PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 11, November 2017, Pages 4763–4771
http://dx.doi.org/10.1090/proc/13687
Article electronically published on July 28, 2017

EXISTENCE OF AN EXTINCTION WAVE IN THE FISHER EQUATION WITH A SHIFTING HABITAT

HAIJUN HU AND XINGFU ZOU

(Communicated by Yingfei Yi)

ABSTRACT. This paper deals with the existence of traveling wave solutions of the Fisher equation with a shifting habitat representing a transition to a devastating environment. By constructing a pair of appropriate upper/lower solutions and using the method of monotone iteration, we prove that for any given speed of the shifting habitat edge, this reaction-diffusion equation admits a monotone traveling wave solution with the speed agreeing to the habitat shifting speed, which accounts for an extinction wave. This predicts not only how fast but also in what manner a biological species will die out in such a shifting habitat.

1. INTRODUCTION

The classical reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.1}$$

was introduced by R. A. Fisher [4] as a model for the spread of an advantageous gene in a population of diploid individuals. This equation and its various extensions are also found in many models arising from physics, chemistry and spatial ecology. For example, in the context of ecology, this equation is often used to study the spread of a mutant of an existing species or a new species in a homogeneous environment. The spatial dynamics of (1.1) including longtime behavior, traveling wave and asymptotic speed of propagation have been well studied; see [3, 5, 7, 12] and the references cited therein.

Recently, due to the threats associated with global climate change and the worsening of the environment resulting from industrialization, the effects of climate and environment changes on the survival of ecological species have attracted much attention from the scientific community, including mathematical modellers and analysts; see for example, Berestycki et al. [1, 2], Li et al. [8], and Hu-Li [6]. In particular, Li et al. [8] considered the following model:

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ur(x-ct) - u^2, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.2}$$

where the birth rate r is assumed to be space-time dependent, changing in the form of shifting with constant speed $c > 0$, while the density dependent death rate has

Received by the editors October 3, 2016.
2010 *Mathematics Subject Classification.* Primary 34C05, 34D20; Secondary 92D40, 92D25.
The first author was partially supported by NNSF of China (Nos. 11326116, 11401051) and Hunan Provincial Natural Science Foundation (Nos. 2015JJ3013, 2016JJ1001). The second author was partially supported by NSERC of Canada (No. RGPIN-2016-04665).

been rescaled to the current form. Obviously, $r(x-ct)$ offers a spatially varying baseline or a historic rate of population growth. Here the function $r(\xi)$ is defined on $\mathbb{R}$, and it is assumed to satisfy the following condition:

(H) $r(\xi)$ is continuous, nondecreasing and bounded with $r(-\infty) < 0 < r(\infty)$.

Due to the sign change property of $r(\cdot)$, at any given time t, $r(x-ct)$ naturally divides the spatial domain into two shifting regions: one is a good quality habitat suitable for growth (i.e., $r(x-ct) > 0$), and the other is a poor quality habitat unsuitable for growth (i.e., $r(x-ct) < 0$). The main concern of [8] is the speed of spatial spread of the species and its relation with the persistence/extinction of the species. The main result in [8] states that the persistence and spreading dynamics depend on the speed of the shifting habitat edge c and the number $c^*(\infty) = 2\sqrt{dr(\infty)}$. More precisely, if $c > c^*(\infty)$, then the species will die out in the whole habitat, and if $c < c^*(\infty)$, then the species will survive and spread along the shifting habitat gradient with an asymptotic spreading speed $c^*(\infty)$.

We point out that the persistence considered in [8] is not location-wise; instead, it is in the sense of "by moving" or "on the go", meaning that the species will spread/move toward the better resource with speed $c^*(\infty)$ which is greater than the shifting speed c. However, at any given location x, as time goes, $r(x-ct)$ will become negative when t is sufficiently large (because $r(-\infty) < 0$) and the population at this location will go to extinction, regardless of whether or not the species can persist "by moving". It is then interesting and important to understand the point-wise "die-out dynamics" of the species, and this motivates us to consider the traveling wave front connecting the extinction steady state $u = 0$, which is not discussed in [8] and has not been considered elsewhere, to the best knowledge of the authors.

For many spatially diffusive population models *with spatially homogeneity in the whole space*, traveling wave fronts (TWF) have played an important role in describing/understanding the *spatial-temporal dynamics* of the population. It is well known that, if the nonlinearity in the reaction-diffusion model system is of the monostable type, then there exists a minimal speed c_* for TWFs connecting the two equilibria, in the sense that for any $c \geq c_*$ there is a TWF with speed c and there is no such TWF with speed $c < c_*$; moreover, this minimal wave speed often coincides with the (asymptotic) spread speed of the population if the spatial system is monotone (see, e.g., Liang-Zhao [9]). Note that the model system (1.2) is *not homogeneous* since r depends on x and t, and the heterogeneity is in the form of "spatial shifting" ($r = r(x-ct)$) with a constant shifting speed. Therefore, it is very natural to consider the traveling wave solution with this shifting speed.

Letting $u(x,t) = \varphi(\xi)$ with $\xi = x - ct$, and plugging this into (1.2) leads to

$$d\varphi''(\xi) + c\varphi'(\xi) + \varphi(\xi)(r(\xi) - \varphi(\xi)) = 0. \tag{1.3}$$

This equation for the profile of the traveling waves should be associated with some boundary conditions at $\xi = \pm\infty$. As we explained above, the condition $r(-\infty) < 0$ leads to point-wise distinction, and this suggests the boundary condition $\varphi(-\infty) = 0$ for the profile function φ. Note that unlike for homogeneous monostable systems in which there is another constant steady state, heterogeneity of r in (1.2) prevents it from having the other constant steady state, and the shifting feature also prevent it from having any other steady state. Hence, the boundary condition at the other end $\xi = \infty$ needs a different perspective. Now, based on the condition $r(\infty) > 0$, it seems that we should pose the condition $\varphi(\infty) = r(\infty)$ at $\xi = \infty$. Based on the above consideration, we associate to (1.3) the following boundary conditions:

$$\lim_{\xi\to-\infty} \varphi(\xi) = 0, \quad \lim_{\xi\to\infty} \varphi(\xi) = r(\infty). \tag{1.4}$$

We will explore whether or not, and for what value of c, the boundary value problem (1.3)-(1.4) has a positive solution. Such a solution accounts for an extinction wave, explaining the spatial-temporal dynamics of extinction.

Note that (1.3) is a *nonautonomous* ordinary differential equation, and the nonautonomous feature makes the existence problem challenging. In the rest of this paper, we will tackle this problem by the approach used by Wu and Zou in [14]: the combination of upper-lower solutions and monotone iterations. In Section 2, we present some preliminaries on a differential operator and its inverse. In Section 3, we construct a pair of upper-lower solutions for the problem, by which an invariant set for the profile of the problem (1.3)-(1.4) is obtained. Using the results in Sections 2 and 3, the existence problem is transformed to a fixed point problem for an operator in the profile set, and the fixed point is confirmed in Section 4 by establishing a monotone iteration scheme, leading to the existence of an extinction wave, as stated in the following main theorem:

Theorem 1.1. *Assume that* (**H**) *holds. For any given* $c > 0$, (1.2) *admits a monotone traveling wave solution* $u(x,t) = \varphi(\xi)$ *with* $\xi = x - ct$ *such that* $\lim_{\xi\to-\infty} \varphi(\xi) = 0$ *and* $\lim_{\xi\to\infty} \varphi(\xi) = r(\infty)$.

We remark that by this theorem, as long as the habitat is shifting according to (**H**), regardless of its magnitude, there will be an extinction wave with the speed that is precisely the same as the shifting speed. This extinction wave contains information of the spatial-temporal dynamics of the population, and may help one estimate the extinction speed at any given location in the habitat. As such, this result is complementary to those in [8], toward a full understanding of this model equation (1.2) with a shifting habitat.

2. Preliminary results

We first introduce some notation. Let $L^\infty(\mathbb{R})$ denote the vector space of all essentially bounded functions from $\mathbb{R}$ to $\mathbb{R}$, $BC(\mathbb{R})$ denote the vector space of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and $BC_+ = \{\phi \in BC(\mathbb{R}) : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$. For any $\psi, \phi \in BC(\mathbb{R})$, we write $\psi \leq \phi$ or $\phi \geq \psi$ if $\phi - \psi \in BC_+$.

Similar to [11], we will introduce the second-order linear differential operator Δ_* and its inverse Δ_*^{-1}. Let

$$\alpha = 2r(\infty) - r(-\infty).$$

It is easy to see that the equation

$$-d\lambda^2 - c\lambda + \alpha = 0$$

has two real roots

$$\lambda_1 = \frac{-c - \sqrt{c^2 + 4d\alpha}}{2d} < 0, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4d\alpha}}{2d} > 0.$$

The second-order linear differential operator Δ_* is defined by

$$\Delta_* h := -dh'' - ch' + \alpha h. \tag{2.1}$$

We can also define the corresponding integral operator Δ_*^{-1} by

$$(\Delta_*^{-1}h)(\xi) = \frac{1}{d(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-x)}h(x)\mathrm{d}x + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-x)}h(x)\mathrm{d}x\right]. \tag{2.2}$$

Clearly, the integral $\Delta_*^{-1}h$ is well defined for any $h \in BC(\mathbb{R})$. In fact, the following equality holds:

$$\Delta_*^{-1}\hat{K} = \frac{K}{\alpha}, \tag{2.3}$$

for any nonnegative constant K, where $\hat{K}$ denotes the constant function on $\mathbb{R}$ taking the value K. Similar to Lemma 3.1 in [11], we have that

$$\Delta_*(\Delta_*^{-1}h) = h \tag{2.4}$$

for any $h \in BC(\mathbb{R})$, and

$$\Delta_*^{-1}(\Delta_* h) = h \tag{2.5}$$

for any $h \in BC(\mathbb{R})$ such that $h', h'' \in BC(\mathbb{R})$. Thus, Δ_*^{-1} is actually the inverse operator of Δ_* in some sense. In fact, we have the following more general conclusion, which is similar to Lemma 2.5 in [10].

Lemma 2.1. *Assume that $h \in BC(\mathbb{R})$ satisfies the following conditions: (i) h'' is continuous on $\mathbb{R}\backslash\{\xi_j\}$ where $\{\xi_j\}$ is a finite increasing sequence, and $h', h'' \in L^\infty(\mathbb{R})$; (ii) $h'(\xi_j+)$ and $h'(\xi_j-)$ exist. Then $\Delta_*^{-1}(\Delta_* h)$ is a continuous function on $\mathbb{R}$ and*

$$[\Delta_*^{-1}(\Delta_* h)](\xi) = h(\xi) + \frac{1}{\lambda_2-\lambda_1}\left(\sum_{\xi_j \ge \xi} \beta_j \mathrm{e}^{\lambda_2(\xi-\xi_j)} + \sum_{\xi_j<\xi} \beta_j \mathrm{e}^{\lambda_1(\xi-\xi_j)}\right) \tag{2.6}$$

for all $\xi \in \mathbb{R}$, where $\beta_j = h'(\xi_j+) - h'(\xi_j-)$.

Proof. The assumptions on h imply that $\Delta_* h = -dh'' - ch' + \alpha h \in L^\infty(\mathbb{R})$. Thus, $\Delta_*^{-1}(\Delta_* h)$ is well defined, and the function $[\Delta_*^{-1}(\Delta_*)h](\xi)$ is continuous with respect to ξ on $\mathbb{R}$. The identity (2.6) follows by direct calculation. □

3. Construction of an invariant set

In order to construct a profile set, we first establish two lemmas. Let

$$x_1 = \varphi, \quad x_2 = \varphi';$$

then equation (1.3) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}\xi} = x_2, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}\xi} = -\dfrac{c}{d}x_2 - \dfrac{r(\xi)}{d}x_1 + \dfrac{1}{d}x_1^2. \end{cases} \tag{3.1}$$

The vector form of (3.1) is

$$x' = Ax + B(\xi)x + f(x), \tag{3.2}$$

where $x = (x_1, x_2)^T$ and

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{r(\infty)}{d} & -\frac{c}{d} \end{pmatrix}, \quad B(\xi) = \begin{pmatrix} 0 & 0 \\ \frac{r(\infty)-r(\xi)}{d} & 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ \frac{1}{d}x_1^2 \end{pmatrix}.$$

It is easy to see that the eigenvalues of A all have negative real parts, $\lim_{\xi\to\infty}\|B(\xi)\| = 0$, and $f(x)$ is Lipschitz continuous in a neighbourhood of $x = 0$ with the property that $f(x) = o(x)$ as $x \to 0$. By Theorem 7.1 in [13], there exist positive constants K, ξ_0, δ, μ such that $\|x_0\| \le \delta$ implies

$$\|x(\xi;\xi_0,x_0)\| \le K\|x_0\|\mathrm{e}^{-\mu(\xi-\xi_0)} \tag{3.3}$$

for $\xi \ge \xi_0$, where $x(\xi;\xi_0,x_0) = (x_1(\xi;\xi_0,x_0), x_2(\xi;\xi_0,x_0))^T$ denotes the solution of (3.2) through x_0 at $\xi = \xi_0$, namely, $x(\xi_0;\xi_0,x_0) = x_0$.

Lemma 3.1. *Let θ^* be the positive root of the equation $d\theta^2 + c\theta + r(-\infty) = 0$. Then there exist $\epsilon > 0$ and $\theta_0 > \theta^*$ such that the function*

$$\underline{\phi}(\xi) = \begin{cases} \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}, & \xi < \xi_0, \\ x_1(\xi;\xi_0,\epsilon(1,\theta_0)^T), & \xi \ge \xi_0, \end{cases}$$

is continuously differentiable on $\mathbb{R}$, and satisfies $\underline{\phi} \le r(\infty)$ and there holds the following differential inequality:

$$d\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + \underline{\phi}(\xi)(r(\xi) - \underline{\phi}) \ge 0 \tag{3.4}$$

for any $\xi \ne \xi_0$.

Proof. Let

$$\epsilon = \min\left\{\frac{\delta}{\sqrt{1+4\theta^{*2}}}, \frac{r(\infty)}{K\sqrt{1+4\theta^{*2}}}, 3d\theta^{*2} + c\theta^*, r(\infty)\right\},$$

and let θ_0 be the positive root of the equation

$$d\theta^2 + c\theta + r(-\infty) - \epsilon = 0.$$

It is easily seen that $\theta^* < \theta_0 \le 2\theta^*$. Moreover, $\underline{\phi}(\xi_0) = x_1(\xi_0;\xi_0,\epsilon(1,\theta_0)^T) = \epsilon$ and $\underline{\phi}'(\xi_0) = x_2(\xi_0;\xi_0,\epsilon(1,\theta_0)^T) = \epsilon\theta_0$. Therefore, $\underline{\phi}$ is continuously differentiable in $\mathbb{R}$.

If $\xi > \xi_0$, then $\underline{\phi}(\xi) = x_1(\xi;\xi_0,\epsilon(1,\theta_0)^T)$. From (3.3), we have

$$\underline{\phi}(\xi) \le |x_1| \le \sqrt{x_1^2 + x_2^2} \le K\epsilon\sqrt{1+\theta_0^2}\mathrm{e}^{-\mu(\xi-\xi_0)} \le r(\infty)\mathrm{e}^{-\mu(\xi-\xi_0)} \le r(\infty).$$

Since $\underline{\phi}'(\xi) = x_1'(\xi;\xi_0,\epsilon(1,\theta_0)^T) = x_2(\xi;\xi_0,\epsilon(1,\theta_0)^T)$, $\underline{\phi}''(\xi) = x_2'(\xi;\xi_0,\epsilon(1,\theta_0)^T)$, we have

$$d\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + \underline{\phi}(\xi)(r(\xi) - \underline{\phi}) = 0.$$

Therefore, (3.4) holds for $\xi > \xi_0$.

If $\xi < \xi_0$, then $\underline{\phi}(\xi) = \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}$. We first have $\underline{\phi}(\xi) \le \epsilon \le r(\infty)$. Recalling the choice of ϵ and θ_0, we have

$$\begin{aligned} &d\underline{\phi}''(\xi) + c\underline{\phi}'(\xi) + \underline{\phi}(\xi)(r(\xi) - \underline{\phi}) \\ &= \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}[d\theta_0^2 + c\theta_0 + r(\xi) - \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}] \\ &= \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}[r(\xi) - r(-\infty) + \epsilon - \epsilon\mathrm{e}^{\theta_0(\xi-\xi_0)}] \\ &\ge 0. \end{aligned}$$

Hence, (3.4) also holds for $\xi < \xi_0$. This completes the proof. ∎

Lemma 3.2. *Let $\xi_1 \le \xi_0$ be given such that $r(\xi_1) < 0$, and let θ_1 be the positive root of the equation $d\theta^2 + c\theta + r(\xi_1) = 0$. Then the function*

$$\overline{\phi}(\xi) = \min\{r(\infty)e^{\theta_1(\xi-\xi_1)}, r(\infty)\}$$

satisfies the following differential inequality:

$$d\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + \overline{\phi}(\xi)(r(\xi) - \overline{\phi}) \le 0 \tag{3.5}$$

for any $\xi \ne \xi_1$. Moreover, $\overline{\phi} \ge \underline{\phi}$.

Proof. For $\xi > \xi_1$, $\overline{\phi}(\xi) = r(\infty) \ge \underline{\phi}(\xi)$. Clearly, (3.5) holds for $\xi > \xi_1$.

For $\xi < \xi_1$, $\overline{\phi}(\xi) = r(\infty)e^{\theta_1(\xi-\xi_1)}$. Noting that $r(-\infty) \le r(\xi_1)$, it is easily seen that $0 < \theta_1 \le \theta^* < \theta_0$. Together with the fact that $\xi_1 \le \xi_0$ and $\epsilon \le r(\infty)$, we obtain that for $\xi < \xi_1$,

$$r(\infty)e^{\theta_1(\xi-\xi_1)} \ge \epsilon e^{\theta_1(\xi-\xi_1)} \ge \epsilon e^{\theta_0(\xi-\xi_1)} \ge \epsilon e^{\theta_0(\xi-\xi_0)}.$$

Therefore, $\overline{\phi}(\xi) \ge \underline{\phi}(\xi)$ for any $\xi < \xi_1$. In addition,

$$\begin{aligned}
&d\overline{\phi}''(\xi) + c\overline{\phi}'(\xi) + \overline{\phi}(\xi)(r(\xi) - \overline{\phi})\\
&= r(\infty)e^{\theta_1(\xi-\xi_1)}[d\theta_1^2 + c\theta_1 + r(\xi)] - r^2(\infty)e^{2\theta_1(\xi-\xi_1)}\\
&\le r(\infty)e^{\theta_1(\xi-\xi_1)}[d\theta_1^2 + c\theta_1 + r(\xi_1)]\\
&= 0.
\end{aligned}$$

Hence, (3.5) also holds for $\xi < \xi_1$. This completes the proof. □

Using these two functions $\underline{\phi}$ and $\overline{\phi}$, we can define the profile set

$$\Gamma := \{\phi \in BC(\mathbb{R}) : \underline{\phi} \le \phi \le \overline{\phi}\}. \tag{3.6}$$

Next, we define $H(\phi)$ by

$$H(\phi)(\xi) = \phi(\xi)[r(\xi) - \phi(\xi)] + \alpha\phi(\xi)$$

for any $\phi \in \Gamma$, and denote by F the composite of H and Δ_*^{-1}, that is,

$$F(\phi) = \Delta_*^{-1}H(\phi). \tag{3.7}$$

Since $H(\phi) \in BC(\mathbb{R})$ for any $\phi \in \Gamma$, then the operator F is also well defined on Γ. About the operator F, we have the following properties.

Lemma 3.3. *F is a monotone operator and maps Γ into Γ. Furthermore, if $\phi \in \Gamma$ and ϕ is nondecreasing, then $F(\phi)(\xi)$ is nondecreasing with respect to ξ.*

Proof. We first verify that, for any $\psi, \phi \in \Gamma$ with $\phi \ge \psi$,

$$\begin{aligned}
H(\phi)(\xi) - H(\psi)(\xi) &= [\phi(\xi) - \psi(\xi)][\alpha + r(\xi) - \phi(\xi) - \psi(\xi)]\\
&= [\phi(\xi) - \psi(\xi)][2r(\infty) - r(-\infty) + r(\xi) - \phi(\xi) - \psi(\xi)]\\
&\ge 0.
\end{aligned}$$

Thus, $F(\phi)(\xi) \ge F(\psi)(\xi)$ for $\xi \in \mathbb{R}$. On the one hand, it follows from (3.4) in Lemma 3.1 that

$$F(\underline{\phi}) = \Delta_*^{-1}H(\underline{\phi}) \ge \Delta_*^{-1}(\Delta_*\underline{\phi}).$$

Note that $\underline{\phi}$ is continuously differentiable on $\mathbb{R}$ satisfying all of the assumptions in Lemma 2.1, and $\underline{\phi}'(\xi_0-) = \underline{\phi}'(\xi_0+)$. By (2.6) we obtain that $\Delta_*^{-1}(\Delta_*\underline{\phi}) = \underline{\phi}$. Hence, $F(\underline{\phi}) \ge \underline{\phi}$. On the other hand, it follows from (3.5) in Lemma 3.2 that

$$F(\overline{\phi}) = \Delta_*^{-1}H(\overline{\phi}) \le \Delta_*^{-1}(\Delta_*\overline{\phi}).$$

Note that $\overline{\phi}$ is continuous on $\mathbb{R}$ satisfying all of the assumptions in Lemma 2.1, and $\overline{\phi}'(\xi_1-) > \overline{\phi}'(\xi_1+) = 0$. By (2.6) we get $\Delta_*^{-1}(\Delta_*\overline{\phi}) \leq \overline{\phi}$. Thus, $F(\overline{\phi}) \leq \overline{\phi}$. Therefore, $F(\Gamma) \subseteq \Gamma$.

If $\phi \in \Gamma$ is nondecreasing, then $\phi \geq 0$, and for any $x \in \mathbb{R}$ and $s > 0$ we have

$$\begin{aligned}
&H(\phi)(x+s) - H(\phi)(x) \\
&= \phi(x+s)[r(x+s) - \phi(x+s)] + \alpha[\phi(x+s) - \phi(x)] - \phi(x)[r(x) - \phi(x)] \\
&\geq [\phi(x+s) - \phi(x)][\alpha + r(x) - \phi(x+s) - \phi(x)] \\
&= [\phi(x+s) - \phi(x)][2r(\infty) - r(-\infty) + r(x) - \phi(x+s) - \phi(x)] \\
&\geq 0.
\end{aligned}$$

Thus,

$$[\Delta_*^{-1}H(\phi)(x+s)](\xi) \geq [\Delta_*^{-1}H(\phi)(x)](\xi)$$

for any $\xi \in \mathbb{R}$. Noting that $[\Delta_*^{-1}h(x+s)](\xi) = [\Delta_*^{-1}h(x)](\xi+s)$ for any $h \in BC(\mathbb{R})$, we have

$$\begin{aligned}
F(\phi)(\xi+s) &= [\Delta_*^{-1}H(\phi)(x)](\xi+s) \\
&= [\Delta_*^{-1}H(\phi)(x+s)](\xi) \\
&\geq [\Delta_*^{-1}H(\phi)(x)](\xi) \\
&= F(\phi)(\xi).
\end{aligned}$$

This completes the proof of the lemma. □

4. Proof of the main result

We construct the following iteration:

$$\phi_1 = F(\overline{\phi}), \quad \phi_{n+1} = F(\phi_n), \quad n \geq 1.$$

Since $\overline{\phi} \in \Gamma$ is nondecreasing on $\mathbb{R}$, by Lemma 3.3, we know that $\phi_n \in \Gamma$, $\phi_n(\xi)$ is nondecreasing with respect to ξ for each fixed $n = 1, 2, \cdots$, and

$$\underline{\phi}(\xi) \leq \phi_{n+1}(\xi) \leq \phi_n(\xi) \leq \overline{\phi}(\xi), \quad \xi \in \mathbb{R},\ n \geq 1.$$

Let

$$\phi(\xi) = \lim_{n\to\infty} \phi_n(\xi).$$

Clearly,

$$\underline{\phi}(\xi) \leq \phi(\xi) \leq \overline{\phi}(\xi), \tag{4.1}$$

and ϕ is a nondecreasing and positive function defined on $\mathbb{R}$. Next, we will prove $\phi(\xi)$ is a fixed point of F.

Since ϕ_n converges point-wise to ϕ, then $H(\phi_n)$ converges point-wise to $H(\phi)$. Note that $|H(\phi_n)| \leq 2r^2(\infty) + \alpha r(\infty)$ for all $n = 1, 2, \cdots$. By (2.3) and the

Lebesgue's dominated convergence theorem, we then have

$$\begin{aligned}
\phi(\xi) &= \lim_{n\to\infty} \phi_n(\xi) \\
&= \lim_{n\to\infty} F(\phi_{n-1}(\xi)) \\
&= \lim_{n\to\infty} \frac{1}{d(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-x)} H(\phi_{n-1})(x)\mathrm{d}x \right. \\
&\quad \left. + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-x)} H(\phi_{n-1})(x)\mathrm{d}x\right] \\
&= \frac{1}{d(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-x)} H(\phi)(x)\mathrm{d}x + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-x)} H(\phi)(x)\mathrm{d}x\right] \\
&= F(\phi)(\xi).
\end{aligned}$$

Thus, from (2.4), we have

$$\Delta_* \phi = H\phi, \tag{4.2}$$

i.e., there exists $\phi \in BC(\mathbb{R})$ satisfying (1.3). It remains to show that ϕ satisfies the boundary condition (1.4). Since

$$\lim_{\xi\to-\infty} \underline{\phi}(\xi) = \lim_{\xi\to-\infty} \overline{\phi}(\xi) = 0,$$

by (4.1) we obtain that

$$\lim_{\xi\to-\infty} \phi(\xi) = 0.$$

Noting that ϕ is nondecreasing and $\phi \le \overline{\phi} \le r(\infty)$, let

$$\lim_{\xi\to\infty} \phi(\xi) = A.$$

By the positivity and monotonicity of ϕ, we have $0 < A \le r(\infty)$. Since

$$\lim_{\xi\to\infty} H(\phi)(\xi) = A[r(\infty) - A] + \alpha A,$$

recalling the definition of Δ_*^{-1}, by L'Hôpital's rule, we have

$$\begin{aligned}
A &= \lim_{\xi\to\infty} \phi(\xi) \\
&= \lim_{\xi\to\infty} \Delta_*^{-1}(H(\phi))(\xi) \\
&= \lim_{\xi\to\infty} \frac{1}{d(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_1(\xi-x)} H(\phi)(x)\mathrm{d}x + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_2(\xi-x)} H(\phi)(x)\mathrm{d}x\right] \\
&= \lim_{\xi\to\infty} \frac{1}{d(\lambda_2-\lambda_1)}\left(\frac{H(\phi)(\xi)}{-\lambda_1} + \frac{H(\phi)(\xi)}{\lambda_2}\right) \\
&= \frac{A[r(\infty)-A]}{\alpha} + A.
\end{aligned}$$

This yields $A = r(\infty)$. Therefore, $\phi(\xi)$ satisfies (1.4). This completes the proof of Theorem 1.1.

Acknowledgment

This work was completed when the first author was visiting the University of Western Ontario as a visiting scholar sponsored by the China Scholarship Council (Grant No. 201408430035), and he would like to thank the staff of the Department of Applied Mathematics for their help and thank the university for its excellent facilities and support during his stay.

References

[1] H. Berestycki, O. Diekmann, C. J. Nagelkerke, and P. A. Zegeling, *Can a species keep pace with a shifting climate?*, Bull. Math. Biol. **71** (2009), no. 2, 399–429, DOI 10.1007/s11538-008-9367-5. MR2471053

[2] H. Berestycki, L. Desvillettes, and O. Diekmann, *Can climate change lead to gap formation?*, Ecological Complexity, 20 (2014), pp. 264–270.

[3] J. Billingham and D. J. Needham, *A note on the properties of a family of travelling-wave solutions arising in cubic autocatalysis*, Dynam. Stability Systems **6** (1991), no. 1, 33–49, DOI 10.1080/02681119108806105. MR1012593

[4] R. A. Fisher, *The wave of advance of advantageous genes*, Annals of eugenics, 7(4)(1937), pp. 355–369.

[5] Xiaojie Hou, Yi Li, and Kenneth R. Meyer, *Traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities*, Discrete Contin. Dyn. Syst. **26** (2010), no. 1, 265–290, DOI 10.3934/dcds.2010.26.265. MR2552787

[6] Changbing Hu and Bingtuan Li, *Spatial dynamics for lattice differential equations with a shifting habitat*, J. Differential Equations **259** (2015), no. 5, 1967–1989, DOI 10.1016/j.jde.2015.03.025. MR3349426

[7] A. Kolmogoroff, I. Petrovsky, and N. Piscounoff, *Étude de l'équations de la diffusion avec croissance de la quantité de matière et son application a un problème biologique*, Bull. Univ. Moscow, Ser. Internat., Sec. A, 1(1937), pp. 1–25.

[8] Bingtuan Li, Sharon Bewick, Jin Shang, and William F. Fagan, *Persistence and spread of a species with a shifting habitat edge*, SIAM J. Appl. Math. **74** (2014), no. 5, 1397–1417, DOI 10.1137/130938463. MR3259481

[9] Xing Liang and Xiao-Qiang Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math. **60** (2007), no. 1, 1–40, DOI 10.1002/cpa.20154. MR2270161

[10] Shiwang Ma, *Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem*, J. Differential Equations **171** (2001), no. 2, 294–314, DOI 10.1006/jdeq.2000.3846. MR1818651

[11] Xiang-Sheng Wang, Haiyan Wang, and Jianhong Wu, *Traveling waves of diffusive predator-prey systems: disease outbreak propagation*, Discrete Contin. Dyn. Syst. **32** (2012), no. 9, 3303–3324, DOI 10.3934/dcds.2012.32.3303. MR2912077

[12] H. F. Weinberger, *Long-time behavior of a class of biological models*, SIAM J. Math. Anal. **13** (1982), no. 3, 353–396, DOI 10.1137/0513028. MR653463

[13] Ferdinand Verhulst, *Nonlinear differential equations and dynamical systems*, 2nd ed., Universitext, Springer-Verlag, Berlin, 1996. Translated from the 1985 Dutch original. MR1422255

[14] Jianhong Wu and Xingfu Zou, *Traveling wave fronts of reaction-diffusion systems with delay*, J. Dynam. Differential Equations **13** (2001), no. 3, 651–687, DOI 10.1023/A:1016690424892. MR1845097

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, People's Republic of China, 410114

E-mail address: huhaijun2000@163.com

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada, N6A 5B7

E-mail address: xzou@uwo.ca