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ON SPATIAL-TEMPORAL DYNAMICS OF A FISHER-KPP EQUATION WITH A SHIFTING ENVIRONMENT

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Abstract. We consider a generalized Fisher-KPP equation with the growth function being time and space dependent in the form of "shifting with constant speed". The main concerns are extinction and persistence, as well as spatial-temporal dynamics. By employing a new method relating to semigroup and some subtle estimates, we not only extend the main results in Li et al. [SIAM J. Appl. Math. 74 (2014), pp. 1397-1417] to a scenario when the growth function may have no sign change, but also improve the main results there by dropping some restrictions on the initial functions.

1. Introduction

In recent years, due to increasing threats associated with global climate change and the worsening of the environment caused by industrialization, the effects of climate and environment changes on the survival of biological species have received much attention from some researchers, including mathematical modellers and analysts. Among the various ways/patterns of changes in climate and environment, is the one that "propagates or shifts with constant speed". Corresponding to such a shifting way of change, some parameters in a mathematical model become time and location dependent, and they depend on the time and location in this special shifting way. See, e.g., the equations in Potapov and Lewis [8], Berestycki et al. [2], and Li et al. [6] and the related references therein for such models. To be more specific, let us use the model equation in Li et al. [6] to explain. In [6], Li et al. considered the following model equation:

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ur(x-ct) - u^2, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.1}$$

where the function $r(\cdot)$ represents the birth rate and is assumed to be space-time dependent, changing in the form of shifting with constant speed $c > 0$, while the density dependent death rate has been rescaled to the current form. Obviously, $r(x-ct)$ offers a spatially varying baseline or a historic rate of population growth.

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Different properties that the function $r(\cdot)$ possesses correspond to different scenarios in reality. When $r(\cdot)$ is a positive constant, representing a situation with stable (constant) environment, (1.1) reduces to the classic Fisher-KPP equation ([4, 5]), which has been extensively and intensively studied (see, e.g., [1, 10]). An increasing $r(\cdot)$ accounts for a scenario that the environment is worsening with a constant speed $c > 0$, while a decreasing $r(\cdot)$ explains a situation that the environment is improving with a constant speed $c > 0$.

In [6], the authors explored the spatial-temporal dynamics of (1.1) under the following condition on the function $r(\cdot)$:

(**H**) $r(\cdot)$ is continuous, nondecreasing, and bounded with $r(-\infty) < 0 < r(\infty)$.

With the condition (**H**), Li et al. [6] obtained the following main results.

Theorem 1.1 ([6, Theorem 2.1]). *Assume that* (**H**) *holds. Let* $c > c^*(\infty) := 2\sqrt{dr(\infty)}$. *If* $0 \le u_0(x) \le r(\infty)$ *and* $u_0(x) = 0$ *for all sufficiently large* x, *then for every* $\varepsilon > 0$ *there exists* $T > 0$ *such that for* $t \ge T$, *the solution of* (1.1) *with* $u(0, x) = u_0(x)$ *satisfies* $u(t, x) \le \varepsilon$ *for all* x.

Theorem 1.2 ([6, Theorem 2.2]). *Assume that* (**H**) *holds. Let* $0 < c < c^*(\infty)$. *Then the following statements are valid:*

(i) *If* $0 \le u(0, x) \le r(\infty)$, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)} u(t,x)\right] = 0.$$

(ii) *If* $0 \le u(0, x) \le r(\infty)$, *and* $u(0, x) \equiv 0$ *for all sufficiently large* x, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\ge t(c^*(\infty)+\varepsilon)} u(t,x)\right] = 0.$$

(iii) *If* $0 \le u(0, x) \le r(\infty)$, *and* $u(0, x) > 0$ *on a closed interval, then for any* $\varepsilon > 0$ *with* $0 < \varepsilon < (c^*(\infty) - c)/2$,

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)} |r(\infty) - u(t,x)|\right] = 0.$$

Note that the hypothesis (**H**) consists of two major requirements for $r(\cdot)$: monotonicity and sign change. Physically/ecologically, nondecreasing property of $r(\cdot)$ means that, as we explained before, the environment is *gradually becoming worse* (in the sense of shifting with constant speed); and the sign change property $r(-\infty) < 0 < r(\infty)$ further indicates that the worsening is in the *very severe sense*. Indeed, due to this sign change property, at any given location x, when t is sufficiently large, $r(x - ct)$ will become negative meaning that this location will become a poor and unsuitable habitat for a biological species. Mathematically, these two requirements play a crucial role in proving the above results in [6]. For example, under condition (**H**), at any given time t, the whole space $\mathbb{R}$ can be naturally divided into exactly two regions: a good (suitable) region where $r(x - ct) > 0$ and a bad region (unsuitable) region where $r(x - ct) < 0$, and the behaviours of solutions to (1.1) in these two regions can be tracked and analyzed accordingly by such a division.

We point out that some most recent works have extended the results in [6] to some *other equations* with shifting parameter(s); for example, to an equation resulting from replacing the random diffusion in (1.1) with *nonlocal diffusion* in Li et al. [7], and to competitive systems of Lotka-Volterra-type with the same type of growth function(s) in Zhang et al. [14] and Yuan et al. [13] as $r(x-ct)$ in (1.1). In these extensions, the growth functions involved are *all assumed to satisfy* (**H**). Our goal in this paper is to extend (also improve) the results (not the equation) for (1.1) in [6] to *allow a no sign change scenario.* More specifically, we replace (**H**) with the following weaker condition for $r(\cdot)$:

(**H***) $r(\cdot)$ is continuous, nondecreasing, and bounded with $r(-\infty) \leq 0 < r(\infty)$.

With such a relaxation of conditions for $r(\cdot)$, the main idea and method used in [6] and the extension works [7, 13, 14] that depend on the "sign change" implied by $r(-\infty) < 0$ can no longer be applied (at least directly), and we are forced to seek an alternative method which is motivated by [12] (a semigroup approach). By our new method, we are able to obtain some results on the spatial-temporal dynamics of (1.1) under (**H***) which are similar to Theorems 1.1 and 1.2 but also contain some improvement to Theorems 1.1 and 1.2 (see Remark 3.1).

2. Preliminary results

In this section, we will introduce notation and two lemmas. We denote

$$C = C(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R}, \mathbb{R})$$

and

$$C_+ = C(\mathbb{R}, \mathbb{R}_+) \cap L^\infty(\mathbb{R}, \mathbb{R}).$$

For any $\psi, \varphi \in C$, we write $\psi \leq \varphi$ or $\varphi \geq \psi$ if $\varphi - \psi \in C_+$.

For convenience of discussion, we denote by $u^\phi(t, x; r(\cdot))$ the solution of (1.1) with the shifting birth rate function $r(\cdot)$ and initial distribution ϕ. It is known (see, e.g., [1, 11]) that if $r(\cdot) = \hat{r}$ is a positive constant function, then the following results for (1.1) have been obtained, which will be used later in our analysis.

Lemma 2.1. *Let $\hat{r}$ be a positive constant and $\hat{c} := 2\sqrt{d\hat{r}}$. Then the following statements are valid:*

(i) *If $\phi \in C_+$ with $\phi(x) = 0$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} \left[\sup_{x \geq t(\hat{c}+\varepsilon)} u^\phi(t, x; \hat{r}) \right] = 0.$$

(ii) *If $\phi \in C_+ \setminus \{0\}$, then for any $\varepsilon > 0$ with $0 < \varepsilon < \hat{c}$,*

$$\lim_{t\to\infty} \left[\sup_{|x| \leq t(\hat{c}-\varepsilon)} |\hat{r} - u^\phi(t, x; \hat{r})| \right] = 0.$$

By the Phragmén-Lindelöf type maximum principle in [9], we can also easily establish the following comparison principle for (1.1).

Lemma 2.2. *Assume that $r, \tilde{r} \in C$ are nondecreasing with $r \geq \tilde{r}$. Let $\psi, \phi \in C_+$ with $\phi \leq \psi$. Then we have*

(i) $0 \leq u^\phi(t, x; \tilde{r}(\cdot)) \leq u^\psi(t, x; r(\cdot)) \leq u^\psi(t, x; r(\infty))$ *for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

(ii) $\limsup_{t\to\infty} ||u^\phi(t, \cdot; r(\cdot))||_{L^\infty} \leq r(\infty)$.

3. The main results

Let $c^*(\infty) := 2\sqrt{dr(\infty)}$, which represents the spread speed of the limit equation

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + u(r(\infty) - u), \qquad x \in \mathbb{R}, \quad t \geq 0 \tag{3.1}$$

in the sense of Lemma 2.1. The next theorem shows that if the environmental worsening speed is less than this limit speed ($c < c^*(\infty)$), then the species can persist by moving toward the direction of better environment (i.e., toward the right hand side) with the moving persistence region given by

$$D_t = \{x \in \mathbb{R} : c\,t < x < c^*(\infty)\,t\}$$

which is expanding as time goes.

Theorem 3.1. *Assume that* (**H***) *is satisfied. Let* $0 \leq c < c^*(\infty)$. *Then the following statements are valid:*

(i) *If* $\phi \in C_+$, *and* $\phi \equiv 0$ *for all sufficiently large* x, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c^*(\infty)+\varepsilon)} u^{\phi}(t,x;r(\cdot))\right] = 0.$$

(ii) *If* $\phi \in C_+ \setminus \{0\}$ *and* $\varepsilon \in (0, (c^*(\infty) - c)/2)$, *then*

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x\leq t(c^*(\infty)-\varepsilon)} |r(\infty) - u^{\phi}(t,x;r(\cdot))|\right] = 0.$$

Proof. For (i), by Lemma 2.1(i), we get

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c^*(\infty)+\varepsilon)} u^{\phi}(t,x;r(\infty))\right] = 0.$$

Lemma 2.2(i) implies $0 \leq u^{\phi}(t,x;r(\cdot)) \leq u^{\phi}(t,x;r(\infty))$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus,

$$0 \leq \sup_{x\geq t(c^*(\infty)+\varepsilon)} u^{\phi}(t,x;r(\cdot)) \leq \sup_{x\geq t(c^*(\infty)+\varepsilon)} u^{\phi}(t,x;r(\infty))$$

for all $t \in \mathbb{R}_+$. Therefore,

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c^*(\infty)+\varepsilon)} u^{\phi}(t,x;r(\cdot))\right] = 0.$$

For (ii), take a nondecreasing function $\tilde{r}$ and $\psi \in C_+ \setminus \{0\}$ with $\tilde{r} \leq r$, $\tilde{r}(\infty) = r(\infty)$, $\tilde{r}(-\infty) = \min\{-r(\infty), r(-\infty)\}$, $\psi \leq r(\infty)$ and $\psi \leq \phi$. It follows from Theorem 1.2(iii) that

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x\leq t(c^*(\infty)-\varepsilon)} |r(\infty) - u^{\psi}(t,x;\tilde{r}(\cdot))|\right] = 0.$$

Lemma 2.1(ii) implies that

$$\lim_{t\to\infty}\left[\sup_{|x|\leq t(c^*(\infty)-\varepsilon)} |r(\infty) - u^{\phi}(t,x;r(\infty))|\right] = 0.$$

In particular, we have

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)}|r(\infty)-u^\phi(t,x;r(\infty))|\right]=0.$$

By Lemma 2.2(i), we obtain that $u^\psi(t,x;\tilde r(\cdot))\le u^\phi(t,x;r(\cdot))\le u^\phi(t,x;r(\infty))$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. Thus,

$$\begin{aligned}&\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)}|r(\infty)-u^\phi(t,x;r(\cdot))|\\ &\le\max\Bigg\{\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)}|r(\infty)-u^\phi(t,x;r(\infty))|,\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)}|r(\infty)\\ &\qquad\qquad -u^\psi(t,x;\tilde r(\cdot))|\Bigg\}\end{aligned}$$

for all $t\in\mathbb{R}_+$. Therefore,

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c^*(\infty)-\varepsilon)}|r(\infty)-u^\phi(t,x;r(\cdot))|\right]=0.$$

This completes the proof. □

Let $S_\mu(t)$ be the semigroup generated by the following linear system:

$$\begin{cases}\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2}-\mu u, & t>0,\\ u(t,0) = 0, & t\in\mathbb{R}_+,\\ u(0,x) = \phi(x), & x\in\mathbb{R},\end{cases}$$

that is, for $(x,\phi)\in\mathbb{R}\times C$,

$$\begin{cases} S_\mu(0)[\phi](x) = \phi(x),\\ S_\mu(t)[\phi](x) = \frac{\exp(-\mu t)}{\sqrt{4\pi dt}}\int_{\mathbb{R}}\phi(y)\exp\left(-\frac{(x-y)^2}{4dt}\right)\mathrm{d}y, \quad t>0.\end{cases}\tag{3.2}$$

For any given $c>0$, by making use of this semigroup, we can establish the following result, which complements Theorem 3.1.

Theorem 3.2. *Let* (**H***) *hold, and $c>0$ any constant. Then for any $\phi\in C_+$ and $\varepsilon>0$, there holds*

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)}u^\phi(t,x;r(\cdot))\right]=0.$$

Proof. Fix $\phi\in C_+$ and $\varepsilon>0$. By Lemma 2.2(ii), we may assume, without loss of generality, that $0\le u^\phi(t,x;r(\cdot))\le M_0:=r(\infty)+1$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. Let

$$U(\tau)=\limsup_{t\to\infty}\left[\sup_{x\le t(c-\tau)}u^\phi(t,x;r(\cdot))\right]\text{ for all }\tau\in(0,\varepsilon].$$

Then $U(\cdot)$ is nonincreasing in $(0,\varepsilon]$, and by Lemma 2.2(ii) we have $0\le U(\tau)\le r(\infty)$ for all $\tau\in(0,\varepsilon]$.

Now it suffices to prove $U(\varepsilon)=0$; otherwise, the monotonicity of U implies that there is $\tau_0\in(0,\varepsilon)$ such that U is continuous at τ_0 and $\mu:=\frac{U(\tau_0)}{2}>0$.

By $r(-\infty) \le 0$, there is $\xi_0 > 0$ such that $r(\xi) < \mu$ for all $\xi \le -\xi_0$. Let $u(t,x) := u^{\phi}(t,x;r(\cdot))$. It follows from (1.1) that for any $(t_0,t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with $t > t_0 \ge \frac{\xi_0}{\tau_0}$, we have

$$\begin{aligned}
u(t,x) &= S_\mu(t-t_0)[u(t_0,\cdot)](x) \\
&\quad + \int_{t_0}^{t} S_\mu(t-s)[(\mu + r(\cdot - cs))u(s,\cdot) - u^2(s,\cdot))](x)\mathrm{d}s \\
&= S_\mu(t-t_0)[u(t_0,\cdot)](x) + \mu \int_{t_0}^{t} S_\mu(t-s)[\mu](x)\mathrm{d}s \\
&\quad - \int_{t_0}^{t} S_\mu(t-s)[(u(s,\cdot)-\mu)^2](x)\mathrm{d}s \\
&\quad + \int_{t_0}^{t} S_\mu(t-s)[(r(\cdot - cs)-\mu)u(s,\cdot)](x)\mathrm{d}s \\
&\le M_0 e^{-\mu(t-t_0)} + \mu[1-e^{-\mu(t-t_0)}] + \int_{t_0}^{t} S_\mu(t-s)[(r(\cdot - cs)-\mu)u(s,\cdot)](x)\mathrm{d}s \\
&= \mu + (M_0-\mu)e^{-\mu(t-t_0)} \\
&\quad + \int_{t_0}^{t} \frac{\exp(-\mu(t-s))}{\sqrt{4\pi d(t-s)}} \int_{\mathbb{R}} (r(y-cs)-\mu)u(s,y)\exp\left(-\frac{(x-y)^2}{4d(t-s)}\right)\mathrm{d}y\mathrm{d}s \\
&\le \mu + (M_0-\mu)e^{-\mu(t-t_0)} \\
&\quad + \int_{t_0}^{t} \frac{\exp(-\mu(t-s))}{\sqrt{4\pi d(t-s)}} \int_{s(c-\tau_0)}^{\infty} (r(y-cs)-\mu)u(s,y)\exp\left(-\frac{(x-y)^2}{4d(t-s)}\right)\mathrm{d}y\mathrm{d}s \\
&\le \mu + (M_0-\mu)e^{-\mu(t-t_0)} \\
&\quad + (r(\infty)-\mu)M_0 \int_{t_0}^{t} \frac{\exp(-\mu(t-s))}{\sqrt{4\pi d(t-s)}} \int_{s(c-\tau_0)}^{\infty} \exp\left(-\frac{(x-y)^2}{4d(t-s)}\right)\mathrm{d}y\mathrm{d}s \\
&= \mu + (M_0-\mu)e^{-\mu(t-t_0)} \\
&\quad + (r(\infty)-\mu)M_0 \int_{t_0}^{t} \frac{\exp(-\mu(t-s))}{\sqrt{4\pi d(t-s)}} \int_{s(c-\tau_0)-x}^{\infty} \exp\left(-\frac{y^2}{4d(t-s)}\right)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Fix $\tau \in (\tau_0, \varepsilon)$. Let

$$\theta := \begin{cases} \frac{1}{3} & \text{if } \tau_0 = c, \\ \max\left\{\frac{1}{3},\ 1 - \frac{2(\tau-\tau_0)}{3|\tau_0 - c|}\right\} & \text{else.} \end{cases} \tag{3.3}$$

Obviously $\frac{1}{3} \le \theta < 1$, and hence $\theta t < t$ for $t > 0$. Therefore, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ with $x \le (c-\tau)t$ and $t \ge \frac{\xi_0}{\theta\tau_0}$, we have

$$\begin{aligned}
u(t,x) &\le \mu + (M_0-\mu)e^{-\mu(t-\theta t)} \\
&\quad + (r(\infty)-\mu)M_0 \int_0^{t-\theta t} \frac{\exp(-\mu s)}{\sqrt{4\pi d s}} \int_{(t-s)(c-\tau_0)-x}^{\infty} \exp\left(-\frac{y^2}{4ds}\right)\mathrm{d}y\mathrm{d}s \\
&\le \mu + (M_0-\mu)e^{-\mu(t-\theta t)} \\
&\quad + (r(\infty)-\mu)M_0 \int_0^{t-\theta t} \frac{\exp(-\mu s)}{\sqrt{4\pi d s}} \int_{\frac{\tau-\tau_0}{3}t}^{\infty} \exp\left(-\frac{y^2}{4ds}\right)\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{3.4}$$

In the last inequality in (3.4), we have used the fact that

$$\begin{aligned}(t-s)(c-\tau_0)-x &= (c-\tau)t - x + t(\tau-\tau_0) + s(\tau_0 - c)\\ &\geq t(\tau-\tau_0)+s(\tau_0-c)\\ &\geq t(\tau-\tau_0)-(t-\theta t)|\tau_0-c|\\ &= t(\tau-\tau_0)-t(1-\theta)|\tau_0-c|\\ &\geq t(\tau-\tau_0)-\frac{2}{3}t(\tau-\tau_0)\\ &= \frac{\tau-\tau_0}{3}t.\end{aligned}$$

Note that from [12, Lemma 2.1-(vi)] one obtains for any $y\in\mathbb{R}$,

$$\int_{\mathbb{R}_+}\frac{\mu e^{-\mu t}}{\sqrt{4d\pi t}}\exp\left(-\frac{y^2}{4dt}\right)\mathrm{d}t=\frac{\mu}{\sqrt{4d\mu}}e^{-\sqrt{\frac{\mu}{d}}|y|}. \tag{3.5}$$

Now, combining (3.4) and (3.5), we conclude that for any $\tau\in(\tau_0,\varepsilon)$, and $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ with $x\leq(c-\tau)t$ and $t\geq\frac{\xi_0}{\theta\tau_0}$ there holds

$$\begin{aligned}u(t,x)&\leq \mu+(M_0-\mu)e^{-\mu(1-\theta)t}\\ &\quad+(r(\infty)-\mu)M_0\int_{\frac{\tau-\tau_0}{3}t}^{\infty}\frac{1}{\sqrt{4d\mu}}e^{-\sqrt{\frac{\mu}{d}}|y|}\mathrm{d}y\\ &=\mu+(M_0-\mu)e^{-\mu(1-\theta)t}\\ &\quad+\frac{(r(\infty)-\mu)M_0}{2\mu}e^{-\sqrt{\frac{\mu}{9d}}(\tau-\tau_0)t}.\end{aligned}$$

This implies $U(\tau)\leq\mu$ for all $\tau\in(\tau_0,\varepsilon)$. Consequently as $\tau\to\tau_0$, we obtain $U(\tau_0)\leq\mu=\frac{U(\tau_0)}{2}$, a contradiction. Therefore,

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c-\varepsilon)}u^{\phi}(t,x;r(\cdot))\right]=0.$$

The proof is complete. □

When the worsening speed c is larger than the limit spreading speed of (1.1) $c^*(\infty)=2\sqrt{d\,r(\infty)}$, one naturally predicts that the species will eventually become extinct in space under some very general conditions on the initial function ϕ. More precisely, we can prove the following result which is similar to Theorem 1.1, although the condition (**H**) on $r(\cdot)$ is relaxed to (**H***), allowing both cases of $r(-\infty)=0$ and $r(-\infty)<0$.

Theorem 3.3. *Assume that* (**H***) *holds. Let* $c^*(\infty)<c$. *If* $\phi\in C_+$ *and* $\phi(x)\equiv 0$ *for all sufficiently large* x, *then*

$$\lim_{t\to\infty}||u^{\phi}(t,x;r(\cdot))||_{L^\infty}=0. \tag{3.6}$$

Proof. On the one hand, by Lemma 2.1(i) we have, for any $\varepsilon\in(0,c-c^*(\infty))$,

$$0\leq\lim_{t\to\infty}\left[\sup_{x\geq t(c-\varepsilon)}u^{\phi}(t,x;r(\cdot))\right]\leq\lim_{t\to\infty}\left[\sup_{x\geq t(c-\varepsilon)}u^{\phi}(t,x;r(\infty))\right]=0.$$

On the other hand, Theorem 3.2 gives

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)} u^{\phi}(t,x;r(\cdot))\right]=0 \quad \text{for any} \quad \varepsilon\in(0,c-c^*(\infty)).$$

Combining the above, we obtain (3.6), completing the proof of the theorem. □

Remark 3.1. Comparing our results in Theorems 3.1, 3.2, and 3.3 with Theorems 1.1 and 1.2, we see that not only the conditions on the growth function $r(\cdot)$ are relaxed, but also the conditions on the initial function ϕ are less demanding, that is, we do not require the initial functions be bounded above by $r(\infty)$. Such an improvement is attributed to our new approach of semigroup arguments.

As far as "no sign change" is concerned for $r(\cdot)$, we have only considered the critical case for $r(-\infty)$ (i.e., $r(-\infty)=0$) which needs special consideration due to its critical nature. There are other scenarios of "no sign change", for example, assuming $r(-\infty)>0$ in (**H**), and even removing the monotonicity of $r(\cdot)$. For the former, with $r(-\infty)>0$ replacing $r(-\infty)<0$, (1.1) has two KPP-type limit equations, each defining a spreading speed. These two speeds may have complicated interplays with the shifting speed c in determining the spatial-temporal dynamics of (1.1). We leave this case for a future research topic.

To conclude this paper, we remark that this paper is mainly a follow-up of [6] in the sense that we only deal with the issues of extinction or persistence as is in [6], and for the latter (in the case of $c<c^*(\infty)$), we identify the asymptotic region of persistence given by $D_t=\{x\in\mathbb{R}: ct<x<c^*(\infty)\,t\,\}$. We point out that, although not autonomous, due to its special way of dependence on time and location, equation (1.1) under (**H***) can also allow existence of traveling wave solutions with the environment forced speed c (hence, such a traveling wave can be referred to as a forced traveling wave). We do not discuss this topic here and thus, we choose to omit those vast literatures on this topic to save space, except for the most recent one by Berestycki and Fang [3] from which a reader can find rich references on the topic of forced traveling waves in equations with shifting habitats.

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