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GLOBAL STABILITY IN A MODEL FOR INTERACTIONS BETWEEN TWO STRAINS OF HOST AND ONE STRAIN OF PARASITE

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ABSTRACT. We prove a stronger version of the conjecture from [**1**] by which a model for interactions between two host strains and one parasite strain is shown to have global threshold dynamics in terms of the model's basic reproduction number $\mathcal{R}_1$: when $\mathcal{R}_1 < 1$, the parasite-free equilibrium U_0 for the model (1.2) is globally asymptotically stable; when $\mathcal{R}_1 > 1$, U_0 becomes unstable and there exists a unique positive equilibrium $\bar{U}$ for the model which is globally asymptotically stable.

1 Introduction In a recent work [**1**], a model was proposed to describe the interactions between two strains of host and two strains of parasite. The model is given by the following system of ordinary differential equations

$$
(1.1) \qquad \begin{cases} \dfrac{d}{dt}S_k = \Lambda_k - \rho_{k1}P_1S_k - \rho_{k2}P_2S_k - \mu S_k, \\ \dfrac{d}{dt}I_{k1} = \rho_{k1}P_1S_k - \rho'_{k2}P_2I_{k1} - (\mu + \delta_{k1})I_{k1}, \\ \dfrac{d}{dt}I_{k2} = \rho_{k2}P_2S_k - \rho'_{k1}P_1I_{k2} - (\mu + \delta_{k2})I_{k2}, \\ \dfrac{d}{dt}I_{k12} = \rho'_{k1}P_1I_{k2} + \rho'_{k2}P_2I_{k1} - (\mu + \delta_{k12})I_{k12}, \\ P_i = \displaystyle\sum_{k=a,b}(c_{ki}I_{ki} + d_{ki}I_{k12}), \quad k = a, b;\ i = 1, 2. \end{cases}
$$

Here S_a and S_b are the populations of the susceptible host of genotypes a and b respectively; I_{ki} is the population of host of genotype k infected

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by the parasite of strain i for $k = a, b$ and $i = 1, 2$; I_{k12} is the host population of genotype k infected by both parasite of both strains; P_i is the density of parasite of strain i for $i = 1, 2$. The parameters in the model are self-explanatory but a reader is referred to [**1**] for details, especially for the underlining assumptions for this model.

The main concern of [**1**] is the evolutionary implications of the interactions between the two host strains and the two parasite strains. To explore the topic, the authors of [**1**] started with a reduced model with a single-strain of parasite but with two strains of host, which is obtained by setting $I_{k2} = 0$ and $I_{k12} = 0$ in (1.1) (assuming the second strain of parasite is absent). This reduced model is give by the following system:

$$
\begin{cases}
\frac{d}{dt}S_a = \Lambda_a - \rho_{a1}P_1S_a - \mu S_a, \\
\frac{d}{dt}S_b = \Lambda_b - \rho_{b1}P_1S_b - \mu S_b, \\
\frac{d}{dt}I_{a1} = \rho_{a1}P_1S_a - (\mu + \delta_{a1})I_{a1}, \\
\frac{d}{dt}I_{b1} = \rho_{b1}P_1S_b - (\mu + \delta_{b1})I_{b1}, \\
P_1 = c_{a1}I_{a1} + c_{b1}I_{b1}.
\end{cases}
\tag{1.2}
$$

Let $\mathcal{R}_1$ be the basic reproduction number for parasite strain 1, given by

$$
\mathcal{R}_1 = \frac{c_{a1}\rho_{a1}}{\mu + \delta_{a1}}\frac{\Lambda_a}{\mu} + \frac{c_{b1}\rho_{b1}}{\mu + \delta_{b1}}\frac{\Lambda_b}{\mu}.
$$

The authors of [**1**] obtained the following result.

Result 2 in [1]: *The parasite-free equilibrium $U_0 = (\Lambda_a/\mu, \Lambda_b/\mu, 0, 0)$ for model (1.2) is globally asymptotically stable if $\mathcal{R}_1 < 1$, and unstable if $\mathcal{R}_1 > 1$. The interior equilibrium $\overline{U} = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$ exists and is unique if and only if $\mathcal{R}_1 > 1$. Moreover, $\overline{U}$ is stable if the following conditions (conditions (34) in [**1**]) hold:*

$$
C_1C_2 - C_3 > 0, \qquad C_1C_2C_3 - C_3^2 - C_1^2C_4 > 0. \tag{H}
$$

where C_i, $i = 1, 2, 3$ depend on the parameters in the model by the following lengthy formulas.

$$
C_1 = 2\mu + \mu_{\delta a} + \mu_{\delta b} + \bar{P}_1\rho_{a1} + \bar{P}_1\rho_{b1} - c_{a1}\rho_{a1}\bar{S}_a - c_{b1}\rho_{b1}\bar{S}_b,
$$

$$
C_2 = \mu^2 + 2\mu\mu_{\delta_a} + 2\mu\mu_{\delta_b} + \mu_{\delta_a}\mu_{\delta_b} + \mu\bar{P}_1\rho_{a1} + \mu_{\delta_a}\bar{P}_1\rho_{a1}
$$

$$+ \mu_{\delta_b}\bar{P}_1\rho_{a1} + \mu\bar{P}_1\rho_{b1} + \mu_{\delta_a}\bar{P}_1\rho_{b1} + \mu_{\delta_b}\bar{P}_1\rho_{b1} + \bar{P}^2\rho_{a1}\rho_{b1}$$
$$- 2c_{a1}\mu\rho_{a1}\bar{S}_a - c_{a1}\mu_{\delta_b}\rho_{a1}\bar{S}_a - c_{a1}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_a$$
$$- 2c_{b1}\mu\rho_{b1}\bar{S}_b - c_{b1}\mu_{\delta_a}\rho_{b1}\bar{S}_b - c_{b1}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_b,$$
$$\begin{aligned}C_3 = {} & \mu^2\mu_{\delta_a} + \mu^2\mu_{\delta_b} + 2\mu\mu_{\delta_a}\mu_{\delta_b} + \mu\mu_{\delta_a}\bar{P}_1\rho_{a1} + \mu\mu_{\delta_b}\bar{P}_1\rho_{a1} \\ & + \mu_{\delta_a}\mu_{\delta_b}\bar{P}_1\rho_{a1} + \mu\mu_{\delta_a}\bar{P}_1\rho_{b1} + \mu\mu_{\delta_b}\bar{P}_1\rho_{b1} \\ & + \mu_{\delta_a}\mu_{\delta_b}\bar{P}_1\rho_{b1} + \mu_{\delta_a}\bar{P}_1^2\rho_{a1}\rho_{b1} + \mu_{\delta_b}\bar{P}_1^2\rho_{a1}\rho_{b1} - c_{a1}\mu^2\rho_1\bar{S}_a \\ & - 2c_{a1}\mu_{\delta_b}\rho_{a1}\bar{S}_a - c_{a1}\mu\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_a - c_{a1}\mu_{\delta_b}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_a \\ & - c_{b1}\mu^2\rho_{b1}\bar{S}_b - 2c_{b1}\mu\mu_{\delta_a}\rho_{b1}\bar{S}_b - c_{b1}\mu\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_b \\ & - c_{b1}\mu_{\delta_a}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_b,\end{aligned}$$
$$\begin{aligned}C_4 = {} & \mu^2\mu_{\delta_a}\mu_{\delta_b} + \mu\mu_{\delta_a}\mu_{\delta_b}\bar{P}_1\rho_{a1} + \mu\mu_{\delta_a}\mu_{\delta_b}\bar{P}_1\rho_{b1} + \mu_{\delta_a}\mu_{\delta_b}\bar{P}_1^2\rho_{a1}\rho_{b1} \\ & - c_{a1}\mu^2\mu_{\delta_b}\rho_{a1}\bar{S}_a - c_{a1}\mu\mu_{\delta_b}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_a - c_{b1}\mu^2\mu_{\delta_a}\rho_{b1}\bar{S}_b \\ & - c_{b1}\mu\mu_{\delta_a}\bar{P}_1\rho_{a1}\rho_{b1}\bar{S}_b.\end{aligned}$$

This result was used in [**1**] to further consider possible establishment of parasite strain 2 assuming the establishment of parasite strain 1, and thus, it plays a crucial role for possible co-invasion of both strains of parasite. However, the two conditions in (H) for this result do not seem to be possible to be be verified (except numerically) since they involve a large amount of computations. On the other hand, these two conditions have no biological explanation(s). Hence, even the authors of [**1**] themselves suspected that conditions in (H) are unnecessary and conjectured that $\overline{U}$ is stable whenever it exists. In this note, we theoretically confirm that this conjecture is correct. Moreover, we prove that $\overline{U}$ is *globally* asymptotically stable (stronger than local stability) if $\mathcal{R}_1 > 1$. In other words, the reduced model (1.2) does demonstrate the global threshold dynamics in terms of the basic reproduction number $\mathcal{R}_1$, as described in the following improved "New Reresult 2."

Theorem. *The parasite-free equilibrium* $U_0 = (\Lambda_a/\mu, \Lambda_b/\mu, 0, 0)$ *for model* (1.2) *is globally asymptotically stable if* $\mathcal{R}_1 < 1$*, and unstable if* $\mathcal{R}_1 > 1$*. In the latter case, there exists a unique interior equilibrium* $\overline{U} = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$ *for* (1.2) *which is globally asymptotically stable.*

Biologically, this new result precisely reflects what the basic reproduction number means for the model, and hence, is of biological significance.

This work also complements [**1**] in a timely fashion because of the the important role of the *Result 2* in [**1**]. The proof is given in Section 2 by constructing a Lyapunov function together with careful and subtle estimates of the derivative of this function. We point out that the Lyapunov function we use here is not new. Indeed it has been adopted for Lotka-Volterra type systems and recently it has been successfully applied to several disease models to prove the globally asymptotic stability of the endemic equilibria of these model systems (see, e.g., [**2, 3, 4**] and the references therein). We would especially draw readers' attention to the recent work [**3**] where graph theory is amazingly employed to help optimally group the terms of the derivative of the Lyapunov function along the model system. In our proof, the grouping is also guided by the results in [**3**].

2 Proof of the theorem In this section we give a proof to the Theorem. We only need to prove the global asymptotical stability as the rest has been proved in [**1**]. Therefore, in the rest of the paper we always assume $\mathcal{R}_1 > 1$, and thus, the interior (positive) equilibrium $\overline{U} = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$ exists and is unique.

Substituting the last equation in (1.2) into the other four leads to

$$\begin{cases} \dfrac{d}{dt}S_a = \Lambda_a - \rho_{a1}c_{a1}I_{a1}S_a - \rho_{a1}c_{b1}I_{b1}S_a - \mu S_a, \\ \dfrac{d}{dt}S_b = \Lambda_b - \rho_{b1}c_{a1}I_{a1}S_b - \rho_{b1}c_{b1}I_{b1}S_b - \mu S_b, \\ \dfrac{d}{dt}I_{a1} = \rho_{a1}c_{a1}I_{a1}S_a + \rho_{a1}c_{b1}I_{b1}S_a - (\mu + \delta_{a1})I_{a1}, \\ \dfrac{d}{dt}I_{b1} = \rho_{b1}c_{a1}I_{a1}S_b + \rho_{b1}c_{b1}I_{b1}S_b - (\mu + \delta_{b1})I_{b1}. \end{cases} \tag{2.1}$$

To simplify notations, we set

$$\begin{aligned} \beta_{11} &= \rho_{a1}c_{a1}, & \beta_{12} &= \rho_{a1}c_{b1}, \\ \beta_{21} &= \rho_{b1}c_{a1}, & \beta_{22} &= \rho_{b1}c_{b1}. \end{aligned} \tag{2.2}$$

Then (2.1) becomes

$$
\begin{cases}
\dfrac{d}{dt}S_a = \Lambda_a - \beta_{11}I_{a1}S_a - \beta_{12}I_{b1}S_a - \mu S_a,\\
\dfrac{d}{dt}S_b = \Lambda_b - \beta_{21}I_{a1}S_b - \beta_{22}I_{b1}S_b - \mu S_b,\\
\dfrac{d}{dt}I_{a1} = \beta_{11}I_{a1}S_a + \beta_{12}I_{b1}S_a - (\mu+\delta_{a1})I_{a1},\\
\dfrac{d}{dt}I_{b1} = \beta_{21}I_{a1}S_b + \beta_{22}I_{b1}S_b - (\mu+\delta_{b1})I_{b1}.
\end{cases} \tag{2.3}
$$

By a standard argument for population models, one can show that for given positive initial values $S_a(0) > 0$, $S_b(0) > 0$, $I_{a1}(0) > 0$, $I_{b1}(0) > 0$, system (2.3) has a unique solution $U(t) = (S_a(t), S_b(t), I_{a1}(t), I_{b1}(t))$ which exists for $t \in (0, \infty)$, remains positive and bounded for all $t \in [0, \infty)$.

Define

$$
\begin{aligned}
\mathbf{V}(S_a, S_b, I_{a1}, I_{b1})
&= v_1\left(S_a - \bar{S}_a - \bar{S}_a \ln\frac{S_a}{\bar{S}_a} + I_{a1} - \bar{I}_{a1} - \bar{I}_{a1}\ln\frac{I_{a1}}{\bar{I}_{a1}}\right)\\
&\quad + v_2\left(S_b - \bar{S}_b - \bar{S}_b \ln\frac{S_b}{\bar{S}_b} + I_{b1} - \bar{I}_{b1} - \bar{I}_{b1}\ln\frac{I_{b1}}{\bar{I}_{b1}}\right)
\end{aligned} \tag{2.4}
$$

where

$$
v_1 = \beta_{21}\bar{I}_{a1}\bar{S}_b \quad \text{and} \quad v_2 = \beta_{12}\bar{I}_{b1}\bar{S}_a. \tag{2.5}
$$

By calculus of multi-variable functioms, it can be easily seen that $\mathbf{V}(S_a, S_b, I_a, I_b)$ has a global minimun attained at $(S_a, S_b, I_{a1}, I_{b1}) = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$. Thus $\mathbf{V}(S_a, S_b, I_a, I_b) \geq \mathbf{V}(\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1}) = 0$ for all $(S_a, S_b, I_a, I_b) \in R^4_+$.

The derivative of $\mathbf{V}(t)$ along the positive solution of (2.3) is given by

$$
\begin{aligned}
\mathbf{V}'(t) &= v_1\left[\left(1 - \frac{\bar{S}_a}{S_a}\right)S_a' + \left(1 - \frac{\bar{I}_{a1}}{I_{a1}}\right)I_{a1}'\right]\\
&\quad + v_2\left[\left(1 - \frac{\bar{S}_b}{S_b}\right)S_b' + \left(1 - \frac{\bar{I}_{b1}}{I_{b1}}\right)I_{b1}'\right]\\
&= v_1\Big[\Lambda_a - \beta_{11}I_{a1}S_a - \beta_{12}I_{b1}S_a - \mu S_a - \Lambda_a\frac{\bar{S}_a}{S_a}
\end{aligned} \tag{2.6}
$$

$$+ \beta_{11} I_{a1} \bar{S}_a + \beta_{12} I_{b1} \bar{S}_a + \mu \bar{S}_a + \beta_{11} I_{a1} S_a$$
$$+ \beta_{12} I_{b1} S_a - (\mu + \delta_{a1}) I_{a1} - \beta_{11} \bar{I}_{a1} S_a - \beta_{12} I_{b1} S_a \frac{\bar{I}_{a1}}{I_{a1}}$$
$$+ (\mu + \delta_{a1}) \bar{I}_{a1} \Big] + v_2 \Big[\Lambda_b - \beta_{21} I_{a1} S_b - \beta_{22} I_{b1} S_b$$
$$- \mu S_b - \Lambda_b \frac{\bar{S}_b}{S_b} + \beta_{21} I_{a1} \bar{S}_b + \beta_{22} I_{b1} \bar{S}_b$$
$$+ \mu \bar{S}_b + \beta_{21} I_{a1} S_b + \beta_{22} I_{b1} S_b - (\mu + \delta_{b1}) I_{b1}$$
$$+ (\mu + \delta_{b1}) \bar{I}_{b1} - \beta_{21} I_{a1} S_b \frac{\bar{I}_{b1}}{I_{b1}} - \beta_{22} \bar{I}_{b1} S_b \Big].$$

By making use of the equilibrium equations for (2.3) and regrouping the terms, (2.6) can be rewritten as

$$\begin{aligned}
\mathbf{V}'(t) = {} & v_1 \Big[\mu \bar{S}_a \left(2 - \frac{\bar{S}_a}{S_a} - \frac{S_a}{\bar{S}_a} \right) + (\beta_{11} I_{a1} \bar{S}_a + \beta_{12} I_{b1} \bar{S}_a \\
& - (\mu + \delta_{a1}) I_{a1}) + \Big(2\beta_{11} \bar{I}_{a1} \bar{S}_a + 2\beta_{12} \bar{I}_{b1} \bar{S}_a \\
& - \beta_{11} \bar{I}_{a1} \frac{\bar{S}_a^2}{S_a} - \beta_{12} \bar{I}_{b1} \frac{\bar{S}_a^2}{S_a} - \beta_{11} I_{a1} S_a \frac{\bar{I}_{a1}}{I_{a1}} \\
& - \beta_{12} I_{b1} S_a \frac{\bar{I}_{a1}}{I_{a1}} \Big) \Big] + v_2 \Big[\mu \bar{S}_b \left(2 - \frac{\bar{S}_b}{S_b} - \frac{S_b}{\bar{S}_b} \right) \\
& + (\beta_{21} I_{a1} \bar{S}_b + \beta_{22} I_{b1} \bar{S}_b - (\mu + \delta_{b1}) I_{b1}) \\
& + \Big(2\beta_{21} \bar{I}_{a1} \bar{S}_b + 2\beta_{22} \bar{I}_{b1} \bar{S}_b - \beta_{21} \bar{I}_{a1} \frac{\bar{S}_b^2}{S_b} \\
& - \beta_{22} \bar{I}_{b1} \frac{\bar{S}_b^2}{S_b} - \beta_{21} I_{a1} S_b \frac{\bar{I}_{b1}}{I_{b1}} - \beta_{22} I_{b1} S_b \frac{\bar{I}_{b1}}{I_{b1}} \Big) \Big] \\
= {} & K_1 + K_2 + v_1 \mu \bar{S}_a \left(2 - \frac{\bar{S}_a}{S_a} - \frac{S_a}{\bar{S}_a} \right) \\
& + v_2 \mu \bar{S}_b \left(2 - \frac{\bar{S}_b}{S_b} - \frac{S_b}{\bar{S}_b} \right),
\end{aligned} \tag{2.7}$$

where K_1 and K_2 are defined and estimated below. Firstly,

$$K_1 = v_1 \left[\beta_{11} I_{a1} \bar{S}_a + \beta_{12} I_{b1} \bar{S}_a - (\mu + \delta_{a1}) I_{a1} \right] \tag{2.8}$$

$$
\begin{aligned}
&+ v_2\left[\beta_{21} I_{a1}\bar{S}_b + \beta_{22} I_{b1}\bar{S}_b - (\mu+\delta_{b1}) I_{b1}\right] \\
&= I_{a1}\left[\beta_{11}\bar{S}_a v_1 + v_2\beta_{21}\bar{S}_b - v_1(\mu+\delta_{a1})\right] \\
&\quad + I_{b1}\left[\beta_{12} v_1\bar{S}_a + \beta_{22} v_2\bar{S}_b - v_2(\mu+\delta_{b1})\right] \\
&= I_{a1}[\beta_{11}\bar{S}_a\beta_{21}\bar{I}_{a1}\bar{S}_b + \beta_{21}\bar{S}_b\beta_{12}\bar{I}_{b1}\bar{S}_a \\
&\quad - (\mu+\delta_{a1})\,\beta_{21}\bar{I}_{a1}\bar{S}_b] + I_{b1}[\beta_{12}\bar{S}_a\beta_{21}\bar{I}_{a1}\bar{S}_b \\
&\quad + \beta_{22}\bar{S}_b\beta_{12}\bar{I}_{b1}\bar{S}_a - (\mu+\delta_{b1})\,\beta_{12}\bar{I}_{b1}\bar{S}_a] \\
&= \beta_{21}\bar{S}_b I_{a1}\left[\beta_{11}\bar{S}_a\bar{I}_{a1} + \beta_{12}\bar{I}_{b1}\bar{S}_a - (\mu+\delta_{a1})\bar{I}_{a1}\right] \\
&\quad + I_{b1}\beta_{12}\bar{S}_a\left[\beta_{21}\bar{I}_{a1}\bar{S}_b + \beta_{22}\bar{I}_{b1}\bar{S}_b - (\mu+\delta_{b1})\bar{I}_{b1}\right].
\end{aligned}
$$

Thus, from the equilibrium equations, we actually get $K_1 \equiv 0$. Next

$$
\begin{aligned}
K_2 &= v_1\left(2\beta_{11}\bar{I}_{a1}\bar{S}_a + 2\beta_{12}\bar{I}_{b1}\bar{S}_a - \beta_{11}\bar{I}_{a1}\frac{\bar{S}_a^2}{S_a} - \beta_{12}\bar{I}_{b1}\frac{\bar{S}_a^2}{S_a}\right. \\
&\quad \left.-\beta_{11} I_{a1} S_a\frac{\bar{I}_{a1}}{I_{a1}} - \beta_{12} I_{b1} S_a\frac{\bar{I}_{a1}}{I_{a1}}\right) \\
&\quad + v_2\left(2\beta_{21}\bar{I}_{a1}\bar{S}_b + 2\beta_{22}\bar{I}_{b1}\bar{S}_b - \beta_{21}\bar{I}_{a1}\frac{\bar{S}_b^2}{S_b} - \beta_{22}\bar{I}_{b1}\frac{\bar{S}_b^2}{S_b}\right. \\
&\quad \left.-\beta_2 I_{a1} S_b\frac{\bar{I}_{b1}}{I_{b1}} - \beta_{22} I_{b1} S_b\frac{\bar{I}_{b1}}{I_{b1}}\right) \\
&= v_1\beta_{11}\bar{I}_{a1}\bar{S}_a\left(2 - \frac{\bar{S}_a}{S_a} - \frac{S_a}{\bar{S}_a}\right) \\
&\quad + v_1\beta_{12}\bar{I}_{b1}\bar{S}_a\left(2 - \frac{\bar{S}_a}{S_a} - \frac{I_{b1}}{\bar{I}_{b1}}\frac{\bar{I}_{a1}}{I_{a1}}\frac{S_a}{\bar{S}_a}\right) \\
&\quad + v_2\beta_{22}\bar{I}_{b1}\bar{S}_b\left(2 - \frac{\bar{S}_b}{S_b} - \frac{S_b}{\bar{S}_b}\right) \\
&\quad + v_2\beta_{21}\bar{I}_{a1}\bar{S}_b\left(2 - \frac{\bar{S}_b}{S_b} - \frac{I_{a1}}{\bar{I}_{a1}}\frac{\bar{I}_{b1}}{I_{b1}}\frac{S_b}{\bar{S}_b}\right) \\
&= v_1\beta_{11}\bar{I}_{a1}\bar{S}_a\left(2 - \frac{\bar{S}_a}{S_a} - \frac{S_a}{\bar{S}_a}\right) + v_2\beta_{22}\bar{I}_{b1}\bar{S}_b\left(2 - \frac{\bar{S}_b}{S_b} - \frac{S_b}{\bar{S}_b}\right) \\
&\quad + v_1 v_2\left(4 - \frac{\bar{S}_b}{S_b} - \frac{I_{a1}}{\bar{I}_{a1}}\frac{\bar{I}_{b1}}{I_{b1}}\frac{S_b}{\bar{S}_b} - \frac{\bar{S}_a}{S_a} - \frac{I_{b1}}{\bar{I}_{b1}}\frac{\bar{I}_{a1}}{I_{a1}}\frac{S_a}{\bar{S}_a}\right) \\
&\le 0
\end{aligned}
\tag{2.9}
$$

with the equality holding if and only if $S_k = \bar{S}_k, I_{k1} = \bar{I}_{k1},\ k = a, b$.

Finally, for the third and fourth terms on the last line of (2.7), we similarly have

$$v_1 \mu \bar{S}_a \left(2 - \frac{\bar{S}_a}{S_a} - \frac{S_a}{\bar{S}_a} \right) \leq 0 \tag{2.10}$$

with the equality holding if and only if $S_a = \bar{S}_a$, and

$$v_2 \mu \bar{S}_b \left(2 - \frac{\bar{S}_b}{S_b} - \frac{S_b}{\bar{S}_b} \right) \leq 0 \tag{2.11}$$

with the equality holding if and only $S_b = \bar{S}_b$.

From (2.7)–(2.11), it follows that $\mathbf{V}^{'}(t) \leq 0$ with the equality holding if and only if $(S_a, S_b, I_{a1}, I_{b1}) = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$. Therefore, the positive equilibrium $\overline{U} = (\bar{S}_a, \bar{S}_b, \bar{I}_{a1}, \bar{I}_{b1})$ is globally asymptotically stable in the sense that it attracts all positive solutions.

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