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A GENERALIZATION OF DARBOUX-FRODA THEOREM AND ITS APPLICATIONS

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ABSTRACT. In real analysis, the Darboux-Froda theorem states that all discontinuities of a real-valued monotone functions of a real variable are at most countable. In this paper, we extend this theorem to *a family* of monotone *real vector-valued functions* of a real variable arising from dynamical systems. To this end, we explore some essential characteristics of countable and uncountable sets by the notions of strong cluster points, upper and lower strong cluster points, and establish the existence of strong cluster point sets, upper and lower strong cluster point sets for an uncountable set. With the help of these strong cluster point sets, we establish a jump lemma that helps characterize the discontinuities of the family of monotone vector-functions. Then we introduce the notion of distinction set and prove the existence of a distinction set. Making use of the upper and lower strong cluster points of the distinction set and the jump lemma, we prove the Darboux-Froda extension theorem. Moreover, we also present two applications of the generalized Darboux-Froda theorem.

1. INTRODUCTION

In real analysis, there is a well-known result about the discontinuities of monotone real-valued functions of a real variable, which states that all discontinuities of a monotone function in a real interval are necessarily jump discontinuities and they are at most countable. This result is important in real analysis as it helps characterize monotone functions and is closely related to other important theorems in real analysis. Although not explicitly stated, this result was included in the famous French mathematician Darboux's work in 1875 [3]; later in 1929, Froda gave an explicit statement of this result and provided a clear proof in his dissertation [7]. Nowadays, this result as a theorem and its various versions of proof can be found in many text books (e.g., [1, 8, 9, 12]). Following Wikipedia [20], we also refer this theorem as Darboux-Froda theorem.

For a real vector-valued function of a real variable $\boldsymbol{\phi}(s) = (\phi_1(s), \cdots, \phi_n(s))^T$, note the $\boldsymbol{\phi}(s)$ is continuous at s if and only if $\phi_i(s)$ is continuous at s for all $i = 1, \cdots, n$; and hence, the set of discontinuities of such a vector-function is nothing but the union of discontinuities of all $\phi_i(s)$, $i = 1, \cdots, n$. Thus, if each

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$\phi_i(s)$ is monotone, then the set of discontinuities of $\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^n$ is also at most countable. For convenience, we call $\boldsymbol{\phi}(s) = (\phi_1(s), \cdots, \phi_n(s))$ a *monotone vector-function* if each $\phi_i(s)$ is monotone. Thus, Darboux-Froda theorem also holds for monotone vector-functions of a real variable.

Very often, one needs to consider a family of monotone vector-functions, denoted by $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ where $s \in \mathbb{R}$ and the element/parameter $\boldsymbol{\theta}$ belongs to certain type of set denoted by $\mathcal{M}$. If $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is monotone in s for each $\boldsymbol{\theta} \in \mathcal{M}$, then by the above Darboux-Froda theorem, the set of discontinuities of $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$, denoted by $\Sigma_{\boldsymbol{\theta}}$ is at most countable, implying that $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous in $s \in \mathbb{R} \setminus \Sigma_{\boldsymbol{\theta}}$. The set of discontinuities of the family $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is then given by $\Sigma = \bigcup_{\boldsymbol{\theta} \in \mathcal{M}} \Sigma_{\boldsymbol{\theta}}$. This set is also at most countable if the set $\mathcal{M}$ is countable, but it may not be countable if $\mathcal{M}$ is uncountable. Particularly, in the study of traveling waves for evolutionary systems including nonlocal dispersal equations with time delay and reaction-diffusion equations in a cylinder and so forth, a family of monotone vector-functions associated with traveling waves is still denoted by $\boldsymbol{\phi} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n$ with $\mathcal{M}$ being $[-\tau, 0]$ in the case of the nonlocal diffusion equations with a time delay τ, or the projection region of a cylinder in the base subspace in the case of reaction-diffusion equations in a cylinder. A *natural question* is whether there are suitable conditions on $\boldsymbol{\phi}$ such that $\bigcup_{\boldsymbol{\theta} \in \mathcal{M}} \Sigma_{\boldsymbol{\theta}}$ is at most countable?

In this paper, we will address this question. Indeed, we can prove (main theorem of the paper) that if (i) for each $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is monotone in $s \in \mathbb{R}$ and (ii) $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous in $\boldsymbol{\theta}$ for each s, then $\bigcup_{\boldsymbol{\theta} \in \mathcal{M}} \Sigma_{\boldsymbol{\theta}}$ is at most countable.

To prove the above mentioned generalized Darboux-Froda theorem for $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$, we develop a novel method with the key lying in describing the characteristic of uncountable sets in $\mathcal{M} \times \mathbb{R}$ and establishing a *jump lemma*. Specifically, we wish to find a subset of an uncountable set such that every point in it is a cluster point, and yet, a cluster point is not necessarily a cluster point of the cluster point set. For example, let $\mathcal{I} = \{\boldsymbol{\theta}_0\} \times ([-1, 0] \cup \{1 + \frac{1}{n}\})$ and $\mathcal{J} = \{\boldsymbol{\theta}_0\} \times (\{1\} \cup [-1, 0])$ with $\boldsymbol{\theta}_0 \in \mathcal{M}$, then the cluster point set of $\mathcal{I}$ is $\mathcal{J}$, but $\{\boldsymbol{\theta}_0\} \times \{1\}$ is not a cluster point of $\mathcal{J}$. In general, it is difficult to find the cluster point set without isolated points if we are confined only to the cluster point set. Based on the essential difference between countable and uncountable sets in $\mathcal{M} \times \mathbb{R}$, we introduce the definitions of strong cluster points as well as upper and lower strong cluster points. It is worth noting that strong cluster points are special kinds of cluster points: each strong cluster point is a strong cluster point in the whole set of strong cluster points (i.e., the strong cluster point set).

In the rest of this paper, we first introduce, in Section 2, the notions of strong cluster points and as well as upper and lower strong cluster point, and use them to describe the clustering features of uncountable sets. Section 3 deals with the continuity/discontinuity of the family of monotone (in s) vector-functions $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$. By establishing a *jump lemma* and confirming the existence of a distinction set, we prove the aforementioned generalized Darboux-Froda theorem. In Section 4, as two direct applications of the generalized Darboux-Froda theorem, we present two propositions, one of them describes the equivalence of functions in $\mathcal{M} \times (\mathbb{R} \setminus \Sigma)$, and the other confirms the continuity of the translation operator associated to a class of *non-compact semiflows* arising from the study of traveling wave solutions to some dynamical systems in [2]. These two propositions are crucial in overcoming the lack of compactness in our forthcoming project on the existence of traveling waves for some dynamical systems that are *only point asymptotically smooth but not necessarily asymptotically smooth.*

2. Strong cluster point

In this section, we first introduce some notations. Let $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^n_+$ be the sets of all natural numbers, positive integers, reals, nonnegative reals, n-dimensional real vectors, and n-dimensional nonnegative real vectors, where n is a positive integer. Let $(\mathcal{M}, d)$ be a compact metric space. For any given $(\boldsymbol{\theta}, \delta) \in \mathcal{M} \times (0, \infty)$, we use $U_\delta(\boldsymbol{\theta})$ to denote the open ball in $\mathcal{M}$ with center $\boldsymbol{\theta}$ and radius δ.

To characterize the clustering features of uncountable sets in $\mathcal{M}\times\mathbb{R}$, we introduce the notions of (upper or lower) strong cluster point and (upper or lower) strong cluster point set, which are crucial for studying the continuity of the family of monotone vector-functions.

Definition 2.1. Suppose $\mathcal{I} \subseteq \mathcal{M} \times \mathbb{R}$ is non-empty. A point $p \triangleq (\boldsymbol{\theta}, s) \in \mathcal{I}$ is said to be

(i) a strong cluster point w.r.t $\mathcal{I}$ if for any $\delta > 0$, $[U_\delta(\boldsymbol{\theta}) \times (s-\delta, s+\delta)] \cap \mathcal{I}$ is uncountable;

(ii) a upper (resp. lower) strong point w.r.t. $\mathcal{I}$ if for any $\delta > 0$, $[U_\delta(\boldsymbol{\theta}) \times (s, s+\delta)] \cap \mathcal{I}$ (resp. $[U_\delta(\boldsymbol{\theta}) \times (s-\delta, , s)] \cap \mathcal{I}$) is uncountable.

A subset $\mathcal{I}$ is called a (resp. upper or lower) strong cluster point set if each point of $\mathcal{I}$ is a (resp. upper or lower) strong cluster point w.r.t $\mathcal{I}$.

Remark 2.1. A strong cluster point must be a cluster point, but a cluster point may not be a strong cluster point. An upper or lower strong cluster point (set) must be a strong cluster point (set).

Next, we give the following properties according to the definitions of strong cluster points, as well as upper and lower strong cluster points.

Proposition 2.1. *Suppose that $\mathcal{I} \subseteq \mathcal{M} \times \mathbb{R}$ is an uncountable subset, and $\mathcal{J} \subseteq \mathcal{I}$. Then the following statements are valid:*

(i) *A (resp. upper or lower) strong cluster point w.r.t. to $\mathcal{J}$ must be a (resp. upper or lower) strong cluster point w.r.t. $\mathcal{I}$;*

(ii) *If $\mathcal{I} \setminus \mathcal{J}$ is at most countable, then a (resp. upper or lower) strong cluster point w.r.t. $\mathcal{I}$ must be a (resp. upper or lower) strong cluster point w.r.t. $\mathcal{J}$.*

Proof. (i) is obvious, so we just need to prove (ii) below. We only give the proof for the case of strong cluster point, and other cases can be proved by similar arguments. If the conclusion does not hold, then there exist $(\boldsymbol{\theta}_0, s_0) \in \mathcal{J}$ and $\delta_0 > 0$ such that $[U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap \mathcal{J}$ is at most countable. Since $(\boldsymbol{\theta}_0, s_0)$ is a strong cluster point with respect to $\mathcal{I}$, we see that $[U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap \mathcal{I}$ is uncountable. Then, the set decomposition relation

$$\begin{aligned}&[U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap \mathcal{I} \\ &= ([U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap \mathcal{J}) \cup ([U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap (\mathcal{I} \setminus \mathcal{J}))\end{aligned}$$

implies that $[U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0 + \delta_0)] \cap (\mathcal{I} \setminus \mathcal{J})$ is uncountable, a contradiction, which completes the proof of statement (ii) for the strong cluster point case. □

Definition 2.2. If for any $(\boldsymbol{\theta}, s), (\tilde{\boldsymbol{\theta}}, \tilde{s}) \in \mathcal{I}$ with $(\boldsymbol{\theta}, s) \neq (\tilde{\boldsymbol{\theta}}, \tilde{s})$, we have $\boldsymbol{\theta} \neq \tilde{\boldsymbol{\theta}}$ and $s \neq \tilde{s}$, then we call $\mathcal{I}$ a distinction set.

Remark 2.2. A strong cluster point of the distinction set must be an upper strong cluster point or a lower strong cluster point.

Lemma 2.1. *Suppose that $\mathcal{I} \subseteq \mathcal{M} \times \mathbb{R}$ is an uncountable subset. Then there exists a strong cluster point w.r.t. $\mathcal{I}$.*

Proof. Otherwise, for any $p := (\boldsymbol{\theta}, s) \in \mathcal{I}$, there exists $\delta_p > 0$ such that $[U_{\delta_p}(\boldsymbol{\theta}) \times (s - \delta_p, s + \delta_p)] \cap \mathcal{I}$ is at most countable. Since $\mathcal{M}$ is a compact metric space, it follows from [10, Theorem 5.5-(d)] that $\mathcal{M}$ is a second countable space. This, combined with the fact that $\mathbb{R}$ is a second countable space, implies that $\mathcal{M} \times \mathbb{R}$ is a second countable space, and hence $\mathcal{I}$ is a Lindelöff space by [10, Theorem 1.15]. Therefore, there are countable points $p_i := (\boldsymbol{\theta}_i, s_i) \in \mathcal{I}$ such that $\mathcal{I} \subseteq \bigcup_{i=1}^{\infty} U_{\delta_{p_i}}(\boldsymbol{\theta}_i) \times (s_i - \delta_{p_i}, s_i + \delta_{p_i})$. Since $[U_{\delta_{p_i}}(\boldsymbol{\theta}_i) \times (s_i - \delta_{p_i}, s_i + \delta_{p_i})] \cap \mathcal{I}$ is at most countable, we see that $\mathcal{I}$ is at most countable, a contradiction. This completes the proof. □

Lemma 2.2. *Suppose that $\mathcal{I} \subseteq \mathcal{M} \times \mathbb{R}$ is an uncountable subset and let $\mathcal{J}$ be the set of all strong cluster points w.r.t. $\mathcal{I}$. Then $\mathcal{I} \setminus \mathcal{J}$ is at most countable and every point in $\mathcal{J}$ is indeed also a cluster point w.r.t. $\mathcal{J}$.*

Proof. By Lemma 2.1, $\mathcal{J}$ is non-empty. We first show that $\mathcal{I} \setminus \mathcal{J}$ is at most countable. Otherwise, $\mathcal{I} \setminus \mathcal{J}$ is uncountable. This, combined with Lemma 2.1, implies that there exists $p_0 \in \mathcal{I} \setminus \mathcal{J}$ such that p_0 is a strong cluster point with respect to $\mathcal{I} \setminus \mathcal{J}$. Thus, in virtue of Proposition 2.1-(i), p_0 is a strong cluster point with respect to $\mathcal{I}$. It follows from the definition of $\mathcal{J}$ that $p_0 \in \mathcal{J}$, a contradiction. This proves the claim. The second conclusion follows from this statement and Proposition 2.1-(ii). □

Proposition 2.2. *Suppose $\mathcal{I} \subseteq \mathcal{M} \times \mathbb{R}$ is a distinction subset. If $\mathcal{I}$ is uncountable, then there exists a subset $\mathcal{K} \subseteq \mathcal{I}$ such that $\mathcal{K}$ is an upper or lower strong cluster point set.*

Proof. Let $\mathcal{J}_+$ (resp. $\mathcal{J}_-$) be the set of all upper (resp. lower) strong cluster points with respect to $\mathcal{J}$, where $\mathcal{J}$ is shown in Lemma 2.2. Since $\mathcal{J} \subset \mathcal{I}$, $\mathcal{J}$ is also a distinction set. It then follows from Remarks 2.1 and 2.2 that $\mathcal{J} = \mathcal{J}_- \cup \mathcal{J}_+$. There are two cases to be considered below.

Case I. $\mathcal{J}_-$ or $\mathcal{J}_+$ is at most countable. Without loss of generality, we assume $\mathcal{J}_-$ is at most countable. The conclusions can be proved by similar arguments if $\mathcal{J}_+$ is at most countable. In this case, $\mathcal{J} \setminus \mathcal{J}_+$ is at most countable. In virtue of the definition of $\mathcal{J}_+$ and Proposition 2.1-(ii), we see that $\mathcal{J}_+$ is an upper strong cluster point set. Hence, $\mathcal{K} := \mathcal{J}_+$ is what is needed for this proposition.

Case II. Both $\mathcal{J}_-$ and $\mathcal{J}_+$ are uncountable. We proceed with the two subcases below.

Case (II-1). $\mathcal{J}_+ \setminus \mathcal{J}_-$ is at most countable. Evidently, $\mathcal{J} \setminus \mathcal{J}_- = \mathcal{J}_+ \setminus \mathcal{J}_-$ is at most countable from Remark 2.2. Again, by Proposition 2.1-(ii), we find that $\mathcal{J}_-$ is a lower strong cluster point set from the definition of $\mathcal{J}_-$. Thus $\mathcal{K} := \mathcal{J}_-$ is what is required for this proposition.

Case (II-2). $\mathscr{J}_+ \setminus \mathscr{J}_-$ is uncountable. Let $\tilde{\mathscr{J}} = \mathscr{J}_+ \setminus \mathscr{J}_-$, and denote all the upper (resp. lower) strong cluster points with respect to $\tilde{\mathscr{J}}$ by $\tilde{\mathscr{J}}_+$ (resp. $\tilde{\mathscr{J}}_-$). It then follows from the contraposition of Proposition 2.1-(i) that $\tilde{\mathscr{J}}_- = \emptyset$, and hence $\tilde{\mathscr{J}} = \tilde{\mathscr{J}}_+$. That is, $\tilde{\mathscr{J}}$ consists of upper strong cluster point; and therefore, $\mathcal{K} := \tilde{\mathscr{J}}$ is what is needed for this proposition.

□

3. A generalized Darboux-Froda theorem

Making use of the notions and properties of upper and lower strong cluster point sets established in the preceding section, we now explore some crucial properties for a family of monotone vector-functions in this section.

Let $Y = C(\mathcal{M}, \mathbb{R}^n)$ be the set of all bounded and continuous functions from $\mathcal{M}$ to $\mathbb{R}^n$ equipped with the usual supremum norm $\|\cdot\|_Y$, and $Y_+ = C(\mathcal{M}, \mathbb{R}^n_+)$. Denote $\mathscr{D} = \{\boldsymbol{\phi} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n | \boldsymbol{\phi}(\boldsymbol{\theta}, s) \text{ is continuous in } \boldsymbol{\theta} \text{ and nonincreasing in } s\}$. Here, we adopt the standard partial order "$\leq$" in $\mathbb{R}^n$ when referring to the monotonicity of $\boldsymbol{\phi}(\cdot, s)$ in s. In particular, for any $\boldsymbol{\phi} \in \mathscr{D}$, it can be regarded as a monotone function from $\mathbb{R}$ to Y. If $\boldsymbol{\phi} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n$, and $\boldsymbol{\phi}(\cdot, s)$ is nonincreasing in s, then for any $(\boldsymbol{\theta}, s) \in \mathcal{M} \times \mathbb{R}$, the left and right limits of $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ at s exist, denoted by $\boldsymbol{\phi}(\boldsymbol{\theta}, s^-)$ and $\boldsymbol{\phi}(\boldsymbol{\theta}, s^+)$, respectively.

According to the monotonicity of $\boldsymbol{\phi}$ in s, we can easily obtain the following results.

Lemma 3.1. *Suppose that $\boldsymbol{\phi} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n$, and $\boldsymbol{\phi}(\cdot, s)$ is nonincreasing in s. Then the following statements are valid:*

(i) $\boldsymbol{\phi}(\boldsymbol{\theta}, s^-) \geq \boldsymbol{\phi}(\boldsymbol{\theta}, s^+)$, $\forall (\boldsymbol{\theta}, s) \in \mathcal{M} \times \mathbb{R}$.
(ii) *For any fixed $(\boldsymbol{\theta}, s) \in \mathcal{M} \times \mathbb{R}$, $\boldsymbol{\phi}(\boldsymbol{\theta}, s^-) > \boldsymbol{\phi}(\boldsymbol{\theta}, s^+)$ if and only if $\boldsymbol{\phi}(\boldsymbol{\theta}, \cdot)$ is discontinuous at s.*

The key to the proof of the generalized Darboux-Froda theorem is to establish the following *jump lemma* from the monotonicity in s and the continuity in $\boldsymbol{\theta}$ of $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ as well as the existence of strong cluster points of an uncountable set.

Lemma 3.2. *Let $k_0 \in \mathbb{N}_+$ and $\boldsymbol{\phi} \in \mathscr{D}$ be given. Suppose $\mathcal{K} \subseteq \{(\boldsymbol{\theta}, s) \in \mathcal{M} \times \mathbb{R} : \boldsymbol{\phi}(\boldsymbol{\theta}, s_-) - \boldsymbol{\phi}(\boldsymbol{\theta}, s^+) > \frac{1}{k_0}\}$ and $(\boldsymbol{\theta}_0, s_0) \in \mathcal{K}$. Then the following statements are valid:*

(i) *If $\mathcal{K}$ is a lower strong cluster point set, then there exists $(\boldsymbol{\theta}^*, s^*) \in \mathcal{I}$ such that $s^* < s_0$ and $\boldsymbol{\phi}(\boldsymbol{\theta}^*, s^*) - \boldsymbol{\phi}(\boldsymbol{\theta}_0, s_0) > \frac{1}{3k_0}$.*
(ii) *If $\mathcal{K}$ is an upper strong cluster point set, then there exists $(\boldsymbol{\theta}^*, s^*) \in \mathcal{I}$ such that $s^* > s_0$ and $\boldsymbol{\phi}(\boldsymbol{\theta}_0, s_0) - \boldsymbol{\phi}(\boldsymbol{\theta}^*, s^*) > \frac{1}{3k_0}$.*

Proof. According to the continuity of $\boldsymbol{\phi}(\boldsymbol{\theta}, s_0)$ at $\boldsymbol{\theta}_0$, we can find $\delta_0 := \delta_0(\boldsymbol{\theta}_0, s_0) > 0$ such that

$$\|\boldsymbol{\phi}(\boldsymbol{\theta}, s_0) - \boldsymbol{\phi}(\boldsymbol{\theta}_0, s_0)\| \leq \frac{1}{3k_0}, \ \forall \boldsymbol{\theta} \in U_{\delta_0}(\boldsymbol{\theta}_0).$$

Note that $A := [U_{\delta_0}(\boldsymbol{\theta}_0) \times (s_0 - \delta_0, s_0)] \cap \mathcal{K}$ is uncountable because $\mathcal{K}$ is a lower strong cluster point set. For any $(\boldsymbol{\theta}, s) \in A$, by the continuity of $\boldsymbol{\phi}(\cdot, s)$ at $\boldsymbol{\theta}$, there exists $\delta_{\boldsymbol{\theta},s} > 0$ such that $U_{\delta_{\boldsymbol{\theta},s}}(\boldsymbol{\theta}) \subseteq U_{\delta_0}(\boldsymbol{\theta}_0)$, and

$$\|\boldsymbol{\phi}(\tilde{\boldsymbol{\theta}}, s) - \boldsymbol{\phi}(\boldsymbol{\theta}, s)\| \leq \frac{1}{3k_0}, \ \forall \tilde{\boldsymbol{\theta}} \in U_{\delta_{\boldsymbol{\theta},s}}(\boldsymbol{\theta}).$$

Let $\widetilde{\mathcal{M}} = [0,\delta_0]\times\mathcal{M}$ and $\mathcal{I} = \{(\delta,\boldsymbol{\theta},s)\in\widetilde{\mathcal{M}}\times\mathbb{R} : (\boldsymbol{\theta},s)\in A \text{ and } \delta=\delta_{\boldsymbol{\theta},s}\}$. Then $\widetilde{\mathcal{M}}$ is a compact metric space and $\mathcal{I}$ is an uncountable subset of $\widetilde{\mathcal{M}}\times\mathbb{R}$. Due to Lemma 2.1, we can find $(\delta_{\boldsymbol{\theta}_1,s_1},\boldsymbol{\theta}_1,s_1)\in\mathcal{I}$ such that $(\delta_{\boldsymbol{\theta}_1,s_1},\boldsymbol{\theta}_1,s_1)$ is a strong cluster point with respect to $\mathcal{I}$. Thus, there exists $(\delta_{\boldsymbol{\theta}_2,s_2},\boldsymbol{\theta}_2,s_2)\in\mathcal{I}$ such that $|\delta_{\boldsymbol{\theta}_2,s_2}-\delta_{\boldsymbol{\theta}_1,s_1}|<\frac{\delta_{\boldsymbol{\theta}_1,s_1}}{2}$, $d(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)<\frac{\delta_{\boldsymbol{\theta}_1,s_1}}{2}$. So $\boldsymbol{\theta}_1,\boldsymbol{\theta}_2\in U_{\delta_{\boldsymbol{\theta}_1,s_1}}(\boldsymbol{\theta}_1)\cap U_{\delta_{\boldsymbol{\theta}_2,s_2}}(\boldsymbol{\theta}_2)$. Without loss of generality, we assume that $s_2\geq s_1$. It follows from the monotonicity of $\boldsymbol{\phi}$ and the condition of $\mathcal{K}$ that

$$\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_1)-\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_0)\geq\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_2^-)-\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_2^+)>\frac{1}{k_0}.$$

Consequently,

$$\begin{aligned}\boldsymbol{\phi}(\boldsymbol{\theta}_1,s_1)-\boldsymbol{\phi}(\boldsymbol{\theta}_0,s_0) &= \boldsymbol{\phi}(\boldsymbol{\theta}_1,s_1)-\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_1)+\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_1)-\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_0)+\boldsymbol{\phi}(\boldsymbol{\theta}_2,s_0)\\ &\quad -\boldsymbol{\phi}(\boldsymbol{\theta}_0,s_0)\\ &> -\frac{1}{3k_0}+\frac{1}{k_0}-\frac{1}{3k_0}\\ &= \frac{1}{3k_0}.\end{aligned}$$

Taking $(\boldsymbol{\theta}^*,s^*)=(\boldsymbol{\theta}_1,s_1)$, (i) is then proved.

The proof of (ii) is similar and is thus omitted here. □

By establishing the existence of the distinction set, we can obtain its upper and lower strong cluster point sets. Then the following statement follows from the *jump lemma*.

Proposition 3.1. *Suppose $\boldsymbol{\phi}\in\mathscr{D}$, and let $S=\{(\boldsymbol{\theta},s)\in\mathcal{M}\times\mathbb{R} : \boldsymbol{\phi}(\boldsymbol{\theta},s^-)>\boldsymbol{\phi}(\boldsymbol{\theta},s^+)\}$ and $\Sigma=\{s\in\mathbb{R} :$ there exists $\boldsymbol{\theta}\in\mathcal{M}$ such that $(\boldsymbol{\theta},s)\in S\}$. Then Σ is at most countable.*

Proof. We distinguish two cases to complete the proof.

Case 1. $\boldsymbol{\phi}(\mathcal{M}\times\mathbb{R})$ is a bounded subset of $\mathbb{R}^n$. Otherwise, Σ is uncountable. We claim that there exists an uncountable subset of S such that S is a distinction set. Indeed, for any $s\in\Sigma$, we can choose $\boldsymbol{\theta}_s\in\mathcal{M}$ such that $(\boldsymbol{\theta}_s,s)\in S$ by the definition of Σ. Set $\tilde{S}=\{(\boldsymbol{\theta}_s,s)\in\mathcal{M}\times\mathbb{R} : s\in\Sigma\}$. Then $\tilde{S}\subseteq S$, and $\tilde{S}$ is uncountable. Let $M=\{\boldsymbol{\theta}_s\in\mathcal{M} : s\in\Sigma\}$, and for any $\boldsymbol{\theta}\in M$, we define $\Sigma_{\boldsymbol{\theta}}=\{s\in\mathbb{R} : (\boldsymbol{\theta},s)\in\tilde{S}\}$. Clearly, $\bigcup_{\boldsymbol{\theta}\in M}\Sigma_{\boldsymbol{\theta}}=\Sigma$, and $\Sigma_{\boldsymbol{\theta}}$ is at most countable from Lemma 3.1-(ii) and the monotonicity of $\boldsymbol{\phi}(\boldsymbol{\theta},\cdot)$. This, combined with the fact that Σ is uncountable, implies that M is an uncountable set. It follows from the definition of M that for any $\boldsymbol{\theta}\in M$, there exists $s_{\boldsymbol{\theta}}\in\mathbb{R}$ such that $(\boldsymbol{\theta},s_{\boldsymbol{\theta}})\in\tilde{S}$. Denote $\mathbb{S}=\{(\boldsymbol{\theta},s_{\boldsymbol{\theta}}) : \boldsymbol{\theta}\in M\}$. Then $\mathbb{S}\subseteq\tilde{S}$ is uncountable, and $\mathbb{S}$ is a distinction set. Thus, the claim holds true.

For any $k\in\mathbb{N}_+$, let $\mathbb{S}_k=\{(\boldsymbol{\theta},s)\in\mathbb{S} : \boldsymbol{\phi}(\boldsymbol{\theta},s^-)-\boldsymbol{\phi}(\boldsymbol{\theta},s^+)>\frac{1}{k}\}$. Then $\mathbb{S}=\bigcup_{k\in\mathbb{N}_+}\mathbb{S}_k$. Hence, there is $k_0\in\mathbb{N}_+$ such that $\mathcal{I}=:\mathbb{S}_{k_0}$ is uncountable. According to Proposition 2.2, there exists an uncountable subset $\mathcal{K}\subseteq\mathcal{I}$ such that $\mathcal{K}$ is an upper or lower strong cluster point set. Without loss of generality, we may assume that $\mathcal{K}$ is a lower strong cluster point set. Choose $(\boldsymbol{\theta}_0,s_0)\in\mathcal{K}$. It then follows from Lemma 3.2-(i) that there exists $(\boldsymbol{\theta}_1,s_1)\in\mathcal{K}$ such that $s_1<s_0$ and $\boldsymbol{\phi}(\boldsymbol{\theta}_1,s_1)-\boldsymbol{\phi}(\boldsymbol{\theta}_0,s_0)>\frac{1}{3k_0}$. Again, applying Lemma 3.2-(i) to $(\boldsymbol{\theta}_1,s_1)\in\mathcal{K}$, there exists $(\boldsymbol{\theta}_2,s_2)\in\mathcal{K}$ such that $s_2 < s_1$, and $\boldsymbol{\phi}(\boldsymbol{\theta}_2, s_2) - \boldsymbol{\phi}(\boldsymbol{\theta}_1, s_1) > \frac{1}{3k_0}$. In a similar way, we can choose the sequence $\{(\boldsymbol{\theta}_i, s_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{K}$ such that $s_i < s_{i-1}$ and $\boldsymbol{\phi}(\boldsymbol{\theta}_i, s_i) - \boldsymbol{\phi}(\boldsymbol{\theta}_{i-1}, s_{i-1}) > \frac{1}{3k_0}$ with $i \in \mathbb{N}$. Therefore, $\lim_{i\to\infty} \boldsymbol{\phi}(\boldsymbol{\theta}_i, s_i) = \infty$, which contradicts the fact that $\boldsymbol{\phi}(\mathcal{M}\times\mathbb{R})$ is a bounded subset of $\mathbb{R}^n$. This completes the proof.

Case 2. $\boldsymbol{\phi}(\mathcal{M} \times \mathbb{R})$ is an unbounded subset of $\mathbb{R}^n$. For any $m \in \mathbb{N}_+$, define $\boldsymbol{\phi}_m : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^n$ by

$$\boldsymbol{\phi}_m(\boldsymbol{\theta}, s) = \begin{cases} \boldsymbol{\phi}(\boldsymbol{\theta}, m), & \forall(\boldsymbol{\theta}, s) \in \mathcal{M} \times (-\infty, -m), \\ \boldsymbol{\phi}(\boldsymbol{\theta}, s), & \forall(\boldsymbol{\theta}, s) \in \mathcal{M} \times [-m, m], \\ \boldsymbol{\phi}(\boldsymbol{\theta}, -m), & \forall(\boldsymbol{\theta}, s) \in \mathcal{M} \times (m, \infty). \end{cases}$$

It is easy to see that $\boldsymbol{\phi}_m \in \mathscr{D}$ and $\boldsymbol{\phi}_m(\mathcal{M} \times \mathbb{R})$ is a bounded subset of $\mathbb{R}^n$. Let $\Sigma_m = \{s \in \mathbb{R} :$ there exists $\boldsymbol{\theta} \in \mathcal{M}$ such that $\boldsymbol{\phi}_m(\boldsymbol{\theta}, s^-) > \boldsymbol{\phi}_m(\boldsymbol{\theta}, s^+)\}$. Thus, applying the statement of Case 1 to $\boldsymbol{\phi}_m$, we obtain that Σ_m is at most countable. In view of the definitions of $\boldsymbol{\phi}_m$ and Σ, we have $\Sigma = \bigcup_{m=1}^{\infty} \Sigma_m$, and hence Σ is at most countable.

Combining Case 1 and Case 2, the proposition is proved. □

With Lemma 3.1-(ii), and Proposition 3.1, we are now in the position to establish the following generalized Darboux-Froda theorem for $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$.

Theorem 3.1. *Suppose $\boldsymbol{\phi} \in \mathscr{D}$. Then there exists a subset $\Sigma \subseteq \mathbb{R}$ such that Σ is at most countable, and $\mathbb{R} \setminus \Sigma \ni s \mapsto \boldsymbol{\phi}(\cdot, s) \in Y$ is continuous. Therefore, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous on $\mathcal{M} \times (\mathbb{R} \setminus \Sigma)$.*

Proof. In virtue of Lemma 3.1-(ii) and Proposition 3.1, we can know that there exists a subset $\Sigma \subseteq \mathbb{R}$ such that Σ is at most countable, and for any $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous with $s \in \mathbb{R} \setminus \Sigma$. For any given $(\boldsymbol{\theta}, s_0) \in \mathcal{M} \times (\mathbb{R} \setminus \Sigma)$, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ converges to $\boldsymbol{\phi}(\boldsymbol{\theta}, s_0)$ as $s \to s_0$. This, combined with the fact that $\boldsymbol{\phi}(\cdot, s_0) \in Y$ and Dini's theorem, implies that $\boldsymbol{\phi}(\cdot, s)$ converges to $\boldsymbol{\phi}(\cdot, s_0)$ in Y; that is, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ converges to $\boldsymbol{\phi}(\boldsymbol{\theta}, s_0)$ uniformly with respect to $\boldsymbol{\theta}$ as $s \to s_0$. Thus, it follows that the map $\mathbb{R} \setminus \Sigma \ni s \mapsto \boldsymbol{\phi}(\cdot, s) \in Y$ is continuous since s_0 is arbitrary. Again, by the arbitrariness of s_0 and $\boldsymbol{\phi} \in \mathscr{D}$, we conclude that $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous on $\mathcal{M} \times (\mathbb{R} \setminus \Sigma)$, completing the proof of the theorem. □

By slightly adapting the proofs of Lemma 3.2, Proposition 3.1, and Theorem 3.1, we can obtain the same statements under weaker conditions, as stated in Theorem 3.2 with its proof omitted.

Theorem 3.2. *Suppose for each $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is nonincreasing in $s \in \mathbb{R}$. Assume that there is an at most countable subset $\Sigma \subseteq \mathbb{R}$ such that for any $s \in \mathbb{R} \setminus \Sigma$, $\boldsymbol{\phi}(\cdot, s) \in Y$. Then there exists a subset $\Sigma_1 \subseteq \mathbb{R}$ such that Σ_1 is at most countable, and $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ is continuous on $\mathcal{M} \times (\mathbb{R} \setminus \Sigma_1)$.*

Remark 3.1. To conclude this section, we point out that if, in addition to the condition $\boldsymbol{\phi} \in \mathscr{D}$, $\boldsymbol{\phi}(\cdot, s^{\pm}) \in Y$ for $s \in \mathbb{R}$, then the conclusion of Theorem 3.1 can be easily obtained by an alternative and simpler method. In fact, for any fixed $s \in \mathbb{R}$, define $J_s = \{\boldsymbol{\theta} \in \mathcal{M} : \boldsymbol{\phi}(\boldsymbol{\theta}, s^-) > \boldsymbol{\phi}(\boldsymbol{\theta}, s^+)\}$ and $\Sigma = \{s \in \mathbb{R} : J_s \neq \emptyset\}$. Note that for any $s \in \mathbb{R}$, J_s is a non-empty open set by the hypothesis and the definition of Σ. It suffices to show that Σ is at most countable. Otherwise, Σ is uncountable. Since $\mathcal{M}$ is a compact metric space, it follows from [10, Theorem 1.14] that there exists a countable dense subset $\mathcal{N} := \{\boldsymbol{\theta}_i : i \in \mathbb{N}_+\}$ of $\mathcal{M}$ such that $\mathcal{N} \cap J_s \neq \emptyset$ for all $s \in \mathbb{R}$. Thus, $\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$, where $\Sigma_i = \{s \in \Sigma : \boldsymbol{\theta}_i \in J_s\}$. Since Σ is uncountable, one of Σ_i is uncountable. Without loss of generality, we assume Σ_1 is uncountable. Therefore for any $s \in \Sigma_1$, we have $\boldsymbol{\phi}(\boldsymbol{\theta}_1, s^-) > \boldsymbol{\phi}(\boldsymbol{\theta}_1, s^+)$, that is, the discontinuities of $\boldsymbol{\phi}(\boldsymbol{\theta}_1, s)$ with s are uncountable. This contradicts the Darboux-Froda theorem. This, together with Dini's theorem, implies that the conclusion of Theorem 3.1 holds. Additionally, we also note that the condition that for any $m \in \mathbb{N}_+$, $\{\boldsymbol{\phi}(\cdot, s) : s \in [-m, m]\}$ is a precompact subset of Y is equivalent to the hypothesis that for any $s \in \mathbb{R}$, $\boldsymbol{\phi}(\cdot, s^{\pm}) \in Y$ from Dini's theorem. Moreover, we also note that the condition that for any $m \in \mathbb{N}_+$, $\{\boldsymbol{\phi}(\cdot, s) : s \in [-m, m]\}$ is a precompact subset of Y is equivalent to the hypothesis that for any $s \in \mathbb{R}$, $\boldsymbol{\phi}(\cdot, s^{\pm}) \in Y$ from Dini's theorem.

4. Applications

In this section, we will apply the generalized Darboux-Froda theorem proved in the preceding section to characterize functions that are equal almost everywhere, and the continuity of monotone mappings with translation invariance in $\mathscr{D}$.

4.1. Equivalence of elements in $\mathscr{D}$.

The main purpose is to show that functions that are equal almost everywhere in $\mathscr{D}$ are actually equal in $\mathbb{R} \setminus \Sigma$, where Σ is at most countable. Such a result is stated in Proposition 4.1 which is useful in the study of existence of traveling waves in monotone function space Chen et al [2].

Proposition 4.1. *Let $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathscr{D}$, and suppose that $\boldsymbol{\phi}(\cdot, s)$ is equal to $\boldsymbol{\psi}(\cdot, s)$ almost everywhere with $s \in \mathbb{R}$. Then there exists a subset $\Sigma \subseteq \mathbb{R}$ such that Σ is at most countable, and for any $(\boldsymbol{\theta}, s) \in \mathcal{M} \times (\mathbb{R} \setminus \Sigma)$, we have $\boldsymbol{\phi}(\boldsymbol{\theta}, s) = \boldsymbol{\psi}(\boldsymbol{\theta}, s)$.*

Proof. In view of $\boldsymbol{\phi}$, $\boldsymbol{\psi} \in \mathscr{D}$, it follows from Theorem 3.1 that we can find a subset $\Sigma \subseteq \mathbb{R}$ such that Σ is at most countable, and for any $\boldsymbol{\theta} \in \mathcal{M}$, both $\boldsymbol{\phi}(\boldsymbol{\theta}, s)$ and $\boldsymbol{\psi}(\boldsymbol{\theta}, s)$ are continuous with $s \in \mathbb{R} \setminus \Sigma$. Since $\boldsymbol{\phi}(\cdot, s) = \boldsymbol{\psi}(\cdot, s)$, a.e. $s \in \mathbb{R}$, there exists a zero test set $\Sigma_0 \subseteq \mathbb{R}$ such that $\boldsymbol{\phi}(\boldsymbol{\theta}, s) = \boldsymbol{\psi}(\boldsymbol{\theta}, s)$ for all $(\boldsymbol{\theta}, s) \in \mathcal{M} \times (\mathbb{R} \setminus \Sigma_0)$. Without loss of generality, we shall assume $(\boldsymbol{\theta}, s) \in \mathcal{M} \times (\mathbb{R} \setminus \Sigma)$. Take a sequence $\{s_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R} \setminus (\Sigma \cup \Sigma_0)$ satisfying $\lim_{k \to \infty} s_k = s$. Therefore, $\boldsymbol{\phi}(\boldsymbol{\theta}, s_k) = \boldsymbol{\psi}(\boldsymbol{\theta}, s_k)$. Then we let $k \to \infty$ to see that $\boldsymbol{\phi}(\boldsymbol{\theta}, s) = \boldsymbol{\psi}(\boldsymbol{\theta}, s)$ from the continuity of $\boldsymbol{\phi}(\boldsymbol{\theta}, \cdot)$ and $\boldsymbol{\psi}(\boldsymbol{\theta}, \cdot)$, which completes the proof. □

4.2. Continuity of monotone mappings with translation invariance.

It is well known that Weinberger [14] established a set of theories about spreading speeds and traveling waves for recursive systems. By the properties of monotone functions and Helly theorem, Yagisita [17] specifically investigated spreading speeds and traveling waves for noncompact and monotone semiflows on a subspace of nonnegative monotone real functions. Fang and Zhao [5] extended the abstract dynamical methods to obtain the existence of traveling waves with weak compactness for monotone semiflows. Subsequently, spreading speeds and travelling wave solutions have also been deeply studied in [4, 6, 11, 15, 16, 18, 19]. Through the above works, we find a class of mappings extracted from recursive systems play a crucial role in studying the existence of traveling waves. In this section, we will discuss the continuity of the relevant monotone mappings with translation invariance.

Let $\mathscr{D}_+ = \{\boldsymbol{\varphi} \in \mathscr{D} : \boldsymbol{\varphi}(\mathcal{M} \times \mathbb{R}) \subseteq \mathbb{R}^n_+\}$, and for any $\boldsymbol{\pi}_+ \in Int(Y_+)$, define $\mathscr{D}_{\boldsymbol{\pi}_+} = \{\boldsymbol{u} \in \mathscr{D} : \mathbf{0} \leq \boldsymbol{\varphi} \leq \boldsymbol{\pi}_+\}$. In what follows, for any given $t \in \mathbb{R}$, we define the translation operator T_t on $\mathscr{D}$ by $T_t[\boldsymbol{\varphi}](\boldsymbol{\theta}, s) = \boldsymbol{\varphi}(\boldsymbol{\theta}, s-t)$ for $(\boldsymbol{\varphi}, \boldsymbol{\theta}, s) \in \mathscr{D} \times \mathcal{M} \times \mathbb{R}$, and assume that $\boldsymbol{Q} = (Q_1, \cdots, Q_k) : \mathscr{D}_{\boldsymbol{\pi}_+} \to \mathscr{D}_{\boldsymbol{\pi}_+}$ satisfies the following hypotheses:

(4.1)
(i) **(Translation invariance)** $\boldsymbol{Q}[\boldsymbol{T}_t[\boldsymbol{u}]] = \boldsymbol{T}_t[\boldsymbol{Q}[\boldsymbol{u}]], \forall (t, \boldsymbol{\phi}) \in \mathbb{R} \times \mathscr{D}_{\boldsymbol{\pi}_+}$;
(ii) **(Monotonicity)** $\boldsymbol{\phi} \leq \boldsymbol{\psi}$ implies that $\boldsymbol{Q}[\boldsymbol{\phi}] \leq \boldsymbol{Q}[\boldsymbol{\psi}]$;
(iii) **(Continuity)** If $\boldsymbol{\phi}_k, \boldsymbol{\phi} \in \mathscr{D}_{\boldsymbol{\pi}_+}$ and $\boldsymbol{\phi}_k$ converges to $\boldsymbol{\phi}$ in $\mathscr{D}_{\boldsymbol{\pi}_+}$ as $k \to \infty$, then $\boldsymbol{Q}[\boldsymbol{\phi}_k](\cdot, s)$ converges to $\boldsymbol{Q}[\boldsymbol{\phi}](\cdot, s)$ almost everywhere with $s \in \mathbb{R}$ in Y.

In order to obtain the continuity of the mapping $\boldsymbol{Q}$, we firstly rely on the chain convergence conclusions for multiple sequences to get the uniform convergence for the sequence in Y as follows.

Lemma 4.1. *Suppose the sequences* $\{\underline{\boldsymbol{\psi}}_{n,k}\}_{n,k\in\mathbb{N}}$, $\{\overline{\boldsymbol{\psi}}_{n,k}\}_{n,k\in\mathbb{N}}$, $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$, $\{\underline{\boldsymbol{\eta}}_n\}_{n\in\mathbb{N}}$, $\{\overline{\boldsymbol{\eta}}_n\}_{k\in\mathbb{N}} \subseteq Y$, *and* $\boldsymbol{\psi} \in Y$ *with* $\underline{\boldsymbol{\psi}}_{n,k} \leq \boldsymbol{\psi}_k \leq \overline{\boldsymbol{\psi}}_{n,k}$. *If for any* $n \in \mathbb{N}$, $\underline{\boldsymbol{\psi}}_{n,k}$ *and* $\overline{\boldsymbol{\psi}}_{n,k}$ *converge to* $\underline{\boldsymbol{\eta}}_n$ *and* $\overline{\boldsymbol{\eta}}_n$ *respectively in* Y *as* $k \to \infty$, *and* $\lim\limits_{n\to\infty} \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}) = \lim\limits_{n\to\infty} \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}) = \boldsymbol{\psi}(\boldsymbol{\theta})$ *for all* $\boldsymbol{\theta} \in \mathcal{M}$, *then* $\boldsymbol{\psi}_k$ *converges to* $\boldsymbol{\psi}$ *in* Y *as* $k \to \infty$.

Proof. First, we claim that for any $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\psi}_k(\boldsymbol{\theta})$ converges to $\boldsymbol{\psi}(\boldsymbol{\theta})$ as $k \to \infty$, and hence $\underline{\boldsymbol{\eta}}_n \leq \boldsymbol{\psi} \leq \overline{\boldsymbol{\eta}}_n$. Clearly, $\underline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}) \leq \boldsymbol{\psi}_k(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathcal{M}$ since $\underline{\boldsymbol{\psi}}_{n,k} \leq \boldsymbol{\psi}_k$. We let $k \to \infty$ to see that $\underline{\boldsymbol{\eta}}_n \leq \liminf\limits_{k\to\infty} \boldsymbol{\psi}_k(\boldsymbol{\theta})$. Passing to the limit as $n \to \infty$, it follows from the conditions of $\underline{\boldsymbol{\psi}}_{n,k}$ and $\underline{\boldsymbol{\eta}}_k$ that for any $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\psi}(\boldsymbol{\theta}) \leq \liminf\limits_{k\to\infty} \boldsymbol{\psi}_k(\boldsymbol{\theta})$. Similarly, we can easily verify that for any $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{\psi}(\boldsymbol{\theta}) \geq \limsup\limits_{k\to\infty} \boldsymbol{\psi}_k(\boldsymbol{\theta})$, and thus $\lim\limits_{k\to\infty} \boldsymbol{\psi}_k(\boldsymbol{\theta}) = \boldsymbol{\psi}(\boldsymbol{\theta})$.

Next, we show that $\boldsymbol{\psi}_k$ converges to $\boldsymbol{\psi}$ in Y as $k \to \infty$. Otherwise, there exist $\varepsilon_0 > 0$, $\{\boldsymbol{\theta}_k\}_{k\in\mathbb{N}} \subseteq \mathcal{M}$, and a subsequence of $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$, still denoted by $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$ such that

$$\|\boldsymbol{\psi}_k(\boldsymbol{\theta}_k) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| \geq \varepsilon_0.$$

As $\mathcal{M}$ is a compact metric space, we shall assume $\lim\limits_{k\to\infty} \boldsymbol{\theta}_k = \boldsymbol{\theta}_0$. According to the above conditions and the claim, it is easy to see that

$$\|\boldsymbol{\psi}_k(\boldsymbol{\theta}_k) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| \leq \max\{\|\overline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\|, \|\underline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\|\}.$$

Using triangle inequality, we have

$$\begin{aligned}\|\overline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\| &\leq \|\overline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\| + \|\overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| + \|\boldsymbol{\psi}(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\| \\ &\leq \|\overline{\boldsymbol{\psi}}_{n,k}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k)\| + \|\overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0)\| + \|\overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_0)\| \\ &\quad + 2\|\boldsymbol{\psi}(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| + \|\underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0)\| + \|\underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_0)\|.\end{aligned}$$

Because of $\lim\limits_{n\to\infty} \underline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0) = \boldsymbol{\psi}(\boldsymbol{\theta}_0)$ and $\lim\limits_{n\to\infty} \overline{\boldsymbol{\eta}}_n(\boldsymbol{\theta}_0) = \boldsymbol{\psi}(\boldsymbol{\theta}_0)$, there exists $N_0 := N_0(\varepsilon, \boldsymbol{\theta}_0)$ such that

$$\|\underline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_0)\| + \|\overline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_0)\| < \frac{\varepsilon_0}{3}.$$

By the fact that $\overline{\boldsymbol{\eta}}_{N_0}$, $\underline{\boldsymbol{\eta}}_{N_0}$, and $\boldsymbol{\psi} \in Y$, there is $K_0 := K_0(\varepsilon_0, N_0) > 0$ such that for any $k \geq K_0$,

$$\|\underline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_0)\| + \|\overline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_0)\| + 2\|\boldsymbol{\psi}(\boldsymbol{\theta}_0) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| < \frac{\varepsilon_0}{3}.$$

Since $\overline{\boldsymbol{\psi}}_{N_0,k}$ converges to $\overline{\boldsymbol{\eta}}_{N_0}$ in Y as $k \to \infty$, there exists $K_1 := K_1(\varepsilon_0, N_0)$ such that for any $k \geq K_1$,

$$\|\overline{\boldsymbol{\psi}}_{N_0,k}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_k)\| < \frac{\varepsilon_0}{3}.$$

Hence, $\|\overline{\boldsymbol{\psi}}_{N_0,k}(\boldsymbol{\theta}_k) - \underline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_k)\| < \varepsilon_0$ if $k \geq \max\{K_0, K_1\}$ is sufficiently large. Similarly, we have $\|\underline{\boldsymbol{\psi}}_{N_0,k}(\boldsymbol{\theta}_k) - \overline{\boldsymbol{\eta}}_{N_0}(\boldsymbol{\theta}_k)\| < \varepsilon_0$ for sufficiently large k.

Consequently, $\|\boldsymbol{\psi}_k(\boldsymbol{\theta}_k) - \boldsymbol{\psi}(\boldsymbol{\theta}_k)\| < \varepsilon_0$ for sufficiently large k, a contradiction, which completes the proof. ∎

Let $\mathcal{D} = BC(\mathcal{M} \times \mathbb{R}, \mathbb{R}^n)$ be the normed vector space of all bounded and continuous functions from $\mathcal{M} \times \mathbb{R}$ to $\mathbb{R}^n$ with the norm $\|\boldsymbol{\varphi}\|_{\mathcal{D}} \triangleq \sum_{m=0}^{\infty} 2^{-m} \sup\{\|\boldsymbol{\varphi}(\boldsymbol{\theta}, s)\| : -m \leq s \leq m,\ \boldsymbol{\theta} \in \mathcal{M}\}$. In what follows, we construct monotone and continuous function sequences and apply the convergence criterion to get a special application of Theorem 3.1.

Proposition 4.2. *Suppose that $\boldsymbol{Q}$ satisfies hypothesis* (4.1), *and $\boldsymbol{\phi}_k$, $\boldsymbol{\phi} \in \mathscr{D}_{\boldsymbol{\pi}_+}$. If $\boldsymbol{\phi}_k(\cdot, s)$ converges to $\boldsymbol{\phi}(\cdot, s)$ almost everywhere with s as $k \to \infty$, then $\boldsymbol{Q}[\boldsymbol{\phi}_k](\cdot, s)$ converges to $\boldsymbol{Q}[\boldsymbol{\phi}](\cdot, s)$ almost everywhere with $s \in \mathbb{R}$ as $k \to \infty$.*

Proof. By $\boldsymbol{\phi} \in \mathscr{D}_{\boldsymbol{\pi}_+}$ and adapting the proof of [17, Proposition 9], we can choose the sequences $\underline{\boldsymbol{\phi}}_m$ and $\overline{\boldsymbol{\phi}}_m \in \mathscr{D}_{\boldsymbol{\pi}_+} \cap \mathcal{D}$ such that for any $(s, m) \in \mathbb{R} \times \mathbb{N}$,

$$\boldsymbol{\phi}(\cdot, s + 2^{-m}) \leq \underline{\boldsymbol{\phi}}_m(\cdot, s) \leq \boldsymbol{\phi}(\cdot, s) \leq \overline{\boldsymbol{\phi}}_m(\cdot, s) \leq \boldsymbol{\phi}(\cdot, s - 2^{-m}). \tag{4.2}$$

This, combined with the fact that $\boldsymbol{\phi}_k(\cdot, s)$ converges to $\boldsymbol{\phi}(\cdot, s)$ almost everywhere with $s \in \mathbb{R}$, implies that for any $m \in \mathbb{N}$, $\min\{\boldsymbol{\phi}_k(\cdot, s), \underline{\boldsymbol{\phi}}_m(\cdot, s)\}$ converges to $\underline{\boldsymbol{\phi}}_m(\cdot, s)$ almost everywhere with $s \in \mathbb{R}$. Further, by the continuity of $\underline{\boldsymbol{\phi}}_m(\cdot, s)$ with s and [5, Lemma 2.3-(ii)], $\min\{\boldsymbol{\phi}_k, \underline{\boldsymbol{\phi}}_m\}$ converges to $\underline{\boldsymbol{\phi}}_m$ in $\mathscr{D}_{\boldsymbol{\pi}_+}$ as $k \to \infty$. Similarly, $\max\{\boldsymbol{\phi}_k, \overline{\boldsymbol{\phi}}_m\}$ converges to $\overline{\boldsymbol{\phi}}_m$ in $\mathscr{D}_{\boldsymbol{\pi}_+}$ as $k \to \infty$. Thus, it follows from the continuity of $\boldsymbol{Q}$ that there exists a zero test set $\Sigma_0 \subseteq \mathbb{R}$ such that for any $s \in \mathbb{R} \setminus \Sigma_0$,

$$\lim_{k \to \infty} \boldsymbol{Q}[\min\{\boldsymbol{\phi}_k, \underline{\boldsymbol{\phi}}_m\}](\cdot, s) = \boldsymbol{Q}[\underline{\boldsymbol{\phi}}_m](\cdot, s),$$

and

$$\lim_{k \to \infty} \boldsymbol{Q}[\max\{\boldsymbol{\phi}_k, \overline{\boldsymbol{\phi}}_m\}](\cdot, s) = \boldsymbol{Q}[\overline{\boldsymbol{\phi}}_m](\cdot, s).$$

Due to the monotonicity and translation invariance of $\boldsymbol{Q}$, and (4.2), we have

$$\boldsymbol{Q}[\min\{\boldsymbol{\phi}_k, \underline{\boldsymbol{\phi}}_m\}](\cdot, s) \leq \boldsymbol{Q}[\boldsymbol{\phi}_k](\cdot, s) \leq \boldsymbol{Q}[\max\{\boldsymbol{\phi}_k, \overline{\boldsymbol{\phi}}_m\}](\cdot, s),$$

and

$$\begin{aligned} \boldsymbol{Q}[\boldsymbol{\phi}](\cdot, s + 2^{-m}) &= \boldsymbol{Q}[\boldsymbol{\phi}(\cdot, \cdot + 2^{-m})](\cdot, s) \\ &\leq \boldsymbol{Q}[\underline{\boldsymbol{\phi}}_m](\cdot, s) \leq \boldsymbol{Q}[\overline{\boldsymbol{\phi}}_m](\cdot, s) \\ &\leq \boldsymbol{Q}[\boldsymbol{\phi}(\cdot, \cdot - 2^{-m})](\cdot, s) = \boldsymbol{Q}[\boldsymbol{\phi}](\cdot, s - 2^{-m}). \end{aligned} \tag{4.3}$$

In view of $\boldsymbol{Q}[\phi] \in \mathscr{D}_{\boldsymbol{\pi}_+}$ and Theorem 3.1, there exists a subset $\Sigma \subseteq \mathbb{R}$ such that Σ is at most countable and for any $\boldsymbol{\theta} \in \mathcal{M}$, $\boldsymbol{Q}[\phi](\boldsymbol{\theta}, s)$ is continuous in $s \in \mathbb{R} \setminus \Sigma$, and hence for any $(\boldsymbol{\theta}, s) \in \mathcal{M} \times (\mathbb{R} \setminus \Sigma)$, we have

$$\lim_{m\to\infty} \boldsymbol{Q}[\phi](\boldsymbol{\theta}, s + 2^{-m}) = \boldsymbol{Q}[\phi](\boldsymbol{\theta}, s) = \lim_{m\to\infty} \boldsymbol{Q}[\phi](\boldsymbol{\theta}, s - 2^{-m}).$$

This, combined with (4.3), implies that for any $(\boldsymbol{\theta}, s) \in \mathcal{M}\times(\mathbb{R}\setminus\Sigma)$, $\lim_{m\to\infty} \boldsymbol{Q}[\underline{\phi}_m](\boldsymbol{\theta}, s) = \lim_{m\to\infty} \boldsymbol{Q}[\overline{\phi}_m](\boldsymbol{\theta}, s) = \boldsymbol{Q}[\phi](\boldsymbol{\theta}, s)$. Thus, applying Lemma 4.1, it follows that for any $s \in \mathbb{R} \setminus (\Sigma \cup \Sigma_0)$, $\boldsymbol{Q}[\phi_k](\cdot, s)$ converges to $\boldsymbol{Q}[\phi](\cdot, s)$ in Y as $k \to \infty$. In other words, $\boldsymbol{Q}[\phi_k](\cdot, s)$ converges to $\boldsymbol{Q}[\phi](\cdot, s)$ almost everywhere with $s \in \mathbb{R}$, which completes the proof. □

In the succeeding work [2], we make use of the conclusions in the above propositions to obtain the existence of traveling waves for monotone mappings with translation invariance under the point asymptotically smooth hypothesis.

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