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Generic Quasi-Convergence for Essentially Strongly Order-Preserving Semiflows

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Abstract. By employing the limit set dichotomy for essentially strongly order-preserving semiflows and the assumption that limit sets have infima and suprema in the state space, we prove a generic quasi-convergence principle implying the existence of an open and dense set of stable quasi-convergent points. We also apply this generic quasi-convergence principle to a model for biochemical feedback in protein synthesis and obtain some results about the model which are of theoretical and realistic significance.

1 Introduction

Hirsch [2] showed that almost all orbits of a cooperative and irreducible system of ordinary differential equations tend to the set of equilibria. Such a generic quasi-convergence property was extended by Hirsch [3] to strongly monotone semiflows in strongly ordered spaces. Matano [7, 8] obtained results parallel to Hirsch's. The results of Hirsch and Matano were later improved by Poláčik [9], Smith and Thieme [14, 15] and Takáč [16]. For a semiflow, the generic quasi-convergence property usually requires two types of conditions on the system:

(I) a certain type of monotonicity on the semiflow;
(II) some compactness assumption on the semiflow.

Along the direction of (I), Yi and Huang [18] recently considered a class of the so-called *essentially* strongly order-preserving semiflows, and extended several principles for strongly order-preserving semiflows to this class of semiflows. By these principles and certain compactness hypotheses, they were able to obtain some results on convergence, quasi-convergence, and stability for such semiflows. When applying the results to a quasi-monotone system of delay differential equations, they observed an obvious advantage: this new order-preserving property for the systems (*i.e.,* essentially strongly order-preserving property) neither requires a dedicated choice of state space nor the technical ignition assumption required in the classical works (see, *e.g.,* Smith [13] for details).

For (II), Hirsch and Smith [4] have recently shown that the strong compactness assumptions on strongly order-preserving semiflows required for the proof of the generic quasi-convergence principle in the aforementioned works can be replaced by the assumption that limit sets have infima and suprema in the state space. As pointed

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out by Hirsch and Smith [4], every compact subset of the space of continuous functions on a compact set with the usual ordering has an infimum and a supremum. Hence, the assumption that limit sets have infima and suprema in the state space is automatically satisfied when the generic quasi-convergence principle in Hirsch and Smith [4] is applied to cooperative and irreducible systems of delay differential equations in the sense of Smith [12]. This means that the difficulty in verifying compactness assumptions in previous works (*i.e.,* Smith and Thieme [15]) can be avoided for systems with bounded delays. For details on this topic, we refer to the monographs by Hirsch and Smith [5], Smith [13] and Wu [17].

This work is motivated by [4] and [18]. The goal is to investigate whether or not the generic quasi-convergence property still holds for essentially strongly order-preserving if the compactness condition is in the sense of [4]. More precisely, we employ the limit set dichotomy for essentially strongly order-preserving semiflows and the assumption that limit sets have infima and suprema in the state space to prove a generic quasi-convergence principle, which naturally has the advantages of both [4] and [18], and thus is of theoretical and realistic significance for various systems with bounded delays.

The rest of this paper is organized as follows. In Section 2, we state some preliminary results from Yi and Huang [18], which also serve as preparation for the statement and proof of the main result. In Section 4, we establish a new generic quasi-convergence principle for essentially strongly order-preserving semiflows. As an illustration, in Section 4 we apply our generic quasi-convergence principle to a model of biochemical feedback in protein synthesis.

2 Preliminary Results

Let X be an ordered metric space with metric d and a closed partial order relation $R \subseteq X \times X$. For any $x, y \in X$, we write $x \leq y$ if and only if $(x, y) \in R$, and $x < y$ if and only if $(x, y) \in R$ and $x \neq y$. Given two subsets A and B of X, we write $A \leq B$ ($A < B$, respectively) whenever $x \leq y$ ($x < y$, respectively) for each choice of $x \in A$ and $y \in B$. For $x \in X$ and $A \subset X$, we have $x \leq A$ if and only if $x \leq a$ for every $a \in A$.

We assume that $\Phi \colon X \times R^1_+ \to X$ is a semiflow on X, that is, Φ is continuous and $\Phi_t(x) \equiv \Phi(x, t)$, satisfying

(i) $\Phi_0(x) = x$ for all $x \in X$;
(ii) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $x \in X$ and $t, s \in R^1_+$.

For $x \in X$, let $O(x) = \{\Phi_t(x) : t \geq 0\}$. If $\overline{O(x)}$ is compact, we define $\omega(x) = \bigcap_{t \geq 0} \overline{O(\Phi_t(x))}$. As is well known, $\omega(x)$ is nonempty, compact, connected, and invariant. Let $E = \{x \in X : \Phi_t(x) = x,\ t \geq 0\}$ be the set of equilibria of Φ. The set of quasi-convergent points is denoted by $Q = \{x \in X : \omega(x) \subset E\}$ and the set of convergent points by $C = \{x \in X : \omega(x) \text{ is a singleton set}\}$.

The semiflow Φ is said to be a continuous monotone semiflow on X, that is, Φ is a continuous semiflow on X and whenever $x, y \in X$ with $x \leq y$, we have $\Phi_t(x) \leq \Phi_t(y)$ for all $t \geq 0$. For any $x, y \in X$ and some constant $t_0 \geq 0$, we write $x \preceq_{t_0} y$ if and only if there exist $\widetilde{x}, \widetilde{y} \in X$ with $\widetilde{x} \leq \widetilde{y}$ such that $\Phi_{t_0}(\widetilde{x}) = x$ and $\Phi_{t_0}(\widetilde{y}) = y$. We shall write "$\preceq$" for "$\preceq_{t_0}$" when no confusion results. We also write $x \prec y$ if and only if $x \preceq y$ and $x \neq y$. For $A,\ B \subset X$, we write $A \preceq B$ to mean "$a \preceq b$ for any $a \in A$ and $b \in B$", and $A \prec B$ to mean "$a \prec b$ for any $a \in A$ and $b \in B$".

The following definition can be found in Yi and Huang [18].

Definition 2.1 The semiflow Φ is said to be essentially strongly order-preserving if for any $x,\ y \in X$ with $x \prec y$, there exist open sets U and V, and some constant $T_0 \geq 0$ such that $x \in U$, $y \in V$ and $\Phi_{T_0}(U) \leq \Phi_{T_0}(V)$.

Remark 2.2. As pointed out in [18], if $T_0 = 0$, then a monotone and essentially strongly order-preserving semiflow is just strongly order-preserving in the sense of Smith and Thieme [15].

Throughout the remainder of this paper, we always assume that the semiflow Φ is monotone and essentially strongly order-preserving on X.

The proofs of the following several results can be found in Yi and Huang [18].

Proposition 2.3 *If $z \in X$, $x \in \omega(z)$ and $x \leq \omega(z)$ ($\omega(z) \leq x$), then $\omega(z) = \{x\}$.*

Proposition 2.4 *Let K and H be two compact subsets of X satisfying $K \prec H$. Then there are two open sets U and V, $K \subset U$, $H \subset V$, and $T_0 \geq 0$, $\varepsilon > 0$ such that*

$$\Phi_{T_0+s}(U) \leq \Phi_{T_0}(V) \quad \textit{and} \quad \Phi_{T_0}(U) \leq \Phi_{T_0+s}(V), \quad 0 \leq s \leq \varepsilon.$$

***Theorem 2.5* (Limit set dichotomy)** *Let $x, y \in X$ satisfy $x \prec y$. Then one of the following holds:*

(i) $\omega(x) < \omega(y)$;

(ii) $\omega(x) = \omega(y) \subset E$.

3 Generic Quasi-Convergence

Assume that $A \subseteq X$ and denote by $L = \{x \in X : x \leq A\}$ the set of the lower bounds of A in X. If there exists $u \in X$ such that $u \in L$ with $L \leq u$, then u is the infimum of A and we denote it by $\inf A$. Similarly, we can define the supremum of A, namely, $\sup A$. It should be pointed out that $\inf A$ and $\sup A$ do not necessarily exist. For $x \in X$, we say that x can be essentially approximated from below (above) in X if for any neighborhood U of x, there exists $y \in U$ such that $y \prec x$ ($x \prec y$). For $p \in E$, we define $C(p) = \{x \in X : \omega(x) = \{p\}\}$. One can observe that $C = \bigcup_{p\in E} C(p)$.

Throughout the rest of this paper, we always assume that every orbit of the semiflow Φ has compact closure in X. Moreover, we introduce the following assumptions.

(ELI) There exists an open and dense subset X_0 of X such that every point of X_0 can be essentially approximated from below and its omega limit set has the infimum.

(ELS) There exists an open and dense subset X_0 of X such that every point of X_0 can be essentially approximated from above and its omega limit set has the supremum.

The following two key lemmas are an improvement of [4, Lemma 3.2].

Lemma 3.1 *Assume that* (ELI) *holds, and let $x \in X_0 \setminus Q$ and $a = \inf\omega(x)$. Then $\omega(a) = \{p\} < \omega(x)$ and $x \in \overline{\operatorname{Int} C(p)}$.*

Proof Suppose that M is a neighborhood of x in X. By Proposition 2.3, we have $a < \omega(x)$. It then follows from the invariance of $\omega(x)$ that $\Phi_t(a) \le \omega(x)$, and hence $\Phi_t(a) \le a$ for all $t \in R^1_+$. So, by [13, Theorem 1.2.1], there exists an equilibrium $p \le a$ such that $\omega(a) = \{p\}$. From $p \le a < \omega(x)$ and Proposition 2.4, it follows that there exist a neighborhood N of $\omega(x)$ and a $T_0 > 0$ such that $p \le \Phi_t(N)$ for all $t \ge T_0$. The definition of $\omega(x)$ implies that there exists $T_1 > 0$ such that $\Phi_{T_1}(x) \in N$ and thus, $p \le \Phi_t(x)$ for all $t \ge T_0 + T_1$. Let

$$V = M \cap (\Phi_{T_1})^{-1}(N).$$

Then V is a neighborhood of x in M and we also have $p \le \Phi_t(V)$ for all $t \ge T_0 + T_1$. This implies that $p \le \omega(v)$ for all $v \in V$. Since x can be essentially approximated from below, it follows that there exists $y \in V$ such that $y \prec x$. Applying Theorem 2.5 and the fact that $x \notin Q$, we obtain $\omega(y) < \omega(x)$. Therefore, by Proposition 2.4, there exist a neighborhood $\widetilde{U}$ of $\omega(y)$ in X and a $T_2 > 0$ such that $\Phi_t(\widetilde{U}) \le \omega(x)$ for all $t \ge T_2$. Again by the definition of $\omega(y)$, there exists $T_3 > 0$ such that $\Phi_{T_3}(y) \in \widetilde{U}$. Let

$$U = (\Phi_{T_3})^{-1}(\widetilde{U}) \cap V.$$

Then U is a neighborhood of y in V and we also have $\Phi_t(U) \le \omega(x)$ for all $t \ge T_2 + T_3$. So, we get $\omega(u) \le \omega(x)$ for all $u \in U$. Thus, $\omega(u) \le a$ for all $u \in U$, and hence $\omega(u) \le \omega(a) = \{p\}$. Again since $p \le \omega(v)$ for all $v \in V$, we have $\omega(u) = \{p\}$ for all $u \in U$. Therefore $U \subseteq C(p) \bigcap M$, hence $x \in \overline{\operatorname{Int} C(p)}$. This completes the proof. ■

Arguing as in the proof of Lemma 3.1, we can also obtain the following result.

Lemma 3.2 *Assume that* (ELS) *holds, and let* $x \in X_0 \setminus Q$ *and* $a = \sup \omega(x)$. *Then* $\omega(a) = \{p\} > \omega(x)$ *and* $x \in \overline{\operatorname{Int} C(p)}$.

Applying Lemmas 3.1 and 3.2 and arguing almost precisely as in the proof of [18, Theorem 3.1], we can obtain the following generic quasi-convergence principle.

Theorem 3.3 *Assume that either* (ELI) *or* (ELS) *holds. Then* $X_0 \setminus Q \subset \overline{\operatorname{Int} C}$ *and the set* Int Q *is dense.*

Proof We may assume that (ELI) ((ELS) resp.) holds. From Lemma 3.1 (Lemma 3.2 resp.), it follows that $X_0 \setminus Q \subseteq \overline{\operatorname{Int} C}$, which implies $X_0 \subseteq Q \cup \overline{\operatorname{Int} Q}$. This proves $X_0 \setminus \overline{\operatorname{Int} Q} \subseteq \operatorname{Int} Q$, and hence $X_0 \subseteq \overline{\operatorname{Int} Q}$. Therefore $X = \overline{X_0} \subseteq \overline{\operatorname{Int} Q}$, completing the proof. ■

4 An Application

In this section, as an application of Theorem 3.3, we consider a class of well-known systems of delay differential equations.

Let $r > 0$ be given and let $C^{(n)} = C([-r, 0], R^n)$ be the Banach space of continuous mappings from $[-r, 0]$ into R^n, equipped with the usual supremum norm. Define $C^{(n)}_+ = C([-r, 0], R^n_+)$. Note that $C^{(n)}_+$ is an order cone in $C^{(n)}$ and induces the usual pointwise ordering: for $\varphi, \psi \in C^{(n)}$, we denote

(i) $\varphi \leq \psi$ if and only if $\psi - \varphi \in C_+^{(n)}$;
(ii) $\varphi < \psi$ if and only if $\varphi \leq \psi$ and $\varphi \neq \psi$;
(iii) $\varphi \ll \psi$ if and only if $\psi - \varphi \in \mathrm{Int}(C_+^{(n)}$.

If $\sigma > 0$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in C([-r, \sigma], R^n)$, then for any $t \in [0, \sigma]$, we let $\varphi_{i,t} \in C([-r, \sigma], R^n)$ be defined by $\varphi_{i,t}(\theta) = \varphi_i(t + \theta)$, $-r \leq \theta \leq 0$. Consider the following biochemical feedback system

$$\begin{cases} x_1'(t) = f(x_{n,t}) - \alpha_1 x_1(t), \\ x_i'(t) = L_{i-1}(x_{i-1,t}) - \alpha_i x_i(t), \quad 2 \leq i \leq n, \end{cases} \tag{4.1}$$

where $\alpha_i > 0$, $L_i(\phi) = \int_{-r}^{0} \phi(\theta) d\eta_i(\theta)$, $\eta_i \colon [-r, 0] \to R^1$ is nondecreasing, $\eta_i(-r) = 0$, and $\eta_i(0) > 0$.

We say that f is essentially cooperative and irreducible on the ordered Banach space $(C^{(1)}, C_+^{(1)})$ in the sense of Yi and Huang [18] if f satisfies the following conditions.

(i) For any $\varphi \in C_+^{(1)}$ with $\varphi(0) = 0$, $f'(\psi)(\varphi) \geq 0$ for all $\psi \in C^{(1)}$.
(ii) $f'(\psi)\widehat{1} > 0$, where $\widehat{1}$ is a constant equal to 1 in $C^{(1)}$.

We can verify that system (4.1) satisfies the assumptions (H) and (I) in Smith [11] but generally fails to satisfy the ignition assumption (R) in [11] (see [18] for more related discussion and example). However, system (4.1) is essentially cooperative and irreducible in the sense of Yi and Huang [18].

In what follows, we assume that for every initial function $\varphi \in C^{(n)}$, (4.1) has a unique solution satisfying the initial condition and existing on R_+^1. For $\varphi \in C^{(n)}$, we use $x_t(\varphi)$ to denote the solution of (4.1) with the initial value $x_0(\varphi) = \varphi$. By [18, Theorem 2.1], $x_t(\,\cdot\,)$ generates an essentially strongly order-preserving semiflow, that is, if $\varphi, \psi \in C^{(n)}$, $\varphi < \psi$ and $t \geq r$, then either $x_t(\varphi) = x_t(\psi)$ or $x_t(\varphi) \ll x_t(\psi)$. Moreover, if $\varphi \ll \psi$, then $x_t(\varphi) \ll x_t(\psi)$ for all $t \geq r$. Hence, every point of $C^{(n)}$ can be essentially approximated from above (below).

System (4.1) has been used to model biochemical feedback in protein synthesis and has been investigated by many authors (see, *e.g.,* [1, 6, 10]). Our next result can be considered as an improvement of a result in Smith [11] where the author showed that choosing a state space properly is necessary for eventually strong monotonicity. But the result established in this paper neither requires the delicate choice of state space nor the technical ignition assumption, and hence gives more easily verifiable conditions and a wider range of applications.

Theorem 4.1 *Suppose f is completely continuous and suppose that each solution of* (4.1) *is bounded. Then there exists an open and dense subset in $C^{(n)}$ such that each orbit with the initial function in the subset converges to the set of equilibria of* (4.1).

Proof By [18, Theorem 2.1], the solution semiflow $x_t(\cdot)$ is essentially strongly order-preserving. According to Hirsch and Smith [4], every compact subset has an infimum and supremum in $(C^{(n)}, C_+^{(n)})$. Hence the hypotheses of Theorem 4.1 are satisfied, and the conclusion follows. ■

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