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ORIGINAL ARTICLE

Wolbachia Dynamics in Mosquitoes with Incomplete CI and Imperfect Maternal Transmission by a DDE System

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Abstract
In this paper, we propose a delay differential equation model to describe the *Wolbachia* infection dynamics in mosquitoes in which the key factor of cytoplasmic incompactibility (CI) is incorporated in a more natural way than those in the literature. By analyzing the dynamics of the model, we are able to obtain some information on the impact of four important parameters: the competition capabilities of the wild mosquitoes and infected mosquitoes, the maternal transmission level and the CI level. The analytic results show that there are ranges of parameters that support competition exclusion principle, and there are also ranges of parameters that allow co-persistence for both wild and infected mosquitoes. These ranges account for the scenarios of failure of invasion, invasion and suppressing the wild mosquitoes, and invasion and replacing the wild mosquitoes. We also discuss some possible future problems both in mathematics and in modeling.

Keywords Delay differential equation · *Wolbachia* Infection · Cytoplasmic incompactibility · Maternal transmission · Intra-species competition

Mathematics Subject Classification 34K20 · 92B05 · 92D25

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1 Introduction

Aedes mosquitoes, including *Aedes aegypti* and *Aedes albopictus*, are capable of spreading viruses such as dengue, Zika, chikungunya and yellow fever. An innovative biological method to combat these mosquito-borne diseases is to use a maternally transmitted bacteria, *Wolbachia*, whose infection in *Aedes* mosquitoes can effectively block the virus replication within mosquitoes (Bian et al. 2013; Iturbe-Ormaetxe et al. 2011). Meanwhile, *Wolbachia* induces cytoplasmic incompatibility (CI), which largely results in early embryonic deaths when uninfected females mate with *Wolbachia*-infected males (Hoffmann and Turelli 1997; Laven 1956). In contrast, *Wolbachia*-infected females produce viable embryos, irrelevant of the paternal mating status (Xi et al. 2005a, b, 2006).

With strong CI and high maternal transmission, *Wolbachia* brings a reproductive advantage for infected female mosquitoes over uninfected ones, together with a reduction in disease transmission potential of female mosquitoes. Based on these observations, the World Mosquito Program's *Wolbachia* method is helping to reduce the occurrence of mosquito-borne diseases by releasing *Wolbachia*-infected mosquitoes. Regarding the *Wolbachia* release, the released males can sterilize uninfected females through CI, and the released females can increase *Wolbachia* infection frequency in mosquito populations through maternal transmission. This leads to two release strategies: (i) only releasing infected males to sterilize wild females, with a goal of population suppression; (ii) releasing both infected females and males so that wild uninfected mosquitoes are replaced by infected ones, aiming at population replacement. The field trials on population suppression and replacement have been implemented in several countries, including Australia, Brazil, China, Colombia, Indonesia and Singapore. On population suppression, since March 2015, by combining the incompatible and sterile insect techniques (IIT-SIT), factory-reared *Wolbachia*-infected male mosquitoes have been released on two islands in Guangzhou city, which enabled near-eradication of wild-type *Aedes albopictus* field populations (Zheng 2019). The first implementation of population replacement strategy was carried out in 2011 in Cairns, Northern Australia, where *Wolbachia* was found to have successfully established and self-sustaining, with no local dengue transmission (Hoffmann et al. 2011; Walker et al. 2011). With promising results internationally, the dynamics of mosquito populations with *Wolbachia* interference have become an important research topic. Various mathematical models have been developed in the forms of difference or differential equations. We refer to Yu (2018); Yu and Li (2020); Zheng et al. (2021); Zheng and Yu (2022) for population suppression models by releasing sterile male mosquitoes as well as *Wolbachia*-infected male mosquitoes and to Hu et al. (2019); Huang et al. (2016); Yu and Zheng (2019); Zhang et al. (2020); Zheng et al. (2021, 2014) for population replacement models by releasing both infected females and males, to cite a few.

In this paper, we propose a delay differential equation model to describe the *Wolbachia* spread dynamics in mosquito populations where the imperfect maternal transmission rate and the incomplete CI are both incorporated. Our model is motivated by an earlier ODE model in the literature (e.g., Farkas and Hinow 2010; Zhang et al. 2015; Zheng et al. 2018) which is given by the following system of ordinary

differential equations

$$\begin{cases} \dfrac{dI}{dt} = \mu b I(t) - (d+D)[I(t)+U(t)]I(t), \\ \dfrac{dU}{dt} = (1-\mu) b I(t) + bU(t)\left[1 - \dfrac{qI(t)}{I(t)+U(t)}\right] - d[I(t)+U(t)]U(t), \end{cases} \tag{1}$$

where $I(t)$ and $U(t)$ denote the population sizes of infected and uninfected female mosquitoes. Here, 1:1 sex ratio and *identical mating rate* for different crossings are assumed. The parameter $\mu \in (0, 1)$ accounts for an *imperfect maternal transmission* of *Wolbachia*, meaning that among the offspring produced from infected females, a fraction μ of them are infected and the remaining $1-\mu$ proportion of them are uninfected. The parameter $b > 0$ is the production rate of mated female mosquitoes, $d > 0$ is the natural death rate, and D is a parameter that gauges the fitness cost for infected mosquitoes. The CI intensity is characterized by $q \in [0, 1]$, with $q = 1$ accounting for the complete CI and $q = 0$ standing for no CI. This can be seen by re-writing

$$bU(t)\left[1 - \frac{qI(t)}{I(t)+U(t)}\right] = bU(t)\frac{U(t)}{I(t)+U(t)} + (1-q)bU(t)\frac{I(t)}{I(t)+U(t)} \tag{2}$$

with the second term on the right side clearly indicating that, due to CI, only a proportion $(1-q)$ of the incompatible crossing can contribute to the uninfected population $U(t)$. We point out that the other birth term $bI(t)$ in (1), split into the fractions μ and $1-\mu$, can be understood by the same way of tracking the two types of matings of infected female mosquitoes:

$$bI(t) = bI(t)\frac{U(t)}{I(t)+U(t)} + bI(t)\frac{I(t)}{I(t)+U(t)}. \tag{3}$$

There are *two obvious omissions* in (1): (i) density-independent death rate and (ii) the maturation period of mosquitoes from adult mating to the emergence of reproductive offspring. In general, density-independent death rate is species specific and is a reflection of the species' genetic disadvantages, while the density-dependent death rate is related to the habitat's condition/environment/resources and is reflective of the intra-species competition. This is why, many previous works considered both density-independent and density-dependent death rates (see, e.g., Lewis and van den Driessche (Lewis and van Den Driessche 1992) for an SIRM model). As for (ii), it is well known that the maturation period of mosquitoes is relatively long compared to their life span. Here by maturation period we mean the sum of the durations of all pre-adult stages. It starts from eggs which hatch into larvae within a few days to months depending mostly on water availability, water temperature and photoperiod (Vinogradova 2007; Zhang et al. 2015). Larvae molt into pupae after developing through four (4) instars, and finally, adult (mature) mosquitoes merge from pupae. Although the maturation time varies and depends on many factors, it is widely accepted that it ranges from 5 to 40 days, comparable to the time that adult mosquitoes live, with male adults mostly

living between 5 and 7 days and female adults mostly living between 7 and 14 days (Wikipedia 2022). On the other hand, CI is a development obstacle that occurs during the development period, leading to a larger mortality rate of the zygotes formed by gametes from *Wolbachia*-infected male and uninfected females.

Based on the above observations, we can modify the above model (1) by incorporating the above two factors to obtain the following system of delay differential equations:

$$\begin{cases} \dfrac{dI}{dt} = e^{-\delta\tau}\mu bI(t-\tau) - [d_1 + d_2(I(t)+U(t))]I(t) \\ \dfrac{dU}{dt} = e^{-\delta\tau}(1-\mu)bI(t-\tau) + e^{-\delta\tau}bU(t-\tau)\dfrac{U(t-\tau)}{I(t-\tau)+U(t-\tau)} \\ \qquad\quad + e^{-(\delta+\theta)\tau}bU(t-\tau)\dfrac{I(t-\tau)}{I(t-\tau)+U(t-\tau)} - [d_3 + d_4(I(t)+U(t))]U(t). \end{cases} \tag{4}$$

Here, $I(t)$ and $U(t)$ now stand for the *adult* populations of infected and uninfected female mosquitoes, respectively, (assuming 1:1 sex ratio). For readers' convenience, we summarize the explanations for the parameters in (4) below:

(p1) $b = b_0 m$ accounts for the production rate where m is the mating rate and b_0 is the per capita birth rate of mated adult mosquitoes (assumed to be the same for all possible crossings);
(p2) τ is the maturation time which sums up the durations in all pre-adult stages;
(p3) δ is the average mortality rate in all immature stages for the offsprings coming from all crossings (assumed to be the same for simplicity) except for those from mating between uninfected females and infected males (ref. (p4));
(p4) $\delta + \theta$ accounts for the average mortality rate of the offspring coming from the mating between *uninfected females and infected males* in all immature stages, and hence, the the extra death rate $\theta \in [0, \infty)$ reflects the level of CI, with $\theta = 0$ corresponding to the case of no CI, while $\theta = \infty$ accounting for the case of complete CI;
(p5) d_1 and d_3 are the *density-independent* death rats of adults of infected and uninfected mosquitoes, respectively;
(p6) d_2 and d_4 are the *density-dependent* death rats of adults of infected and uninfected mosquitoes, respectively;
(p7) $\mu \in [0, 1]$ is the fraction of the offspring produced by infected females that also carry Wolbachia and it measures the imperfective maternal transmission;

With the above explanations of the parameters, one then can easily understand the terms in (4). Indeed, the two negative (losing) terms in (4) account for the deaths corresponding to (p5)-(p6). The positive terms account for the recruitments of infected and uninfected adults. For infected adult, its recruitments come from the productions of *infected female adults* after mating with *infected and uninfected* males, respectively (refer to (p1)), *survived to adult* (τ time units later) with the probability $e^{-\delta\tau}$ (refer

to (p2)-(p3)). Adding these two maturation terms,

$$e^{-\delta\tau}\cdot bI(t-\tau)\cdot\frac{I(t-\tau)}{I(t-\tau)+U(t-\tau)}+e^{-\delta\tau}\cdot bI(t-\tau)\cdot\frac{u(t-\tau)}{I(t-\tau)+U(t-\tau)}$$

gives $e^{-\delta\tau}bI(t-\tau)$. Since we aim to accommodate *imperfect maternal transmission* (refer to (p7)), only a fraction μ of this entry rate into adult contributes to infected adult population (first term on the right side of $I'(t)$ equation in (4)), while the remaining fraction $1-\mu$ goes to the recruitment of the uninfected adult population (first term on the right side of $U'(t)$ equation in (4)). Similarly, the production of *uninfected females* from mating with *uninfected males* results in a recruitment rate for the uninfected adult population given by the second term on the right side of $U'(t)$ equation in (4). However, the offspring of *uninfected females* from mating with *infected males* will suffer a higher mortality rate (refer to (p4)) due to the CI effect (a defect in development) which reduces the survival probability from $e^{-\delta\tau}$ to $e^{-(\delta+\theta)\tau}$, and this is reflected in the third term on the right side of $U'(t)$ equation in (4).

Comparing (4) with (1), we see that the CI effect incorporated in (4) is now *closely related to the development stage* and hence is more reasonable and realistic. Therefore, analyzing (4) may shed some light on how the CI and the imperfect maternal transmission together with the maturation period will affect the mosquito populations and accordingly help us predict the outcomes of some mosquito control strategies.

We point out that the maternal transmission rate of *Wolbachia* depends on both the host and the specific strains. For example, for WB1 in *Aedes aegypti*, $\mu=1$ (Xi et al. 2005b), and for type R in *Drosophila simulans*, $\mu<1$ (Turelli and Hoffmann 1995). We analyze the case of perfect maternal transmission with $\mu=1$ and imperfect case with $\mu\in(0,1)$ in Sects. 3 and 4, respectively. Also, in (4), we have allowed *variances* in the density-dependent death rates and density-independent death rates in the *Wolbachia*-infected and uninfected mosquitoes, as evolution may eventually lead to such variances, which may account for a type of cost or benefit, depending on whether $d_1\geq d_3$ or $d_1\leq d_3$ and whether $d_2\geq d_4$ or $d_2\leq d_4$.

When $d_1=d_3=0$, and letting $d_4=d$ and $D=d_2-d_4$ (hence $d_2=d_4+(d_2-d_4)=d+D$), the model (4) reduces to

$$\begin{cases}\dfrac{dI}{dt}=e^{-\delta\tau}\mu bI(t-\tau)-(d+D)[I(t)+U(t)]I(t)\\ \dfrac{dU}{dt}=e^{-\delta\tau}(1-\mu)bI(t-\tau)+e^{-\delta\tau}bU(t-\tau)\dfrac{U(t-\tau)}{I(t-\tau)+U(t-\tau)}\\ \qquad\quad+e^{-(\delta+\theta)\tau}bU(t-\tau)\dfrac{I(t-\tau)}{I(t-\tau)+U(t-\tau)}-d[I(t)+U(t))]U(t).\end{cases}\tag{5}$$

The structure of equilibrium of (5) is essentially the same as in Zhang et al. (2015), if we replace τ by μ, b by $\bar{b}=be^{-\delta\tau}$, q by $\bar{q}=1-e^{-\theta\tau}$. However, now $\bar{b}$ and $\bar{q}$ both depend on τ, and thus, in a sense, they are related by τ, and therefore, those conditions on b and q in Zhang et al. (2015) need to be carefully checked for $\bar{b}$ and $\bar{q}$ to avoid conflicts. It would be mathematically interesting and biologically meaningful to explore how the imperfect maternal transmission rate μ, the CI level parameter θ and

the maturation delay τ will interplay to affect the population dynamics, qualitatively and quantitatively. Particularly, it is desirable to know if the delay τ will destroy the stability of a positive equilibrium, causing periodic oscillations, and how it will interplay with the other two important parameters, μ and θ. In the rest of this paper, we will address these questions by analyzing model (4), including its well-posedness with properly given initial conditions in Sect. 2, and its equilibria and their stability in Sects. 3 and 4. We conclude the paper in Sect. 5 in which we summarize our main results, discuss the biological implications of the results and some possible future work both in mathematics and in modeling.

2 Well-Posedness and Equilibria

The DDE model system (4) is not defined at origin. Nevertheless, it can be extended continuously and smoothly to the origin by defining $\frac{dU}{dt} = 0$ if $(I(t-\tau), U(t-\tau)) = (0, 0)$. We will maintain this remediation in our following discussion without further notice.

Denote $X = C([-\tau, 0], \mathbb{R}^2)$ which is a Banach space with the norm $||\psi|| = \max_{s\in[-\tau,0]} |\psi(s)|$ for $\psi \in C([-\tau, 0], \mathbb{R}^2)$. Let $X_+ = C([-\tau, 0], \mathbb{R}^2_+)$ be the positive cone of X. Then, we can have the following well-posed results for (4).

Theorem 2.1 *For any initial function $\phi = (\phi_1, \phi_2) \in X_+$, (4) has a unique solution satisfying this initial condition, which remains non-negative. Moreover, if $\phi_1 \not\equiv 0$ and $\phi_2 \not\equiv 0$, then $I(t, \phi) > 0$ and $U(t, \phi) > 0$ for all $t > \tau$.*

This is a result of combining the method of steps and constant-variation method, and we omit the details of the proof.

Theorem 2.2 *If the initial function $\phi = (\phi_1, \phi_2) \in X_+$, then the corresponding solution is bounded and hence exists globally.*

Proof Let

$$K_1 = \max\{||\phi_1||, e^{-\delta}\mu b/d_2\}, \;\; K_2 = \max\{K_1, ||\phi_2||, e^{-\delta\tau} b/d_4\},$$

then we claim that $I(t, \phi) < K_1$ and $U(t, \phi) < K_2$, for all $t \geq 0$. Assume the contrary. Then, there exists a $t_0 > 0$ such that $I(t_0) = K_1$, $I(t) < K_1$ for $t < t_0$ and $I'(t) \geq 0$. However,

$$\begin{aligned} I'(t_0) &= e^{-\delta\tau}\mu b I(t_0 - \tau) - [d_1 + d_2(K_1 + U(t_0))]K_1 \\ &< K_1(e^{-\delta\tau}\mu b - d_2 K_1) \leq 0, \end{aligned}$$

which is a contradiction. Similarly, we can prove that $U(t) < K_2$ for all $t \geq 0$. Assume the contrary. Then, there exists a $t_1 > 0$ such that $U(t_1) = K_2$, $U(t) < K_2$ for $t < t_1$

and $U'(t) \geq 0$. However,

$$\begin{aligned} U'(t_1) &= e^{-\delta\tau}(1-\mu)bI(t_1-\tau) + e^{-\delta\tau}bU(t_1-\tau)\frac{U(t_1-\tau)}{I(t_1-\tau)+U(t_1-\tau)} \\ &\quad + e^{-(\delta+\theta)\tau}bU(t_1-\tau)\frac{I(t_1-\tau)}{I(t_1-\tau)+U(t_1-\tau)} - [d_3 + d_4(I(t_1)+K_2)]\,K_2 \\ &< e^{-\delta\tau}(1-\mu)bI(t_1-\tau) + e^{-\delta\tau}bU(t_1-\tau) - d_4(K_2)^2 \\ &< [e^{-\delta\tau}(1-\mu)b + e^{-\delta\tau}b - d_4K_2]K_2 \leq 0, \end{aligned}$$

which is also a contradiction. The proof is completed. □

Denote

$$R_1 = e^{-\delta\tau}b/d_3,\ \ R_2 = e^{-\delta\tau}b/d_1.$$

If $R_1 < 1$ and $R_2 < 1$, the model (4) only has the trivial equilibrium $E_0 = (0, 0)$, which attracts all feasible (non-negative) solutions. To see this, we first observe that the first equation in (4) has

$$\frac{dI}{dt} \leq e^{-\delta\tau}bI(t-\tau) - d_1I(t)$$

as a comparison equation from above. Note that this comparison scalar DDE is a linear and monotone, and hence, its trivial solution is (globally) asymptotically stable if $e^{-\delta\tau}b - d_1 < 0$ (see, e.g., (Smith 1995, P93,Corollary5.2)) which is equivalent to $R_2 < 1$. Then, the comparison principle for DDEs (see, e.g., Smith 1995) implies $I(t) \to 0$ as $t \to \infty$. This in turn implies that the second equation in (4) has the following limit equation:

$$\frac{dU}{dt} = e^{-\delta\tau}bU(t-\tau) - [d_3 + d_4U(t)]U(t),$$

which has $U = 0$ as a globally asymptotically stable equilibrium if $e^{-\delta\tau}b - d_3 < 0$ which is equivalent to $R_1 < 1$. By the theory of asymptotically autonomous systems (see, e.g., (Mischaikow et al. 1995, Theorem 1.8)), the $U(t)$ component of any nonnegative solution $(I(t), U(t))$ to (4) also approaches zero.

From the above result on E_0, we see that existence of other possible non-trivial equilibria are possible only when $R_1 > 1$ or $R_2 > 1$. The following two lemmas address the existence of the two semi-trivial equilibria.

Lemma 2.3 *If $R_1 > 1$, then model (4) has an infection-free equilibrium $E_1 := (0, \bar{U})$ with $\bar{U} = (e^{-\delta\tau}b - d_3)/d_4$.*

Lemma 2.4 *The model (4) has a total-infection equilibrium $E_2 := (\bar{I}, 0)$ if and only if $\mu = 1$ and $R_2 > 1$ under which $\bar{I} = (e^{-\delta\tau}b - d_1)/d_2$.*

These two equilibria are important because, biologically E_1 accounts for the scenario of *Wolbachia* failing to establish in the mosquitoes, while E_2 reflects the situation that the wild mosquitoes are fully replaced by *Wolbachia*-infected mosquitoes. Mathematically, their stability will affect the global dynamics of the model system (7). In the sequel, we study the stability of these equilibria and this involves linearization of (4) at each of them and analyzing the corresponding characteristics equations which are transcendental equations rather than polynomial equations. Note that if (I^*, U^*) is an equilibrium of (4), then the linearization of (4) at (I^*, U^*) is

$$\begin{pmatrix} \frac{dI(t)}{dt} \\ \frac{dU(t)}{dt} \end{pmatrix} = A \begin{pmatrix} I(t) \\ U(t) \end{pmatrix} + B \begin{pmatrix} I(t-\tau) \\ U(t-\tau) \end{pmatrix}, \tag{6}$$

where

$$A = \begin{pmatrix} -d_1 - 2d_2 I^* - d_2 U^* & -d_2 I^* \\ -d_4 U^* & -d_3 - 2d_4 U^* - d_4 I^* \end{pmatrix} \text{ and } B = \begin{pmatrix} e^{-\delta\tau}\mu b & 0 \\ M & N \end{pmatrix}$$

with

$$\begin{aligned} M &= e^{-\delta\tau}(1-\mu)b - \frac{e^{-\delta\tau} b U^{*2}}{(I^*+U^*)^2} + \frac{e^{-(\delta+\theta)\tau} b U^*}{I^*+U^*} - \frac{e^{-(\delta+\theta)\tau} b I^* U^*}{(I^*+U^*)^2}, \\ N &= \frac{2e^{-\delta\tau} b U^*}{I^*+U^*} - \frac{e^{-\delta\tau} b U^{*2}}{(I^*+U^*)^2} + \frac{e^{-(\delta+\theta)\tau} b I^*}{I^*+U^*} - \frac{e^{-(\delta+\theta)\tau} b I^* U^*}{(I^*+U^*)^2}. \end{aligned}$$

In the remainder of this paper, we focus on a scenario that both the infected and uninfected (by *Wolbachia*) have the same density-independent death rate (i.e., $d_1 = d_3$), but allow distinct *density-dependent death rates* d_2 and d_4. Biologically d_2 and d_4 accounts for intra-species competition for resources, and accordingly, the case $d_2 > d_4$ (resp. $d_2 < d_4$) would mean that the *Wolbachia*-infected mosquitoes are less (resp. more) competitive than uninfected mosquitoes, and hence, $d_2 - d_4$ can be considered as a measurement of fitness cost/benefit of *Wolbachia* infections in competition for resources, as is mentioned in Zhang et al. (2015) (corresponding to $d = d_4$ and $D = d_2 - d_4$ in (1), indicating that in this case D indeed measures the fitness in density-dependent death rate reflected by the competition capability). There may be other factors that can measure the fitness (cost and/or benefit), and we will have some further discussion on this in the end of the paper.

For convenience of notations and statements, we denote by d the identical $d_1 = d_3$ and rewrite (4) for this case as

$$\begin{cases} \dfrac{dI}{dt} = e^{-\delta\tau}\mu b I(t-\tau) - dI(t) - d_2[(I(t)+U(t))]I(t) \\ \dfrac{dU}{dt} = e^{-\delta\tau}(1-\mu)bI(t-\tau) + e^{-\delta\tau} b U(t-\tau)\dfrac{U(t-\tau)}{I(t-\tau)+U(t-\tau)} \\ \qquad + e^{-(\delta+\theta)\tau} b U(t-\tau)\dfrac{I(t-\tau)}{I(t-\tau)+U(t-\tau)} - dU(t) - d_4[I(t)+U(t))]U(t). \end{cases} \tag{7}$$

In the following sections, we will analyze the stability of equilibria and bifurcation of (7), with respect to two cases: the case of perfect maternal transmission ($\mu = 1$) and the case of imperfect maternal transmissions ($\mu \in (0, 1)$). For (7), $R_1 = R_2 =: R$ and under $R > 1$, the formulas for E_1 and E_2 reduce to

$$E_1 := (0, \bar{U}) = (0, (e^{-\delta\tau}b - d)/d_4), \quad E_2 := (\bar{I}, 0) = ((e^{-\delta\tau}b - d)/d_2, 0).$$

3 Dynamics of (7): with Perfect Maternal Transmission ($\mu = 1$)

In this section, we always assume $R > 1$ (and $\mu = 1$) so that both E_1 and E_2 exist. The following theorem deals with the stability of the infection-free equilibrium E_1.

Theorem 3.1 *Assume $R > 1$. Then, the infection-free equilibrium E_1 is locally asymptotically stable if $d_2 > d_4$, and it is unstable if $d_2 < d_4$.*

Proof At E_1, the linearization (6) becomes

$$\begin{pmatrix} \frac{dI(t)}{dt} \\ \frac{dU(t)}{dt} \end{pmatrix} = A_1 \begin{pmatrix} I(t) \\ U(t) \end{pmatrix} + B_1 \begin{pmatrix} I(t-\tau) \\ U(t-\tau) \end{pmatrix},$$

with

$$A_1 = \begin{pmatrix} -d - \frac{d_2}{d_4}(e^{-\delta\tau}b - d) & 0 \\ d - e^{-\delta\tau}b & d - 2e^{-\delta\tau}b \end{pmatrix} \text{ and}$$

$$B_1 = \begin{pmatrix} e^{-\delta\tau}b & 0 \\ e^{-(\delta+\theta)\tau}b - e^{-\delta\tau}b & e^{-\delta\tau}b \end{pmatrix}$$

The corresponding characteristic equation can be calculated as

$$\Delta_1^1 \Delta_2^1 = 0, \tag{8}$$

with

$$\Delta_1^1 = \lambda + d + d_2(e^{-\delta\tau}b - d)/d_4 - e^{-\delta\tau}be^{-\lambda\tau} \text{ and } \Delta_2^1 = \lambda - d + 2e^{-\delta\tau}b - e^{-\delta\tau}be^{-\lambda\tau}.$$

If $\lambda = \mathrm{Re}\lambda + i\mathrm{Im}\lambda$ is the root of $\Delta_2^1 = 0$, then $\mathrm{Re}\lambda$ satisfies the following equation

$$\mathrm{Re}\lambda - d + 2e^{-\delta\tau}b - e^{-\delta\tau}be^{-\tau\mathrm{Re}\lambda}\cos(\tau\mathrm{Im}\lambda) = 0.$$

We claim that $\mathrm{Re}\lambda < 0$. Otherwise, $\mathrm{Re}\lambda \geq 0$ would lead to

$$0 \leq \mathrm{Re}\lambda = d - 2e^{-\delta\tau}b + e^{-\delta\tau}be^{-\tau\mathrm{Re}\lambda}\cos(\tau\mathrm{Im}\lambda) \leq d - e^{-\delta\tau}b = d\,(1 - R)\,,$$

contradicting $R > 1$. Now we discuss the roots of $\Delta_1^1 = 0$. Rewrite Δ_1^1 as the following standard form

$$\Delta_1^1 = \lambda - \alpha_1 - \beta_1 e^{-\lambda\tau},$$

with $\alpha_1 = -d - d_2(e^{-\delta\tau}b - d)/d_4$ and $\beta_1 = e^{-\delta\tau}b$. Note that $\alpha_1 + \beta_1 = (1 - d_2/d_4)(e^{-\delta\tau}b - d)$ which is greater (resp. less) than zero if $d_2 > d_4$ (resp. $d_2 < d_4$). Moreover, $\alpha_1 < 0$ and $\beta_1 > 0$, when $d_2 > d_4$. Then, from (Smith 2011, Proposition 4.6), all roots of $\Delta_1^1 = 0$ have negative real parts $d_2 > d_4$, and there exists at least one real positive root of $\Delta_1^1 = 0$ if $d_2 < d_4$. The proof is competed. □

The stability of the total-infection equilibrium E_2 is more interesting and also more complicated, since it depends not only on d_2 and d_4, but also on the CI level parameter θ, as described in the following two theorems.

Theorem 3.2 *Assume $R > 1$.*

(i) When $\theta = \infty$ (complete CI case), the total infection equilibrium E_2 is locally asymptotically stable;

(ii) When $\theta = 0$ (no CI case), the total infection equilibrium E_2 is locally asymptotically stable if $d_2 < d_4$ and unstable if $d_2 > d_4$.

(iii) When $d_2 < d_4$, the total infection equilibrium E_2 is locally asymptotically stable for any value of $\theta > 0$.

(iv) When $d_2 > d_4$, there exists a $\theta^* > 0$ defined by

$$\theta^* = -\frac{1}{\tau}\ln\frac{d + d_4(e^{-\delta\tau}b - d)/d_2}{b} - \delta, \tag{9}$$

such that the total infection equilibrium E_2 is locally asymptotically stable if $\theta > \theta^*$ and unstable if $\theta < \theta^*$. In fact θ^* is a fold bifurcation point.

Proof At E_2, the linearization (6) becomes

$$\begin{pmatrix} \frac{dI(t)}{dt} \\ \frac{dU(t)}{dt} \end{pmatrix} = \begin{pmatrix} d - 2e^{-\delta\tau}b & d - e^{-\delta\tau}b \\ 0 & -d - \frac{d_4}{d_2}(e^{-\delta\tau}b - d) \end{pmatrix} \begin{pmatrix} I(t) \\ U(t) \end{pmatrix} + \begin{pmatrix} e^{-\delta\tau}b & 0 \\ 0 & e^{-(\delta+\theta)\tau}b \end{pmatrix} \begin{pmatrix} I(t-\tau) \\ U(t-\tau) \end{pmatrix}.$$

The corresponding characteristic equation can then be calculated as

$$\Delta_1^2(\theta)\Delta_2^2(\theta) = 0, \tag{10}$$

where

$$\Delta_1^2(\theta) = \lambda - d + 2e^{-\delta\tau}b - e^{-\delta\tau}be^{-\lambda\tau},$$
$$\Delta_2^2(\theta) = \lambda + d + d_4(e^{-\delta\tau}b - d)/d_2 - e^{-(\delta+\theta)\tau}be^{-\lambda\tau}.$$

Note that $\Delta_1^2(\theta) = \Delta_2^1$ for any fixed θ, it follows from the proof of Theorem 3.1 that all roots of $\Delta_1^2(\theta) = 0$ have negative real parts. The roots of $\Delta_2^2(\theta)$ satisfy

$$\lambda = -d - d_4(e^{-\delta\tau}b - d)/d_2 + e^{-(\delta+\theta)\tau}be^{-\lambda\tau},$$

which is negative when $\theta \to \infty$, proving (i). Notice that $\Delta_2^2(0)$ has the same form as Δ_1^1 but with the positions of d_2 and d_4 switched. Following the proof of Theorem 3.1, we can obtain (ii).

Rewrite $\Delta_2^2(\theta)$ as the following standard form

$$\Delta_2^2(\theta) = \lambda - \alpha_2(\theta) - \beta_2(\theta)e^{-\lambda\tau},$$

with $\alpha_2(\theta) = -d - d_4(e^{-\delta\tau}b - d)/d_2$ and $\beta_2(\theta) = e^{-(\delta+\theta)\tau}b$. Then, $\alpha_2(\theta) + \beta_2(\theta)$ is a strictly decreasing function of θ. Noting that $\alpha_2(0) + \beta_2(0) = (1 - d_4/d_2)(e^{-\delta\tau}b - d) < 0$ if $d_2 < d_4$. Consequently, when $d_2 < d_4$, then $\alpha_2(\theta) + \beta_2(\theta) < 0$ for all $\theta \in (0, \infty)$. Furthermore, $\beta(\theta) > 0$ for any fixed θ. Then, by (Smith 2011, Proposition 4.6), the equilibrium E_2 is locally asymptotically stable for any fixed $\theta \in (0, \infty)$ provided that $d_2 < d_4$.

When $d_2 > d_4$, then $\alpha_2(0) + \beta_2(0) > 0$. This together with the monotonicity of $\alpha_2(\theta) + \beta_2(\theta)$ in θ and the fact that $\alpha_2(\infty) + \beta_2(\infty) < 0$ implies that there exists a unique $\theta^* > 0$ such that

$$\alpha_2(\theta) + \beta_2(\theta) = \begin{cases} < 0, & \theta < \theta^*; \\ = 0, & \theta = \theta^*; \\ > 0, & \theta > \theta^*. \end{cases}$$

Again, by (Smith 2011, Proposition 4.6) we conclude that E_2 is locally asymptotically stable if $\theta > \theta^*$ and unstable if $\theta < \theta^*$, where θ^* satisfies the following equation

$$-d - d_4(e^{-\delta\tau}b - d)/d_2 + e^{-(\delta+\theta)\tau}b = 0.$$

which can be explicitly solve for θ to give the formula (9). □

Remark 3.3 Note that in (9), $\theta^* = 0$ when $d_2 = d_4$. This implies that the conclusion of Theorem 3.2-(iii) also holds if the condition "when $d_2 < d_4$" in replaced by"when $d_2 \leq d_4$." Also, it is easy to see that θ^* depends on τ and it is deceasing in τ, satisfying $\theta^* \to \infty$ as $\tau \to 0^+$ and $\theta^* \to 0$ as $\tau \to \tau_{max} := \frac{1}{\delta} \ln \frac{b}{d}$. Here τ_{max} is the upper bound for τ to ensure $R > 1$. Therefore, when ignoring the maturation delay ($\tau = 0$, hence an ODE model), the total infection equilibrium E_2 is unstable for any $\theta > 0$ provided that $d_2 < d_4$.

Combining the results in Theorems 3.1 and 3.2-(iv), we see that under the condition $d_2 > d_4$, the critical value θ^* is actually a fold bifurcation point, at which a positive (or interior equilibrium) occurs when E_2 loses its stability. This is reflected in the next theorem below which is mainly concerned with the existence of positive equilibrium (or equilibria).

Theorem 3.4 *Assume $R > 1$.*

(i) When $\theta = 0$, all points (I^+, U^+) lying on the straight line $I^+ + U^+ = (e^{-\delta\tau}b - d)/d_2$ are interior equilibria for rm (7) if $d_2 = d_4$, and there is no interior equilibrium if $d_2 \neq d_4$.

(ii) For $\theta > 0$, when $d_2 < d_4$, there is no interior equilibrium for (7); when $d_2 > d_4$, there is a unique interior equilibrium for (7) if $\theta > \theta^*$, and there is no interior equilibrium if $0 < \theta < \theta^*$, where θ^* is defined by (9).

Proof If $E^+ = (I^+, U^+)$ with $I^+, U^+ > 0$ is an equilibrium of (7), then it satisfies

$$\begin{cases} e^{-\delta\tau}b - \left[d + d_2(I^+ + U^+)\right] = 0 \\ e^{-\delta\tau}b\dfrac{U^+}{I^+ + U^+} + e^{-(\delta+\theta)\tau}b\dfrac{I^+}{I^+ + U^+} - \left[d + d_4(I^+ + U^+)\right] = 0 \end{cases} \tag{11}$$

From the first equation of (11), we have

$$I^+ + U^+ = \frac{e^{-\delta\tau}b - d}{d_2} := T.$$

When $\theta = 0$, from the second equation of (11) we have

$$I^+ + U^+ = \frac{e^{-\delta\tau}b - d}{d_4};$$

therefore, (I^+, U^+) is a positive equilibrium if and only if $d_2 = d_4$. When $\theta > 0$, explicitly solving (11), we have

$$I^+ = \frac{T(e^{-\delta\tau}b - d - Td_4)}{e^{-\delta\tau}b(1 - e^{-\theta\tau})} = \frac{(1 - d_4/d_2)(e^{-\delta\tau}b - d)^2}{d_4e^{-\delta\tau}b(1 - e^{-\theta\tau})}$$

which is positive if and only if $d_2 > d_4$, and

$$U^+ = \frac{e^{-\delta\tau}b - d}{d_4}\left[1 - \frac{(1 - d_4/d_2)(e^{-\delta\tau}b - d)}{d_4e^{-\delta\tau}b(1 - e^{-\theta\tau})}\right]$$

which is positive if and only if

$$d_2 > d_4 \text{ and } -d - d_4(e^{-\delta\tau}b - d)/d_2 + e^{-(\delta+\theta^*)\tau}b < 0.$$

Comparing the second inequality with (9), one finds that it is indeed equivalent to $\theta > \theta^*$. The proof is completed. □

4 Dynamics of (7): with Imperfect Maternal Transmission($0 < \mu < 1$)

In this case, by Lemma 2.4, we know there is no total-infection equilibrium. Thus, the main concern is the stability of the infection-free equilibrium E_1 and possible occurrence of positive equilibrium.

Theorem 4.1 *Assume* $R > 1$.

(i) When $d_2 \geq d_4$, *the infection-free equilibrium* E_1 *is asymptotically stable.*
(ii) When $d_2 < d_4$, *there exists a critical value of* μ

$$\mu^* = \frac{d + d_2(e^{-\delta\tau}b - d)/d_4}{be^{-\delta\tau}}, \tag{12}$$

such that the infection-free equilibrium E_1 *is asymptotically stable if* $\mu < \mu^*$ *and is unstable if* $\mu > \mu^*$. *In fact,* μ^* *is a fold bifurcation point.*

Proof At E_1, the linearization (6) becomes

$$\begin{pmatrix} \frac{dI(t)}{dt} \\ \frac{dU(t)}{dt} \end{pmatrix} = A_2 \begin{pmatrix} I(t) \\ U(t) \end{pmatrix} + B_2 \begin{pmatrix} I(t-\tau) \\ U(t-\tau) \end{pmatrix},$$

with

$$A_2 = \begin{pmatrix} -d - \frac{d_2}{d_4}(e^{-\delta\tau}b - d) & 0 \\ d - e^{-\delta\tau}b & d - 2e^{-\delta\tau}b \end{pmatrix} \text{ and}$$
$$B_2 = \begin{pmatrix} e^{-\delta\tau}\mu b & 0 \\ e^{-(\delta+\theta)\tau}b - e^{-\delta\tau}\mu b & e^{-\delta\tau}b \end{pmatrix}.$$

The corresponding characteristic equation is calculated as

$$\Delta_1^3 \Delta_2^3 = 0, \tag{13}$$

with

$$\Delta_1^3 = \lambda + d_1 + d_2(e^{-\delta\tau}b - d)/d_4 - e^{-\delta\tau}\mu b e^{-\lambda\tau} \text{ and } \Delta_2^3 = \lambda - d + 2e^{-\delta\tau}b - e^{-\delta\tau}be^{-\lambda\tau}.$$

Using (Smith 2011, Proposition 4.6), it is easy to prove that all the roots of $\Delta_2^3 = 0$ have negative real parts. Rewrite Δ_1^3 as following standard form

$$\Delta_1^3 = \lambda - \alpha_3(\mu) - \beta_3(\mu)e^{-\lambda\tau},$$

with $\alpha_3(\mu) = -d - d_2(e^{-\delta\tau}b - d)/d_4$ and $\beta_3(\mu) = \mu e^{-\delta\tau}b$. Then, $\alpha_3(\mu) + \beta_3(\mu)$ is a strictly increasing function of μ. Note that $\alpha_3(\mu) < 0$ and $\beta_3(\mu) > 0$ for any fixed μ. Moreover, direct calculation shows that $\alpha_3(\mu) + \beta_3(\mu) = 0$ has a unique root μ^* given by (12). Therefore, from (Smith 2011, Proposition4.6) we know that all roots of $\Delta_1^3 = 0$ have negative real parts when $\mu < \mu^*$, and there exists at least one root with positive real part for $\Delta_1^3 = 0$ when $\mu > \mu^*$. In addition, using $R > 1$, it is easy to see that $\mu^* > 1$, $\mu^* = 1$ and $\mu^* < 1$ if $d_2 > d_4$, $d_2 = d_4$ and $d_2 < d_4$, respectively. The above arguments together with the fact that $0 < \mu < 1$, complete the proof of the theorem. □

Remark 4.2 We note that under $d_2 < d_4$, the critical value μ^* given in (12) depends on τ and is increasing in τ, satisfying $\mu^* \to \frac{d(d_4-d_2)+d_2b}{d_4b}$ when $\tau \to 0$, and $\mu^* \to 1$ when $\tau \to \tau_{max} = \frac{1}{\delta}\ln\frac{b}{d}$.

Theorem 4.1 indicates that when $d_2 < d_4$, increasing μ to pass μ^* will destroy the stability of the infection-free equilibrium E_1. Noting that μ is the maternal transmission probability, and hence, large μ should mean better chance for the *Wolbachia* to establish in the mosquitoes. In other words, one would expect persistence of $I(t)$ when $\mu > \mu^*$, and this expectation is confirmed in the following theorem.

Theorem 4.3 *When $d_2 < d_4$ and $\mu > \mu^*$, the infected population $I(t)$ in (7) is uniformly persistent in the sense that there is an $\eta > 0$ such that every positive solution $\big(I(t), U(t)\big)$ of (7) satisfies*

$$\liminf_{t\to\infty} I(t) \geq \eta. \tag{14}$$

Proof Define $\rho_1 : X_+ \to \mathbb{R}_+$ by $\rho_1(\phi) = ||\phi_1||, \forall \phi \in X_+$. Firstly, we prove that the solution semiflow Φ of (7) is uniformly weakly ρ_1-persistent, that is, there exists some $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} \rho_1(\Phi(t,\phi)) \geq \varepsilon, \ \forall \phi \in X_+, \ \rho_1(\phi) > 0.$$

Consider space $X_\partial := \{\phi \in X_+ : \rho_1(\phi) = 0\}$. Following Chapter 8 of Smith and Thieme (2011), we examine the set $\Omega = \cup_{\phi\in X_\partial}\omega(\phi)$, where $\omega(\phi)$ is the omega-limit set of the orbit starting from ϕ. Then, $\Omega = \{E_0, E_1\}$ which is compact with both $\{E_0\}$ and $\{E_1\}$ invariant and isolated, and Ω is acyclic. Therefore, to apply Theorem 8.17 of Smith and Thieme (2011), we need to show that both E_0 and E_1 are weakly ρ_1-repellers. (Note that a set M is called weakly ρ_1-repeller if there is no $\phi \in X_+$ such that $\rho_1(\phi) > 0$ and $\Phi_t(\phi) \to M$ as $t \to \infty$.)

Assume the contrary. Then, combining Theorem 2.1, there exists a solution $\big(I(t), U(t)\big)$ with $I(t) > 0$ for $t > \tau$ such that $\big(I(t), U(t)\big) \to E_0$ or $\big(I(t), U(t)\big) \to E_1$ as $t \to \infty$. First assume $\big(I(t), U(t)\big) \to E_1$. Since $\mu > \mu^*$ which is equivalent to $\mu be^{-\delta\tau} > d + d_2\bar{U}$, then there is an $\epsilon > 0$ such that $\mu be^{-\delta\tau} > d + d_2\bar{U} + d_2\epsilon$. On the other hand, the assumption $\big(I(t), U(t)\big) \to E_1$ implies that for sufficiently large

t, $I(t) + U(t) < \bar{U} + \epsilon$. Now define

$$w(t) = e^{-\delta\tau}\mu b \int_{t-\tau}^{t} I(s)ds + I(t).$$

Then, $w(t) > 0$, and $w(t) \to 0$ as $t \to \infty$. But differentiating $w(t)$ gives

$$w'(t) = [e^{-\delta\tau}\mu b - d - d_2(I(t) + U(t))]I(t),$$

that is,

$$w'(t) > [e^{-\delta\tau}\mu b - d - d_2(U^* + \epsilon)]I(t) > 0$$

for sufficiently large t, which is a contradiction. Hence, E_1 is a weak ρ_1-repeller. Using the same auxiliary function $w(t)$ and using the same argument, we can prove that E_0 is also a weak ρ_1-repeller. Therefore, by (Smith and Thieme 2011, Theorem8.17), (7) is uniformly weakly ρ_1-persistent. The above established uniform weak ρ_1 persistence together with the dissipativity property of the model system (Theorem 2.2), a compact global attractor exists, and all the conditions of (Smith and Thieme 2011, Theorem 4.5) hold, which guarantees uniform strong persistence with respect to the distance function ρ_1, which means that $I(t)$ variable is uniformly strongly persistent, that is, (14) holds. The proof of the theorem is completed. □

Once $I(t)$ becomes uniformly strongly persistent under $d_2 < d_4$ (hence $\mu^* \in (0, 1)$) and $\mu \in (\mu^*, 1)$, the fact that $1 - \mu > 0$ implies a continued recruitment for $U(t)$, and accordingly, one would expect that $U(t)$ will also be uniformly strongly persistent. This is confirmed in the next theorem.

Theorem 4.4 *When $d_2 < d_4$ and $\mu > \mu^*$, the uninfected population $U(t)$ in (7) is also uniformly persistent in the sense that there is an $\zeta > 0$ such that every positive solution $\big(I(t), U(t)\big)$ of (7) satisfies*

$$\liminf_{t\to\infty} U(t) \geq \zeta. \tag{15}$$

Proof The proof is similar to that of Theorem 4.3, but we need to adopt the following distance to the other boundary piece in X_+: $\rho_2 : X \to \mathbb{R}_+$ by $\rho_2(\phi) = ||\phi_2||, \forall \phi \in C([-\tau, 0], X)$. Then, with respect to this distance ρ_2, the corresponding set $\Omega = \cup_{\phi\in X_\partial}\omega(\phi)$ induced by $X_\partial = \{\phi \in X_+ : \rho_2(\phi) = 0\}$ becomes $\Omega = \{E_0\}$ in this case. As in the proof of Theorem 4.3, to complete the proof, we only need to prove that E_0 is a weakly ρ_2-repeller. Note that in the proof of above theorem, we have already proved that there is no solution $(I(t), U(t))$ with $I(t) > 0$ for $t > \tau$ such that $(I(t), U(t)) \to E_0$. Thus, if E_0 is not a weakly ρ_2-repeller, then there exists a solution $(I(t), U(t))$ with $I(t) \equiv 0$ and $U(t) > 0$ for $t > \tau$ such that $(I(t), U(t)) \to E_0$. This implies that there is an $\epsilon > 0$ such that $U(t) < \epsilon$ for sufficiently large t and $e^{-\delta\tau}b - d > d_4\epsilon$. Now define

$$h(t) = e^{-\delta\tau}b \int_{t-\tau}^{t} U(s)ds + U(t),$$

then $h(t) > 0$, and $h(t) \to 0$ as $t \to \infty$ should hold. But differentiating $h(t)$ gives

$$h'(t) = [e^{-\delta\tau} b - d - d_4 U(t)]U(t),$$

that is

$$h'(t) > (e^{-\delta\tau}\mu b - d - d_4\epsilon)U(t) > 0$$

for sufficiently large t, which is a contradiction. Hence, E_0 is a weak ρ_2-repeller. □

Corollary 4.5 *When $d_2 < d_4$ and $\mu > \mu^*$, the model system (7) is uniformly strongly persistence in the sense that there is an $\eta > 0$ such that every positive solution $\big(I(t), U(t)\big)$ of (7) satisfies*

$$\liminf_{t\to\infty} U(t) \geq \eta \quad \textit{and} \quad \liminf_{t\to\infty} U(t) \geq \eta \tag{16}$$

Moreover, (7) has an interior (positive) equilibrium $E^+ = (I^+, U^+)$ with $I^+ > 0$ and $U^+ > 0$.

Proof The first part is just a direct result of the above two theorems, and the second part (existence of a positive equilibrium) can be concluded by, e.g., (Zhao 1995, Theorem 2.4). □

Remark 4.6 Using similar arguments as in the proofs of the above two theorems, we can also show that, without assuming $d_1 = d_3$, the trivial equilibrium $E_0 = (0, 0)$ is unstable for (4) for any $\mu \in [0, 1]$ and $\theta \geq 0$ provided that $R_1, R_2 > 1$.

5 Conclusion and Numerical Illustrations

We have analyzed the population dynamics of (7). Under the scenario that the trivial equilibrium $E_0 = (0, 0)$ is a repeller ($R > 1$), analytical results are obtained on existence and stability of two semi-trivial equilibria and a possible positive equilibrium. These results are described in terms of the four important parameters: d_2 and d_4 which reflect the competition capability of uninfected and infected mosquitoes, μ which accounts for the maternal transmission level, and θ which represents the CI level. For readers' convenience, we summarize the analytical results in the preceding sections as below, in term of the above mentioned parameters.

(A) When $d_2 > d_4$, meaning that the *Wolbachia*-infected mosquitoes are *less competitive* than uninfected mosquitoes, the infection-free equilibrium $E_1 = (0, \bar{U})$ is always locally asymptotically stable, meaning that the infected mosquitoes cannot establish in the mosquitoes population (hence *Wolbachia* cannot invade) *if only a small number* of infected mosquitoes are brought in. For this case, if the maternal transmission is perfect (i.e., $\mu = 1$), there is a critical CI level θ^* given by (9), such that

1) if $\theta < \theta^*$ (small CI level), then the total-infection equilibrium $E_2 = (\bar{I}, 0)$ is unstable, yielding a full competition exclusion outcome;
2) if $\theta > \theta^*$ (large CI level), then E_2 is also asymptotically stable, leading to a *bistable scenario* which implies the existence of a *unstable* positive (co-persistence) equilibrium $E^+ = (I^+, U^+)$.

(B) When $d_2 < d_4$, meaning that the *Wolbachia*-infected mosquitoes are *more competitive* than uninfected mosquitoes, there exists a critical maternal transmission rate $\mu^* \in (0, 1)$ given by (12) such that

1) if $\mu < \mu^*$ (low maternal transmission rate), then the infection-free equilibrium E_1 is asymptotically stable,
2) if $\mu \in (\mu^*, 1)$ (high maternal transmission rate), then the infection-free equilibrium E_1 is unstable and there exists a *stable* positive equilibrium $E^+ = (I^+, U^+)$;
3) if $\mu = 1$, E_1 is unstable, and E_2 exists which is asymptotically stable.

Now we present some numerical simulations to illustrate these analytic results. To this end, we fix

$$\tau = 5, \quad d = 0.05, \quad b = 0.4, \quad \delta = 0.1, \tag{17}$$

Corresponding to the case (A), we choose $\mu = 1$, $d_2 = 0.2$ and $d_4 = 0.1$. By (9), we calculate $\theta^* = 0.1012$. For $\theta = 0.05 < \theta^*$ and $\theta = 1 > \theta^*$, the solutions of (7) with various initial value are shown in Fig. 1-Left and Fig. 1-Right, respectively, demonstrating the sub-cases (A)-1) and (A)-2), respectively.

Corresponding to the case (B), we choose $\theta = 100$, $d_2 = 0.1$ and $d_4 = 0.2$. Then, from (12) we obtain $\mu^* = 0.6030$. For $\mu = 0.5 < \mu^*$, $\mu = 0.7$ and $\mu = 1$, the solutions of (7) with various initial values are shown in Fig. 2—left, Fig. 2—middle, and Fig. 2—right, respectively, illustrating the sub-cases (B)-1), (B)-2) and (B)-3), respectively.

The case of $d_2 = d_4$ is also worth mentioning. In this case, if $0 < \mu < 1$, the infection-free equilibrium E_1 is asymptotically stable from Theorem 4.1; but if

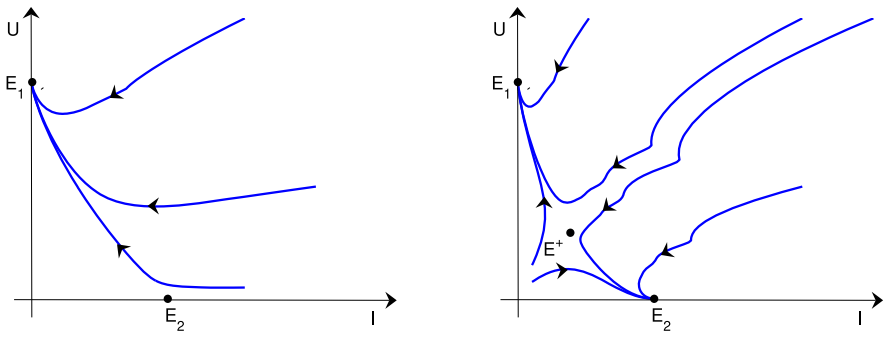

Fig. 1 Phase diagram of (7) with parameters in (17) and $\mu = 1$, $d_2 = 0.2$ and $d_4 = 0.1$ for Case (A). Left: $\theta = 0.05$; right: $\theta = 1$

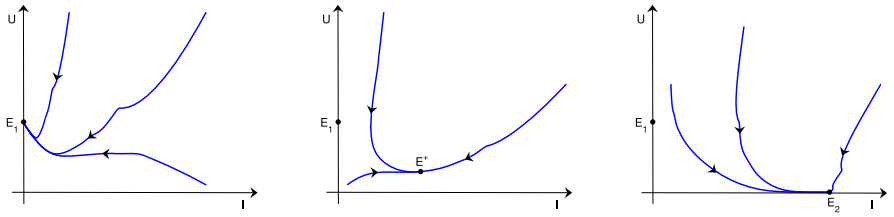

Fig. 2 Phase diagram of (7), with parameters in (17) and $\theta = 100$, $d_2 = 0.1$ and $d_4 = 0.2$ for Case (B). Left: $\mu = 0.5 < \mu^* = 0.6030$; middle: $\mu = 0.7$; right: $\mu = 1$

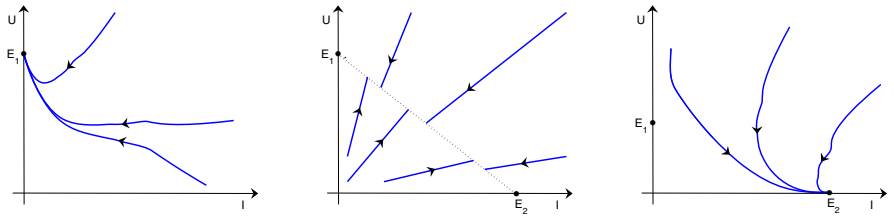

Fig. 3 Phase diagram of (7) with parameters give in (17) and $d_2 = d_4 = 0.1$. Left:$\mu = 0.6$ and $\theta = 100$; middle:$\mu = 1$ and $\theta = 0$; right: $\mu = 1$ and $\theta = 100$

$\mu = 1$, there is a special scenario: for $\theta > 0$, there is *no positive equilibrium* for (7), while for $\theta = 0$ it has *a continuum of equilibria* located on the line $\{(I, U); I + U = (e^{-\delta\tau b} - d)/d_2\}$ (can be seen from (11) in the proof of Theorem 3.4). For the latter, the asymptotic behavior of a solution depends on the initial value. To demonstrate this numerically, we choose $d_2 = d_4 = 0.1$. Figure 3-Left shows that the solutions with various initial value converge to the infection-free equilibrium E_1 when $\mu = 0.6$ and $\theta = 100$. When $\mu = 1$ and $\theta = 0$, Fig. 3—middle shows that solutions with various initial value converge to different equilibria. In Fig. 3—right, we choose $\mu = 1$ and $\theta = 100$; we see that the solutions tend toward the total infection equilibrium E_2.

We point out that these results represent three possible biological outcomes for mosquitoes infected with *Wolbachia* (or for invasion of *Wolbachia* in mosquitoes):

(a) Failure to establish—this happens when the infected mosquitoes are less competitive and the CI level is not high enough [(A)-1)], or CI level is high, but the initial population of infected mosquitoes is low [(A)-2)]; or when the infected mosquitoes are more competitive, the maternal transmission rate is low [(B)-1)];
(b) Establishing and suppressing the wild—this happens when the infected mosquitoes are more competitive, but the maternal transmission is high [(B)-2)];
(c) Establishing and replacing the wild—this happens when the infected mosquitoes are more competitive and maternal transmission is perfect [(B)-3)], or when the infected mosquitoes are less competitive, the CI level is high and the initial population of infected mosquitoes is also high [(A)-2)];

One novelty of our model is the incorporation of maturation delay and relating it to the CI effect. The delay can affect the population dynamics, as seen in the condition $R > 1$ where R depends on τ and explored in Remarks 3.3 and 4.2. These observations offer some insights into the populations dynamics that an ODE model cannot provide.

For example, the much worried/concerned global warming is believed/expected to shorten the maturation time. On the other hand, in the case $d_2 < d_4$ (Wolbachia-infected mosquitoes are more competitive), smaller τ will result in *a smaller* μ^* by Remark 4.2, and by Theorem 4.1-(ii) and Corollary 4.5, this would *enhance the chances* for the Wolbachia to establish in mosquitoes. While in the case $d_2 > d_4$ (Wolbachia-infected mosquitoes are less competitive), smaller τ will lead to *a larger* θ^*, and by Theorem 3.2-(iii)–(iv), this would *decrease the chances* for the Wolbachia to establish. Therefore, we can see that the impact of the global warming on Wolbachia's establishment would depend on the density-dependent death rate d_2 of the infected mosquitoes vs that of uninfected mosquitoes (d_4), which partially reflects the fitness of the infected mosquitoes.

We remark that our results on the stability of E_1 and E_2 are local, leaving the global stability under the respective cases open mathematical problems. Also, for the scenario when there is a positive equilibrium $E^+ = (I^+, U^+)$, we have not discussed its stability/ instability, let alone its global stability. Since the model (7) is a DDE system with both negative and positive delayed feedbacks, it is not an order preserving (monotone) DDE system. This makes the aforementioned problems mathematically challenging, and we leave them as a future research projects. Our numerical simulation results are suggestive and should hive some hints and directions.

6 Some Discussions on the Model

We would like to conclude this paper by some discussions on the modeling aspects. In (7), we have assumed $d_1 = d_3$ but allowed different density-dependent death rates to account for a fitness cost/benefit of immature infected mosquitoes in competition capability for resources. We point out that one may also opt to assume $d_2 = d_4$ and allow different density-independent death rates d_1 and d_3. Since the density-independent death rate is a reflection of genetic characteristics, $d_1 - d_3$ can also be considered as a fitness cost/benefit of the infected mosquitoes. Also, in (4) we have assumed the same death rate δ for the eggs/juveniles produced by both uninfected and infected female mosquitoes. As mentioned above, there may also be a fitness cost/benefit for the infected females in the death rate of immature stage.

Also, as far as competition is concerned, for sexual species like mosquitoes, competition for mating is as important as for resources, sometimes even more important. Therefore, assuming different mating rates for uninfected and infected mosquitoes may allow us to explore another fitness cost/benefit of the infected mosquitoes. In addition, there may also be cost/benefit for *Wolbachia*-infected female mosquitoes in production rate. With regard to these two aspects, the recruitment terms for $U(t)$ and $I(t)$ in (2) and (3) can then be generalized to

$$b_{02}\frac{U(t)}{I(t)+U(t)}\cdot m_2U(t)+(1-q)b_{20}\frac{U(t)}{I(t)+U(t)}\cdot m_1I(t) \tag{18}$$

and

$$b_{01}\frac{I(t)}{I(t)+U(t)}\cdot m_2U(t)+b_01\frac{I(t)}{I(t)+U(t)}\cdot m_1I(t), \tag{19}$$

respectively. Here, b_{02} (resp. b_{01}) is the per capita birth rate rate for uninfected (resp. infected) female mosquitoes; m_2 (resp. m_1) is the mating rate of uninfected (resp. infected) male mosquitoes (assuming that matings are driven by male mosquitoes).

Taking all these aspects into consideration, as well as incorporating the CI and the maternal transmission effects, one then can obtain a more general model corresponding to (4), which is given by the following DDE system

$$\begin{cases} \dfrac{dI}{dt} = \mu e^{-\delta_1\tau} b_{01} I(t-\tau)\dfrac{m_2 U(t-\tau)+m_1 I(t-\tau)}{I(t-\tau)+U(t-\tau)} - [d_1 + d_2(I(t)+U(t))]I(t) \\ \dfrac{dU}{dt} = (1-\mu)e^{-\delta_1\tau} b_{01} I(t-\tau)\dfrac{m_2 U(t-\tau)+m_1 I(t-\tau)}{I(t-\tau)+U(t-\tau)} \\ \qquad + e^{-\delta_2\tau} b_{02} m_2 \dfrac{U^2(t-\tau)}{I(t-\tau)+U(t-\tau)} + e^{-(\delta_2+\theta)\tau} b_{02} m_1 \dfrac{U(t-\tau)I(t-\tau)}{I(t-\tau)+U(t-\tau)} \\ \qquad - [d_3 + d_4(I(t)+U(t))]U(t). \end{cases} \tag{20}$$

This model system can obviously accommodate more aspects for fitness cost/benefit of the infected mosquitoes. When all mating rates are the same ($m_1 = m_2 = m$), production rates are the same ($b_{01} = b_{02}$), and immature death rates are the same ($\delta_1 = \delta_2 = \delta$), (18) and (19) reduce to (2) and (3), respectively, with b denoting bm, and (20) reduces to (4). We believe such a generalized model would bring in more interesting yet challenging problems both in mathematics and in biology and will leave them as possible future research projects.

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