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Stable Periodic Solutions in a Discrete Periodic Logistic Equation

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Abstract—In this paper, we consider a discrete logistic equation

$$x(n+1) = x(n) \exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right],$$

where $\{r(n)\}$ and $\{K(n)\}$ are positive ω-periodic sequences. Sufficient conditions are obtained for the existence of a positive and globally asymptotically stable ω-periodic solution. Counterexamples are given to illustrate that the conclusions in [1] are incorrect.

Keywords—Periodic solution, Logistic equation, Stability.

1. INTRODUCTION

One of the the basic differential equation models for population growth of a single species is the logistic equation

$$\frac{dx(t)}{dt} = r(t)x(t)\left[1 - \frac{x(t)}{K(t)}\right], \qquad t \geq 0, \tag{1.1}$$

where $r(\cdot)$ and $K(\cdot)$ are positive functions in $[0, \infty)$, representing the intrinsic growth rate and the carrying capacity, respectively. When $K(\cdot)$ is constant, the dynamics of (1.1) are completely known: every positive solution converges to the positive equilibrium. In many situations, $r(t)$ and $K(t)$ can be assumed to be nonconstant periodic functions with a common period T to reflect the seasonal fluctuations. In such a periodic case, it has been shown that (1.1) has a positive T-periodic solution $\tilde{x}(t)$ which attracts every positive solution $x(t)$ of (1.1) as $t \to \infty$. See, e.g., [2–4].

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In this paper, we consider a discrete analogue of (1.1),

$$x(n+1) = x(n)\exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right], \qquad n \in N, \tag{1.2}$$

under the assumptions that $x(0) > 0$, $\{r(n)\}$ and $\{K(n)\}$ are strictly positive sequences of real numbers defined for $n \in N = \{0, 1, 2, \dots\}$. In addition, there exist positive constants r_*, r^*, K_*, and K^* such that

$$0 < r_* \le r(n) \le r^*, \quad 0 < K_* \le K(n) \le K^*, \qquad n \in N. \tag{1.3}$$

For a justification of (1.2), we refer to [1].

For (1.2), one may naturally conjecture a parallel conclusion: if $\{r(n)\}$ and $\{K(n)\}$ are both periodic with a common period ω, then (1.2) has a positive ω-periodic solution $\{\tilde{x}(n)\}$, and every positive solution $\{x(n)\}$ of (1.2) tends to $\{\tilde{x}(n)\}$ as $n \to \infty$. However, the following example shows that this cannot be true.

EXAMPLE 1.1. Consider equation (1.2) with

$$\begin{aligned} r(3n) &= 1, & r(3n+1) &= 1.5, & r(3n+2) &= 1,\\ K(3n) &= 1, & K(3n+1) &= 5, & K(3n+2) &= 8, \end{aligned}$$

for $n \in N$. Then (1.2) has a 3-periodic solution $\{\tilde{x}(n)\}$, where

$$\tilde{x}(3n) = 3.2184, \quad \tilde{x}(3n+1) = 0.3501, \quad \tilde{x}(3n+2) = 1.4126, \qquad \text{for } n \in N,$$

and a 6-periodic solution $\{x^*(n)\}$ where

$$\begin{aligned} x^*(6n) &= 5.6940, & x^*(6n+1) &= 0.0521, & x^*(6n+2) &= 0.2299,\\ x^*(6n+3) &= 0.6072, & x^*(6n+4) &= 0.8993, & x^*(6n+5) &= 3.0774, \end{aligned} \qquad \text{for } n \in N.$$

Let

$$f_n(x) = x\exp\left(r(n)\left(1 - \frac{x}{K(n)}\right)\right), \qquad n \in N.$$

Then

$$\prod_{n=0}^{2} f_n'(\tilde{x}(n)) = -1.6348, \qquad \prod_{n=0}^{5} f_n'(x^*(n)) = -0.7921.$$

This implies $\{\tilde{x}(n)\}$ is unstable and $\{x^*(n)\}$ is asymptotically stable.

This example shows that even for very simple models, a stability result for a continuous model does not automatically carry over for the corresponding discrete model.

Recently, Mohamad and Gopalsamy [1] also considered equation (1.2), and obtained the following two main theorems.

THEOREM A. *(See [1, Theorem 3.2].) Assume that $\{r(n)\}$ and $\{K(n)\}$ satisfy (1.3). Then (1.2) is extremely stable in the sense that*

$$\lim_{n\to\infty} |x(n) - y(n)| = 0,$$

for any two solutions $\{x(n)\}$ and $\{y(n)\}$ of (1.2).

THEOREM B. *(See [1, Theorem 4.1].) Assume that $\{r(n)\}$ and $\{K(n)\}$ are almost periodic sequences satisfying (1.3) with $r^* < 2$. Then (1.2) has a unique positive and globally asymptoticly stable almost periodic solution.*

Looking back at the above example, since $\min_{n\in N} |\tilde{x}(n) - x^*(n)| > 0$, it is easy to see that these two theorems are incorrect. Digging into the source of the incorrectness of Theorems A and B in [1], we find that the proofs of these two theorems are based on Lemmas 2.1 and 3.1 in [1], which are stated below.

LEMMA A. *(See [1, Lemma 2.1].) Assume that $\{r(n)\}$ and $\{K(n)\}$ satisfy (1.3). Then for any positive solution of (1.2), there exists $N > 0$ such that*

$$x_{\min} \le x(n) \le x_{\max}, \qquad n \ge N, \tag{1.4}$$

where

$$x_{\max} = \frac{K^*}{r_*}\exp\left(r^* - 1\right), \qquad x_{\min} = x_{\max}\exp\left(r_* - \frac{r^*}{K_*}x_{\max}\right).$$

LEMMA B. *(See [1, Lemma 3.1].) Let $\{r(n)\}$ and $\{K(n)\}$ be strictly positive bounded with*

$$0 < r_* \le r(n) \le r^* < 2, \qquad n \in Z. \tag{1.5}$$

Then for any positive solution $\{x(n)\}$ of (1.2), there exists a positive integer N such that

$$\left|1 - r(n)\frac{x(n)}{K(n)}\right| < 1, \qquad n \ge N, \tag{1.6}$$

$$\left|\left[1 - r(n)\frac{x(n)}{K(n)}\right]\right| \exp\left[r(n)\left(1 - \frac{x(n)}{K(n)}\right)\right] < 1, \qquad n \ge N. \tag{1.7}$$

Unfortunately, the above two lemmas are incorrect as well. To see that Lemma A is invalid, let us consider the following example.

EXAMPLE 1.2. Consider equation (1.2) with

$$r(n) = \frac{1}{2}, \quad K(2n) = \frac{10}{11} = 0.9090909, \quad K(2n+1) = \frac{10}{9}e^{-1/20} = 1.0569216, \qquad n \in N.$$

Clearly, (1.3) holds with

$$K^* = \frac{10}{9}e^{-1/20} = 1.0569216, \quad K_* = \frac{10}{11} = 0.9090909, \qquad r_* = r^* = 0.5.$$

Let $x(0) = 1$, then

$$x(2n) = 1, \quad x(2n+1) = e^{-1/20} = 0.9512294, \qquad \text{for } n \in N.$$

We can calculate $x_{\max}$ and $x_{\min}$ as follows:

$$\begin{aligned} x_{\max} &= \frac{K^*}{r_*}\exp(r^* - 1) = \frac{20}{9}e^{-11/20} = 1.2821107, \\ x_{\min} &= x_{\max}\exp\left(r_* - \frac{r^*}{K_*}x_{\max}\right) \\ &= \frac{20}{9}\exp\left(-\frac{1}{20} - \frac{11}{9}e^{-11/20}\right) \\ &= 1.0443000. \end{aligned}$$

Thus, $x(2n+1) < x_{\min}$ for $n \in N$, which implies that (1.4) in Lemma A is incorrect.

To show Lemma B is incorrect, we consider the following two examples.

EXAMPLE 1.3. Consider equation (1.2) with

$$r(n) = 1, \quad K(3n) = \frac{4}{1+\ln 4}, \quad K(3n+1) = \frac{1}{1-\ln 2}, \quad K(3n+2) = \frac{2}{1-\ln 2}, \qquad n \in N.$$

Clearly, (1.3) and (1.5) hold with

$$r_* = r^* = 1, \qquad K_* = \frac{4}{1+\ln 4}, \quad K^* = \frac{2}{1-\ln 2}.$$

Let $x(0) = 4$, then

$$x(3n) = 4, \quad x(3n+1) = 1, \quad x(3n+2) = 2, \qquad n \in N.$$

Thus,

$$1 - r(3n)\frac{x(3n)}{K(3n)} = -\ln 4 < -1,$$

a contradiction to (1.6) in Lemma B.

EXAMPLE 1.4. Consider equation (1.2) with

$$r(n) = 1, \quad K(2n) = 4, \quad K(2n+1) = \frac{4}{7}e^{3/4} = 1.20971430, \qquad n \in N.$$

Clearly, (1.3) and (1.5) hold with

$$r_* = r^* = 1, \qquad K_* = \frac{4}{7}e^{3/4}, \qquad K^* = 4.$$

Let $x(0) = 1$; then

$$x(2n) = 1, \quad x(2n+1) = e^{3/4}, \qquad \text{for } n \in N.$$

Thus,

$$\left(1 - r(2n)\frac{x(2n)}{K(2n)}\right) \exp\left[r(n)\left\{1 - \frac{x(n)}{K(n)}\right\}\right] = \frac{3}{4}e^{3/4} = 1.58775 > 1,$$

for $n \in N$. This contradicts (1.7) in Lemma B.

In the rest of this paper, we will derive, in Section 3, sufficient conditions under which (1.2) has a unique, positive, and globally asymptotically stable periodic solution. For this purpose, in Section 2, we need to establish a persistence result.

2. PERSISTENCE

In this section, we establish the following persistence result for (1.2), which is a correction of Lemma A.

THEOREM 2.1. *Assume that $\{r(n)\}$ and $\{K(n)\}$ satisfy (1.3). Then any positive solution $\{x(n)\}$ of (1.2) satisfies*

$$u_* \le \liminf_{n\to\infty} x(n) \le \limsup_{n\to\infty} x(n) \le u^*, \tag{2.1}$$

where

$$u^* = \frac{K^*}{r^*}\exp(r^* - 1), \qquad u_* = K_* \exp\left(r^*\left(1 - \frac{u^*}{K_*}\right)\right).$$

PROOF. We first present two cases to show that

$$\limsup_{n\to\infty} x(n) \le u^*. \tag{2.2}$$

CASE 1. There exists a positive integer n_0 such that $x(n_0) < x(n_0+1)$.

From (1.2), we see that $1 - (x(n_0)/K(n_0) > 0)$, this implies

$$x(n_0) < K(n_0) \le K^*.$$

Therefore, by the fact that $\max_{x\in R} x\exp[r(1-x)] = (1/r)\exp(r-1)$ where $r > 0$, we have

$$\begin{aligned} x(n_0+1) &= x(n_0)\exp\left[r(n_0)\left(1-\frac{x(n_0)}{K(n_0)}\right)\right] \\ &\le K(n_0)\frac{x(n_0)}{K(n_0)}\exp\left[r^*\left(1-\frac{x(n_0)}{K(n_0)}\right)\right] \\ &\le \frac{K^*}{r^*}\exp(r^*-1) = u^*. \end{aligned}$$

We claim that

$$x(n) \le u^*, \qquad \text{for } n \ge n_0.$$

In fact, if there exists an integer $m \ge n_0 + 2$ such that $x(m) > u^*$, and letting m^* be the least integer between n_0 and m such that $x(m^*) = \max_{n_0 \le n \le m} x(n)$, then $m^* \ge n_0 + 2$ and $x(m^*) > x(m^*-1)$ which implies $x(m^*) \le u^* < x(m)$. This is impossible.

CASE 2. $x(n) \ge x(n+1)$ for $n \in N$.

By (1.2), we see that

$$1 - \frac{x(n)}{K(n)} \le 0, \qquad n \in N. \tag{2.3}$$

This implies that $x(n) \ge K(n) \ge K_*$ for $n \in N$. Since $\{x(n)\}$ is nonincreasing and has a lower bound K_*, we know $\lim_{n\to\infty} x(n) = \bar{x} \ge K_*$. Letting $n \to \infty$ in (1.2), we get

$$\bar{x} = \lim_{n\to\infty} K(n) \le K^* \le u^*.$$

Therefore, (2.2) holds.

Now, we show that

$$\liminf_{n\to\infty} x(n) \ge u_*. \tag{2.4}$$

In view of (2.2), for each ϵ, there exists a large integer n^* such that

$$x(n) \le u^* + \epsilon, \qquad \text{for } n \ge n^*. \tag{2.5}$$

We consider two cases.

CASE (i). There exists a positive integer $\bar{n}_0 \ge n^*$ such that $x(\bar{n}_0 + 1) < x(\bar{n}_0)$.

Similar to Case 1 in the proof of (2.2), we see that

$$x(n) \ge K_* \exp\left(r^*\left(1 - \frac{u^*+\epsilon}{K_*}\right)\right), \qquad n \ge n^*. \tag{2.6}$$

CASE (ii). $x(n+1) \ge x(n)$ for $n \ge n^*$.

According to (2.5), we know $\lim_{n\to\infty} x(n) = l$. Letting $n \to \infty$ in (1.2) leads to $\lim_{n\to\infty} K(n) = l$. So,

$$l = \lim_{n\to\infty} x(n) = \lim_{n\to\infty} K(n) \ge K_* \ge K_* \exp\left(r^*\left(1 - \frac{u^*+\epsilon}{K_*}\right)\right).$$

Combining Cases (i) and (ii), we see that

$$\liminf_{n\to\infty} x(n) \ge K_* \exp\left(r^*\left(1 - \frac{u^*+\epsilon}{K_*}\right)\right).$$

Since ϵ is arbitrary, we know (2.4) holds.

The proof is completed by combining (2.2) with (2.4).

REMARK 2.1. Since $u^* \le x_{\max}$, where $x_{\max}$ is as in Lemma A, (2.1) gives a better upper bound than (1.4). This also confirms that the right half of (1.4) is valid (the left half is invalid though).

REMARK 2.2. In view of the proof of Theorem 2.1, we see that, if either $\lim_{n\to\infty} K(n)$ does not exist or $r^* \ne 1$, then $u_* \le x(n) \le u^*$ eventually holds.

3. EXISTENCE AND STABILITY OF PERIODIC SOLUTION

Now we consider (1.2) with $\{r(n)\}$ and $\{K(n)\}$ being periodic, and we are concerned with the existence and stability of a periodic solution. First, we have the following existence result.

THEOREM 3.1. *Assume that $\{r(n)\}$ and $\{K(n)\}$ are positive periodic sequences with a common positive period ω, that is,*

$$r(n+\omega) = r(n), \quad K(n+\omega) = K(n), \qquad n \in N. \tag{3.1}$$

Then there exists an ω-periodic solution for equation (1.2).

PROOF. If $K(n) \equiv K$(constant), then $x(n) = K$ is a solution of (1.2) which implies Theorem 3.1 holds.

Now assume that $\{K(n)\}$ is not constant, so $\lim_{n\to\infty} K(n)$ does not exist. By the assumptions, we see (1.3) holds with $r_* = \min_{n\in N}\{r(n)\}$, $r^* = \max_{n\in N}\{r(n)\}$, $K_* = \min_{n\in N}\{K(n)\}$, and $K^* = \max_{n\in N}\{K(n)\}$. According to the proof of Theorem 2.1, it is easy to see that

$$x(0) \in [u_*, u^*] \text{ implies } x(n) \in [u_*, u^*], \qquad \text{for } n \in N. \tag{3.2}$$

Now, we define a mapping F on $[u_*, u^*]$ by $F(x(0)) = x(\omega)$. From (1.2), we see that $x(\omega)$ depends continuously on $x(0)$. Thus, F is continuous and maps the interval $[u_*, u^*]$ into itself. Therefore, F has a fixed-point p. Let $x(0) = p$, then the corresponding solution $\{\tilde{x}(n)\}$ of (1.2) is an ω-periodic solution to (1.2) in $[u_*, u^*]$. This completes the proof.

The next theorem confirms the globally asymptotic stability of the periodic solution obtained in Theorem 3.1, under an additional condition.

THEOREM 3.2. *Assume that (3.1) holds with*

$$\frac{K^*}{K_*}\exp\left(r^* - 1\right) \le 2, \tag{3.3}$$

where $r^* = \max_{n\in N}\{r(n)\}$, $K_* = \min_{n\in N}\{K(n)\}$, and $K^* = \max_{n\in N}\{K(n)\}$. Let $\{\tilde{x}(n)\}$ be a periodic solution of (1.2). Then for every positive solution $\{x(n)\}$ of (1.2), we have

$$\lim_{n\to\infty}\left(x(n) - \tilde{x}(n)\right) = 0. \tag{3.4}$$

PROOF. If $K(n) \equiv K$(constant), since (3.3) implies that $r^* \le 1 + \ln 2 < 2$. By [5], we know that $\lim_{n\to\infty} x(n) = K$, this implies that (3.4) holds with $\tilde{x}(n) = K$.

Now we assume that $\{K(n)\}$ is not constant. Let $x(n) = \tilde{x}(n)\exp(y(n))$. Then (1.2) is transformed to

$$y(n+1) = y(n) - \frac{r(n)}{K(n)}\tilde{x}(n)(\exp(y(n)) - 1).$$

Define $V(n) = y^2(n)$. Then

$$\begin{aligned}
\Delta V(n) &= V(n+1) - V(n) \\
&= (y(n+1) - y(n))(y(n+1) + y(n)) \\
&= -\frac{r(n)}{K(n)}\tilde{x}(n)(\exp(y(n)) - 1)\left(2y(n) - \frac{r(n)}{K(n)}\tilde{x}(n)\right)(\exp(y(n)) - 1) \\
&= -\frac{r(n)}{K(n)}\tilde{x}(n)\exp(\theta y(n))\left(2 - \frac{r(n)}{K(n)}\tilde{x}(n)\exp(\theta y(n))\right)y^2(n),
\end{aligned} \tag{3.5}$$

for some $\theta \in (0,1)$. Since $\tilde{x}(n)\exp(\theta y(n))$ lies between $\tilde{x}(n)$ and $x(n)$, by Theorem 2.1 and Remark 2.2, we know that there exists a positive integer n_1 such that

$$2 - \frac{r(n)}{K(n)}\exp(\theta y(n)) \ge 2 - \frac{r^* u^*}{K_*} = 2 - \frac{K^*}{K_*}\exp\left(r^* - 1\right) \ge 0, \qquad n \ge n_1.$$

This implies that $\{V(n)\}$ is nonincreasing for $n \geq n_1$. So,

$$\lim_{n\to\infty} V(n) = v^* \in [0, \infty). \tag{3.6}$$

We claim $v^* = 0$. In fact, if $v^* > 0$, then $y(n) \geq \sqrt{v^*}$ for $n \geq n_1$. Since $\{K(n)\}$ is not constant, there exists an integer p with $0 \leq p < \omega$ such that $K(p) > K_*$, from (3.5), we have

$$\Delta V(p + n\omega) \leq -\frac{r_*}{K^*} u_* \left(2 - \frac{r^*}{K(p)} u^*\right) v^* < 0, \qquad n \geq n_1.$$

This implies that $\sum_{n=0}^{\infty} \Delta V(n)$ diverges to $-\infty$. But from (3.6), $\sum_{n=0}^{\infty} \Delta V(n) = v^* - V(0)$; this is a contradiction. Therefore, $v^* = 0$. Thus, $\lim_{n\to\infty} y(n) = 0$ and (3.4) holds. The proof is complete.

REMARK 3.1. Theorem 3.2 shows that $\{\tilde{x}(n)\}$ is the global attractor of all positive solutions of (1.2), and hence, $\{\tilde{x}(n)\}$ is the unique ω-periodic positive solution of (1.2).

REMARK 3.2. When $K(n) \equiv K$(constant), [5] has proved that if $r^* \leq 3/2$, then the solution $x(n) = K$ is a global attractor of (1.2). Since, in this case, (3.3) reduces to $r^* \leq 1 + \ln 2 = 1.69314718 > 3/2$, Theorem 3.2 actually improves the corresponding result in [6], since, in this case, (3.3) reduces to $r^* \leq 1 + \ln 2 = 1.69314718$, which is larger than 3/2.

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