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Stability of linear neutral systems with multiple delays: boundary criteria

Guang-Da Hu [a], Guang-Di Hu [b], Xingfu Zou [c,*,1]

[a] *Department of Computer Science, Memorial University of Newfoundland, St. John's, Nfld, Canada A1B 3X5*

[b] *Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, Ont., Canada M5S 3G4*

[c] *Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Nfld, Canada A1C 5S7*

Abstract

The stability of linear neutral delay–differential systems with multiple delays is investigated. The delay-dependent stability criteria are presented through the evaluation of the corresponding harmonic functions on the boundary of a certain half circular region, and such criteria are extensions of some results in literature. An example is given to illustrate the stability criteria.

Keywords: Boundary criteria; Neutral delay system; Multiple delays; Delay-dependent stability criteria

1. Introduction

Consider the asymptotic stability of linear neutral delay–differential systems with multiple delays described by

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m}(B_j x(t-\tau_j) + C_j \dot{x}(t-\tau_j)), \tag{1}$$

* Corresponding author.
E-mail address: xzou@math.mun.ca (X. Zou).

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where $x(t)$ is assumed to be a function $\mathbf{R}_+ \to \mathbf{R}^n$, A, B_j and $C_j \in \mathbf{R}^{n\times n}$ are constant matrices and τ_j ($j = 1, \ldots, m$) stand for positive constant delays.

It is well known that stability criteria for system (1) can be divided into two categories according to their dependence upon the size of delays. The criteria which do not contain information on delays are called delay-independent, whereas those carrying information on delays are viewed as delay-dependent criteria. Delay-independent stability of system (1) has been extensively studied in [3–5,10]. Delay-dependent stability is also discussed in [10]. As a special case of system (1), linear neutral delay–differential system with a single delay, is exploited in [2,6,9,11]. We have recently dealt with stability of linear discrete-delay systems of the type

$$x(t) = \sum_{j=1}^{m} C_j x(t - \tau_j) \tag{2}$$

and obtain two boundary delay-dependent stability criteria for system (2) in [7].

In the present paper, delay-dependent criteria for stability of system (1) are established. The stability criteria are expressed by evaluating a harmonic function on the boundary of some torus region in the complex plane. A necessary and sufficient condition for stability of system (1) is also given by means of *Principle of the Argument.* The criteria for delay-dependent stability of system (1) may be viewed as the complements of the existing stability criteria in [3–5,10].

An outline of the paper is as follows. In Section 2, some lemmas are cited. In Section 3, the main results, i.e. the boundary criteria for stability of system (1) are derived and an example is given to illustrate the criteria.

2. Preliminaries

Let T denote a bounded region of the complex plane. ∂T and $\overline{T}$ stand for the boundary and the closure of T, respectively. We have $\overline{T} = \partial T \cup T$. Let

$$f(s) = f(x,y) = u(x,y) + \mathrm{i}v(x,y) \tag{3}$$

be an arbitrary analytical function for $s \in \overline{T}$. Here, we adopt the notations $i^2 = -1$, $s = x + \mathrm{i}y$, $u(x,y) = \Re f(s)$, $v(x,y) = \Im f(s)$.

The following two lemmas give sufficient conditions for non-existence of zeros of $f(s) \in \overline{T}$, which only require the evaluation of harmonic functions on the boundary ∂T.

Lemma 2.1 [7]. *If for any* $(x,y) \in \partial T$ *the real part* $u(x,y)$ *in* (3) *does not vanish, then*

$$f(x,y) \neq 0 \quad \textit{for any } (x,y) \in \overline{T}.$$

Lemma 2.2 [7]. *If there exists a real constant λ satisfying*

$$u(x,y) + \lambda v(x,y) \neq 0$$

for any $(x,y) \in \partial T$, then

$$f(s) = u(x,y) + \mathrm{i}v(x,y) \neq 0 \quad \textit{for any } (x,y) \in \overline{T}.$$

The characteristic equation of system (1) reads

$$P(s) = \det[sI - A - (B(s) + sC(s))] = 0, \tag{4}$$

where

$$B(s) = \sum_{j=1}^{m} B_j \exp(-s\tau_j)$$

and

$$C(s) = \sum_{j=1}^{m} C_j \exp(-s\tau_j).$$

The following Lemma is a well-known result.

Lemma 2.3 [4]. *If $a_D = \sup\{\Re s : P(s) = 0\}$ and $a_D < 0$, then the system* (1) *is asymptotically stable.*

For a square matrix Q, $|Q|$ stands for the matrix whose entries are replaced by the modulus of the corresponding entries of Q, $\lambda_j(Q)$ and $\rho(Q)$ denote the jth eigenvalue and the spectral radius of Q, respectively. If $Q = \{q_{jk}\}$ and $V = \{v_{jk}\}$ are real matrices, we shall write $Q \geqslant V$ if $q_{jk} \geqslant v_{jk}$ holds for all pairs $\{j,k\}$.

Lemma 2.4 [8]. *Let $Q \in \mathbf{C}^{n\times n}$ and $V \in \mathbf{R}^{n\times n}$. If the inequality $|Q| \leqslant V$ holds, then the inequality $\rho(Q) \leqslant \rho(V)$ holds.*

Lemma 2.5 [8]. *Let $Q \in \mathbf{C}^{n\times n}$. If $\rho(Q) < 1$, then $(I-Q)^{-1}$ exists and*

$$(I-Q)^{-1} = I + Q + Q^2 + \cdots$$

3. Delay-dependent stability criteria

Let

$$W = \sum_{j=1}^{m} |C_j|.$$

If $\rho(W) < 1$, by Lemma 2.5, we can define two matrices X and G by

$$X = (I - W)^{-1}\left(\sum_{j=1}^{m}|C_jA| + \sum_{j=1}^{m}\left(\sum_{k=1}^{m}|C_jB_k|\right)\right)$$

and

$$G = |A| + \sum_{j=1}^{m}|B_j| + X.$$

The following theorem is one of the main results in the present paper.

Theorem 3.1. *Assume* $\rho(W) < 1$. *If* s *is a characteristic root of Eq.* (4) *and* $\Re s \geqslant 0$, *then*

$$|s| \leqslant \rho(G).$$

Proof. By the assumption that s is a characteristic root of Eq. (4) and $\Re s \geqslant 0$, we have

$$\det[sI - A - (B(s) + sC(s))] = 0$$

and

$$|C(s)| \leqslant \sum_{j=1}^{m}|C_j \exp(-s\tau_j)| \leqslant \sum_{j=1}^{m}|C_j| = W.$$

According to Lemma 2.4, the inequality $\rho(C(s)) < \rho(W) < 1$ holds, and thus $(I - C(s))^{-1}$ exists. Since $\det[I - C(s)] \neq 0$, we get

$$\det[sI - (I - C(s))^{-1}(A + B(s))] = 0.$$

This means that s is an eigenvalue of the matrix $(I - C(s))^{-1}(A + B(s))$, and therefore,

$$|s| \leqslant \rho[(I - C(s))^{-1}(A + B(s))].$$

Next, we prove the inequality

$$|[(I - C(s))^{-1}(A + B(s)]| \leqslant G. \tag{5}$$

In fact, according to Lemma 2.5, we obtain

$$\begin{aligned}
&|(I - C(s))^{-1}(A + B(s))| \\
&\quad = |[(I + C(s) + C(s)^2 + \cdots)(A + B(s))]| \\
&\quad = |[(A + B(s)) + (C(s) + C(s)^2 + \cdots)(A + B(s))]| \\
&\quad \leqslant |A| + |B(s)| + |(C(s) + C(s)^2 + \cdots)(A + B(s))| \\
&\quad = |A| + |B(s)| + |(I + C(s) + C(s)^2 + \cdots)(C(s)A + C(s)B(s))| \\
&\quad \leqslant |A| + \sum_{j=1}^{m} |B_j| + (|I| + |C(s)| + |C(s)|^2 + \cdots)(|C(s)A| + |C(s)B(s)|) \\
&\quad \leqslant |A| + \sum_{j=1}^{m} |B_j| + \left(|I| + \sum_{j=1}^{m} |C_j| \right. \\
&\qquad \left. + \left(\sum_{j=1}^{m} |C_j| \right)^2 + \cdots \right) \left(\sum_{j=1}^{m} |C_j A| + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} |C_j B_k| \right) \right) \\
&\quad = |A| + \sum_{j=1}^{m} |B_j| + \left(I - \sum_{j=1}^{m} |C_j| \right)^{-1} \left(\sum_{j=1}^{m} |C_j A| + \sum_{j=1}^{m} \left(\sum_{k=1}^{m} |C_j B_k| \right) \right) \\
&\quad = |A| + \sum_{j=1}^{m} |B_j| + X = G.
\end{aligned}$$

In view of (5) and Lemma 2.4, the proof is completed. □

Theorem 3.2. *Assume* $\rho(W) < 1$. *Then characteristic roots of Eq.* (4) *do not accumulate at* $\pm i\infty$.

Proof. By Theorem 3.1, we only need to consider those characteristic roots s with $\Re s < 0$. Assume that $\{s_n\}$ is a sequence of characteristic roots with $\Re s_n < 0$ and $\Re s_n \to 0_-$ as $\to \infty$. Let $a_n = \Re s_n,\ b_n = \max_j\{\exp(-a_n \tau_j)\}$, for $j = 1, \ldots, m$. Since $a_n \to 0_-$, and $b_n \to 1$ as $n \to \infty$. By

$$|C(s_n)| \leqslant \sum_{j=1}^{m} |C_j \exp(-s_n \tau_j)| \leqslant b_n \sum_{j=1}^{m} |C_j| = b_n W,$$

we know that for sufficiently large n, $\rho(b_n W) < 1$ and thus $\rho(C(s_n)) < 1$ by Lemma 2.4. Hence $(I - C(s_n))^{-1}$ exists when n is sufficiently large. Noting that

$$\begin{aligned}
0 &= \det[s_n I - A - (B(s_n) + s_n C(s_n))] \\
&= \det[I - C(s_n)] \det[s_n I - (I - C(s_n))^{-1}(A + B(s_n))],
\end{aligned}$$

it follows that

$$\det[s_nI - (I - C(s_n))^{-1}(A + B(s_n))] = 0.$$

This implies that s_n is an eigenvalue of the matrix $(I - C(s_n))^{-1}(A + B(s_n))$, and thus

$$|\Im s_n| \leqslant |s_n| \leqslant \rho((I - C(s_n))^{-1}(A + B(s_n))) \leqslant \rho(|(I - C(s_n))^{-1}(A + B(s_n))|). \tag{6}$$

Fix $d > 0$ such that $\rho((1 + d)W) < 1$. Then there exists an $N = N(d) > 0$, such that $b_n \leqslant 1 + d$ for $n \geqslant N$. Therefore for all $n \geqslant N$,

$$\begin{aligned} |(I - C(s_n))^{-1}(A + B(s_n))| &= |(I + C(s_n) + C(s_n)^2 + \cdots)(A + B(s_n))| \\ &\leqslant (I + |C(s_n)| + (|C(s_n)|)^2 + \cdots)(|A| + |B(s_n)|) \\ &\leqslant (I + b_nW + (b_nW)^2 + \cdots)|(|A| + |B(s_n)|) \\ &\leqslant (I + (1 + d)W + ((1 + d)W)^2 + \cdots)|(|A| \\ &\quad + (1 + d)|B(s_n)|) \\ &= (I - (1 + d)W)^{-1}\left(|A| + (1 + d)\sum_{j=1}^{m}|B_j|\right) \\ &=: F. \end{aligned} \tag{7}$$

Combining (6) and (7) with Lemma 2.4 gives

$$|\Im s_n| \leqslant |s_n| \leqslant \rho(F) \tag{8}$$

for $n \geqslant N$. This shows $\{s_n\}$ is bounded away from $\pm\infty$. The proof is completed. □

In what follows, we establish some delay-dependent stability criteria for system (1). By Theorem 3.1, we only need to restrict our attention to the bounded and closed region D defined by

$$D = \{(r, \theta) : 0 \leqslant r \leqslant \mathscr{R}, -\pi/2 \leqslant \theta \leqslant \pi/2\},$$

where $\mathscr{R} = \rho(G)$. The boundary of D is denoted by ∂D. By means of Lemma 2.3, Theorems 3.1 and 3.2, we have

Theorem 3.3. *Assume $\rho(W) < 1$. Then, system* (1) *is asymptotically stable if and only if there are no characteristic roots of* (4) *in the bounded region $\overline{D} = D$.*

Since $P(s)$ is an entire function, there can be only a finite number of zeros $P(s)$ in any bounded region. According to the above theorem and Principle of the Argument, we have the following necessary and sufficient condition for stability of system (1).

Theorem 3.4. *Assume* $\rho(W) < 1$. *Then, system* (1) *is asymptotically stable for* $\rho(W) < 1$ *if and only if*

$$P(s) \neq 0$$

for $s \in \partial D$, *and*

$$\oint_{\partial D} \frac{\dot{P}(s)}{P(s)}\,\mathrm{d}s = 0.$$

When $\rho(W) < 1$, by Lemma 2.3 and Theorem 3.3, in order to test the stability of system (1), it suffices to prove all of roots of $P(s) = 0$ are located in the left half s plane. Using Lemmas 2.1 and 2.2, the following two theorems, which give the delay-dependent stability criteria of system (1), can be derived in a straightforward manner.

Theorem 3.5. *If* $\rho(W) < 1$ *and for any* $s \in \partial D$ *the real part* $U(s)$ *of* $P(s)$ *does not vanish, then the system* (1) *is asymptotically stable.*

Theorem 3.6. *Assume that* $\rho(W) < 1$ *and for any* $s \in \partial D$ *there exists a real constant* λ *satisfying*

$$U(s) + \lambda V(s) \neq 0,$$

where $U(s)$ *and* $V(s)$ *are real and imaginary parts of* $P(s)$, *respectively, then system* (1) *is asymptotically stable.*

Remark 3.1. If $\rho(W) < 1$, the discrete-delay system (2) is asymptotically stable. (see [7]).

Remark 3.2. When $C_j = 0$ for $j = 1, \ldots, m$, the system (1) reduces to the linear delay systems with multiple delays

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} B_j x(t - \tau_j), \tag{9}$$

whose delay-dependent stability has been studied in [1]. The stability criteria obtained in this paper are also valid to system (9). Note that for (9) $W = 0$ and $\rho(W) = 0 < 1$.

Remark 3.3. In the case of system (1) with a single delay, the delay-dependent criteria are given in [6].

Remark 3.4. Theorem 3.4 gives a necessary and sufficient condition for stability of system (1) when $\rho(W) < 1$. Testing Theorem 3.5 or Theorem 3.6, which only requires an evaluation of corresponding harmonic function on the half circular region, is much easier than testing Theorem 3.4. The criteria obtained in this paper are extensions of the results in [1,6,7].

Remark 3.5. In the case of real coefficient matrices the complex function $P(s)$ is symmetric respect with to the real axis. Thus it is sufficient to consider Theorems 3.5 and 3.6 for s only on the upper half part of ∂D.

Remark 3.6. All the criteria in this paper are established under the assumption $\rho(W) < 1$. It still remains an open problem to derive stability criteria for system (1) in the case $\rho(W) \geqslant 1$.

4. An illustrative example

Consider stability of the scalar neutral differential equation

$$\dot{x}(t) = ax(t) + bx(t-1) + c\dot{x}(t-\tau), \tag{10}$$

where $a = -1$, $b = -1.1$, $c = 0.1$ and $\tau = 1$. The characteristic function is

$$P(s) = s + 1 + 1.1\exp(-s) - 0.1s\exp(-s).$$

Set $s = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta$, The real part $U(s)$ of $P(s)$ can be written as follows:

$$U(s) = r\cos\theta + 1 + 1.1\exp(-r\cos\theta)\cos(r\sin\theta) - 0.1r\exp(-r\cos\theta) \times [\cos(r\sin\theta)\cos\theta + \sin(r\sin\theta)\sin\theta].$$

We have $\mathscr{R} = \rho(G) = 2.3334$, where

$$G = |a| + |b| + \frac{|ca| + |cb|}{1 - |c|}.$$

Then

$$D = \{(r,\theta) : r \leqslant 2.3334, -\pi/2 \leqslant \theta \leqslant \pi/2\}.$$

The boundary ∂D of D consists of the two parts: the straight segment

$$\{(r,\theta) : 0 \leqslant r \leqslant 2.3334, \theta = \pm\pi/2\}$$

and the half circumference

$$\{(r,\theta) : r = 2.3334, -\pi/2 \leqslant \theta \leqslant \pi/2\}.$$

By using Matlab, we can easily test that $U(s) > 0$ on the boundary ∂D. According to Theorem 3.5, the Eq. (10) is asymptotically stable because $U(s) > 0$

on the boundary ∂D and $|c| = 0.1 < 1$. Because $|c| + \tau|b| = 1.2 > 1$, Theorem 1 in [10] cannot be applied in Eq. (10). On the other hand, the delay-independent criteria in [3,5,9,10] are not satisfied since $a + |b| > 0$. Hence the criteria of the present paper can implement those in [3,5,9,10].

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