag{28}$$

and let $\mathcal{T}_2 = \mathcal{L}^{-1}\mathcal{S}_2$. Then $\mathcal{T}_2$ is also a strongly positive compact endomorphism of $C_{\mathbf{e}}[0,\pi]$. Since w_0 is a positive eigenvector corresponding to the eigenvalue 1 of the eigenvalue problem (28), we get $r(\mathcal{T}_2) = 1$. However, since $w_0 < \bar{w}$, by (H1), we get $g(w_0(x)) \geq g(\bar{w}(x))$ but $g(w_0(x)) \neq g(\bar{w}(x))$ on $[0,\pi]$. Therefore, $\mathcal{S}_2 w > \mathcal{S}_1 w$ for any $w \in Y$, implying that $\mathcal{T}_2 w > \mathcal{T}_1 w$ for any $w \in Y$. From the monotonicity of the spectral radius, it follows that $1 = r(\mathcal{T}_2) > r(\mathcal{T}_1) = 1$, which is a contradiction. Therefore, we have $w_0(x) \equiv \bar{w}(x)$ for all $x \in [0, L]$, i.e., $w_0 = \bar{w}$, proving uniqueness of positive solution of (4) in the order interval $[w^{(l)}, \hat{M}_1]$. Because $\varepsilon > 0$ is arbitrary, we have actually shown that (4) has a unique positive solution in the order interval $[0, \hat{M}_1]$.

By the fact that g is decreasing, we can exclude positive solutions of (4) beyond the interval $[0, \hat{M}_1]$. Otherwise, we assume that there is a positive solution to (4), say $\tilde{w}$, satisfying $\max_{x\in[0,\pi]} \tilde{w}(x) > M_1$. Let $M_2 = \max_{x\in[0,\pi]} \tilde{w}(x)$. Then $M_2 > M_1$ and hence $g(M_2)\gamma_1 < g(M_1)\gamma_1 < k^2$. Replacing M_1 by M_2 in the above proof, we can actually conclude that (4) has a unique positive solution in the interval $[0, \hat{M}_2]$. But now, both $\bar{w}$ and $\tilde{w}$ are located in $[0, \hat{M}_2]$, a contradiction, which implies the uniqueness. The proof is completed. □

Remark 4. In order to obtain a positive solution to (4) in the ordered interval $[w^{(l)}, \hat{M}_1]$, we do not need η to be the minimum of $b'(w)$ in the whole half line $w \geq 0$. In fact, if we choose $\eta = \min_{w\in[0,M_1]} b'(w)$, then the conclusion of Theorem 4.3 still holds except for the result on the global uniqueness. Also, from the proof of Theorem 4.3, we see that this theorem is still valid if γ_1 is replaced by γ. Since γ_1 has an explicit formula, here and in the sequel, we use γ_1.

By Remark 4, if b is increasing in the interval $[0, M_0]$ and $b(M_0) = \sup_{w\in[0,\infty)} b(w)$, we may take $\eta = 0$. Then (16) and (17) hold automatically. Furthermore, if $1 + k^2 < g(0)e^{-\alpha}$ (necessary condition) and $k^2 > g(M_0)\gamma_1$, then (4) has a unique positive solution. This is just the conclusion of Corollary 1. Since $g(w)$ is decreasing (by (H1)), there may occur the case that $k^2 \leq g(M_0)\gamma_1$ but $k^2 \geq g(M_1)\gamma_1$ for some $M_1 > M_0$. This is true especially when $\lim_{w\to\infty} g(w) = 0$. For such a truly non-monotone case, we have the following corollary.

Corollary 3. *Suppose that there is a positive constant M_0, such that $b'(w) > 0$, for $w < M_0$ and $b'(w) < 0$ for $w > M_0$. Assume, in addition to (H1), that $\lim\limits_{w\to+\infty} g(w) = 0$. If the following inequalities*

$$-\frac{\eta\pi^2}{2}\frac{1}{e^\alpha - 1} < 1, \tag{29}$$

$$1 < g(0)e^{-\alpha} \tag{30}$$

hold simultaneously, then for sufficiently small k, (4) has a unique positive solution.

Proof. By (29), (30) and Remark 2, inequalities (16), (17) and $1 + k^2 < g(0)e^{-\alpha}$ hold simultaneously for sufficiently small k. Since $\lim_{w\to+\infty} g(w) = 0$, for such an appropriately chosen k, there exists $M_1 > M_0$, such that $k^2 > g(M_1)\gamma_1$. By Theorem 4.3, there exists a unique positive solution to (4), and the proof is completed. □

5. **Examples.** In this section, we present two examples to illustrate the applicability of our main results.

First we consider the following equation

$$\frac{\partial w(t,x)}{\partial t} = D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + \int_0^\pi f_\alpha(x,y)b_1(w(t-\tau,y))dy, \quad t \geq 0, \quad x \in (0,\pi) \tag{31}$$

$b_1(w) = \varepsilon pwe^{-qw}$ which is referred to as the Ricker's birth function in population dynamics. Here ε accounts for the probability that newly born individual can survive the immature period of length τ, and hence is of the form $\varepsilon = e^{-\delta\tau}$ with δ being the death rate of immature population (see,[6, 10]).

Transforming the steady state equation of (31), i.e.,

$$-D\frac{d^2w}{dx^2} + dw = \int_0^\pi f_\alpha(x,y)b_1(w(y))dy,$$

into the form of (4), we then obtain $k = \sqrt{d/D}$ and $b(w) = b_1(w)/D = wg(w)$ with $g(w) = (\varepsilon p/D)e^{-qw}$. Calculations show that $g(0) = \varepsilon p/D$, $b'(w) = (\varepsilon p/D)(1 - qw)e^{-qw}$, $b''(w) = (\varepsilon pq/D)(qw-2)e^{-qw}$. Thus, $b'(2/q) = \min_{w\in[0,+\infty)} b'(w) = -\varepsilon p/(De^2)$, $b(1/q) = \max_{w\in[0,+\infty)} b(w) = \varepsilon p/(qeD)$. So we can take $\eta = -\varepsilon p/(De^2)$ and $M_0 = 1/q$.

Combining the above with Theorem 2.2 and Corollary 2, we have

Proposition 1. *If $1 + \frac{d}{D} \geq \frac{\varepsilon p}{D}e^{-\alpha}$, then there is no positive steady state for (5) subject to the homogeneous Dirichlet boundary condition; if $1 + \frac{d}{D} < \frac{\varepsilon p}{D}e^{-\alpha}$, and $d > \frac{2\varepsilon pe^{\alpha-1}}{\pi}\ln\frac{e^{2\alpha}+1}{e^{2\alpha}-1}$, then there exists a unique positive steady state.*

Applying Corollary 3 to this example, we have

Proposition 2. *If $e^\alpha < \frac{\varepsilon p}{D} < \frac{2e^2}{\pi^2}(e^\alpha - 1)$, then for sufficiently small $\frac{d}{D}$, there exists a unique positive steady state for (5) subject to the homogeneous Dirichlet boundary condition.*

The second example is the following nonlocal reaction diffusion equation

$$\frac{\partial w(t,x)}{\partial t} = D\frac{\partial^2 w(t,x)}{\partial x^2} - dw(t,x) + \int_0^\pi f_\alpha(x,y)b_2(w(t-\tau,y))dy, \quad t \geq 0, \quad x \in (0,\pi) \tag{32}$$

where $b_2(w) = \frac{pw}{q+w^m}$, $m > 0, p > 0, q > 0$. This nonlinear function $b_2(w)$ was used as the production function for blood cells in [7], and has since been widely adopted. Again, transforming the steady state equation of (5.1) into the form of (1.4), we obtain $k = \sqrt{\frac{d}{D}}$, $b(w) = wg(w)$, $g(w) = \frac{p}{D(q+w^m)}$. Clearly, $g(0) = \frac{p}{qD}$, $b'(w) = \frac{p}{D}\frac{(1-m)w^m+q}{(q+w^m)^2}$.

The parameter $m > 0$ has a qualitative impact on the shape of $b(w)$. If $m \leq 1$, then $b'(w) \geq 0$ for $w \geq 0$. In this case, one can easily study the existence and uniqueness of positive steady state of (32) by monotone iteration techniques.

If $m > 1$, then $b'(w) > 0$ for $w < M_0$ and $b'(w) < 0$ for $w > M_0$ where $M_0 = (\frac{q}{m-1})^{1/m}$. Since

$$b''(w) = \frac{pmw^{m-1}}{D(m-1)(q+w^m)^3}\left[w^m - \frac{q(m+1)}{m-1}\right],$$

the derivative function $b'(w)$ attains its minimum at $w = M_1 = \left(\frac{q(m+1)}{m-1}\right)^{1/m}$. Hence we can choose $\eta = b'(M_1) = -\frac{(m-1)^2 p}{4mqD}$. By Theorem 2.2 and Corollary 2, we then have the following result for (32) subject to the homogeneous Dirichlet boundary condition.

Proposition 3. *Assume that $m > 1$. If $1 + \frac{d}{D} \geq \frac{p}{qD}e^{-\alpha}$, then there is no positive steady state for (32) subject to the homogeneous Dirichlet boundary condition; if $1 + \frac{d}{D} < \frac{p}{qD}e^{-\alpha}$ and $d > \frac{2p(m-1)e^{\alpha}}{qm\pi}\ln\frac{e^{2\alpha}+1}{e^{2\alpha}-1}$, then (32) has a unique positive steady state subject to the homogeneous Dirichlet boundary condition.*

Applying Corollary 3 to (32), we have

Proposition 4. *Assume $m > 1$. If $e^{\alpha} < \frac{p}{qD} < \frac{2e^2}{\pi^2}(e^{\alpha}-1)$, then for sufficiently small $\frac{d}{D}$, there exists a unique positive steady state for (32) subject to the homogeneous Dirichlet boundary condition.*

Remark 5. The sufficient conditions in Proposition 1-4 are feasible in the sense that within certain ranges of parameters, these conditions will be satisfied. For example, one can verify that the conditions in the second half of Proposition 1 (Proposition 3 as well) are satisfied when $D = 0.1, d = 0.2, \alpha = 1$ and $\varepsilon p = 1$ ($p/q = 1, \forall m > 1$). Feasibility of the sufficient conditions in Proposition 2 and Proposition 4 can also be easily checked.

Acknowledgments. This work was completed when ZG and ZY were visiting the University of Western Ontario. They would like to thank the staff in the Department of Applied Mathematics for their friendly and helpful assistance during their visit, and thank the university for the support from its Visiting University Scholar Program.

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Received March 2011; revised September 2011.

E-mail address: guozm@gzhu.edu.cn
E-mail address: zhichy@yahoo.com.cn
E-mail address: xzou@uwo.ca