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Traveling waves for the diffusive Nicholson's blowflies equation ☆

Joseph W.-H. So [a], Xingfu Zou [b,*]

[a] Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
[b] Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1C 5S7

Abstract

We consider traveling wave front solutions for the diffusive Nicholson's blowflies equation on the real line. The existence of such solutions is proved using the technique developed by J. Wu and X. Zou (J. Dyn. Differ. Equations 13 (3) (2001)). Some numerical simulation using the iteration formula of Wu and Zou [7] is also provided. Crown Copyright © 2001 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the reaction–diffusion equation with a discrete time delay

$$\frac{\partial N(t,x)}{\partial t} = \frac{\partial^2 N(t,x)}{\partial x^2} - \delta N(t,x) + pN(t-\tau,x)e^{-aN(t-\tau,x)}, \qquad (1)$$

where $x \in \mathbb{R}$ and $t \geqslant 0$. Such an equation had been studied in [4,5,8]. For the case when there is no spatial dependence, the corresponding delay equation was referred to as Nicholsons's blowflies equation, cf. [1] after the experiments of Nicholson [2,3] had been extensively studied. Eq. (1) can be derived based on

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* Corresponding author.
E-mail addresses: joseph.so@ualberta.ca (J.W.-H. So), xzou@math.mun.ca (X. Zou).

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first principles by making use of the spatial (this leads to the diffusion term) and age structures (this leads to the discrete delay τ) of the population. The general theory of reaction–diffusion equations with delays can be found in [6].

Assume $p/\delta > 1$. Then there are two equilibria: $N_0 = 0$ and $N_e = 1/a \ln(p/\delta)$. A *traveling wave front* is a solution of (1) of the form $u(t,x) = \phi(x+ct)$, where $c > 0$, $\phi(t)$ is monotone increasing and it satisfies

$$\begin{aligned} &c\phi'(t) = \phi''(t) - \delta\phi(t) + p\phi(t-c\tau)\mathrm{e}^{-a\phi(t-c\tau)}, \\ &\phi(-\infty) = N_0, \quad \phi(+\infty) = N_e. \end{aligned} \tag{2}$$

The main result of this paper is:

Theorem. *If* $1 < (p/\delta) \leqslant e$, *then there exists* $c^* > 0$ *such that for every* $c > c^*$ *there exists a traveling wave front for* (1) *with speed* c.

In Section 2, we will prove this theorem by applying [7, Theorem 3.6], which is for delayed reaction–diffusion *systems*. Since we are considering a scalar equation here, for the convenience of reference and for simplicity, we only need a scalar version of this theorem, which is stated below.

Consider the delayed reaction–diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f(u_t(x)), \tag{3}$$

where $t \geqslant 0,\ x \in \mathbb{R},\ u \in \mathbb{R}$ with $D > 0$, and $f:\ C([-\tau,0],\mathbb{R}) \to \mathbb{R}$ is continuous and $u_t(x)$ is an element in $C([-\tau,0],\mathbb{R})$ parameterized by $x \in \mathbb{R}$ and given by

$$u_t(x)(s) = u(t+s,x), \quad s \in [-\tau,0], \quad t \geqslant 0, \quad x \in \mathbb{R}.$$

Looking for traveling wave solutions of the form $u(t,x) = \phi(x+ct)$ leads to a second-order functional differential equation

$$D\phi''(t) - c\phi'(t) + f_c(\phi_t) = 0, \quad t \in \mathbb{R}, \tag{4}$$

where $f_c:\ X_c = C([-c\tau,0],\mathbb{R}) \to \mathbb{R}$ is defined by

$$f_c(\psi) = f(\psi^c), \quad \psi^c(s) = \psi(cs), \quad s \in [-\tau,0].$$

We assume

(A1) There exists $K > 0$ such that $f_c(\hat{0}) = f_c(\hat{K}) = 0$ and $f_c(\hat{u}) \neq 0$ for $u \in (0,K)$, where $\hat{u}$ denotes the constant function taking the value u on $[-c\tau,0]$.

(A2) (Quasi-monotonicity). There exists $\beta \geqslant 0$ such that

$$f_c(\phi) - f_c(\psi) + \beta[\phi(0) - \psi(0)] \geqslant 0$$

for $\phi,\psi \in X_c$ with $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant K,\ s \in [-c\tau,0]$.

If for some $c > 0$, (4) has a monotone solution ϕ satisfying $\lim_{t\to-\infty}\phi(t) = 0$ and $\lim_{t\to\infty}\phi(t) = K$, then $u(t,x) = \phi(x+ct)$ is called a *traveling wave front* of (3) with speed c.

Next we define the profile set for traveling wave fronts of (3) by

$$\Gamma = \phi \in C(\mathbb{R};\mathbb{R}^n),$$

(i) ϕ is nondecreasing in $\mathbb{R}$,
(ii) $\lim_{t\to-\infty}\phi(t) = 0, \quad \lim_{t\to\infty}\phi(t) = K.$

A function $\phi \in C(\mathbb{R},\mathbb{R})$ is called an *upper* (resp., *lower*) *solution* of (4) if it is differentiable almost everywhere (a.e.) and satisfies

$$c\phi' \geqslant D\phi''(t) + f_c(\phi_t), \quad \text{a.e. in } \mathbb{R},$$

resp.,

$$c\phi' \leqslant D\phi''(t) + f_c(\phi_t), \quad \text{a.e. in } \mathbb{R}.$$

Now we are in the position to state a scalar version of [7, Theorem 3.6].

Theorem A. *Assume that* (A1)–(A2) *hold. Suppose that* (4) *has an upper solution* $\overline{\phi}$ *in* Γ *and a lower solution* $\underline{\phi}$ (*which is not necessarily in* Γ) *with* $0 \leqslant \underline{\phi}(t) \leqslant \overline{\phi}(t) \leqslant K$ *and* $\underline{\phi}(t) \not\equiv 0$ *in* $\mathbb{R}$, *then* (3) *has a traveling wave front*.

2. Proof of existence of traveling waves

Define the functional f_c by $f_c(\phi) = -\delta\phi(0) + p\phi(-c\tau)\mathrm{e}^{-a\phi(-c\tau)}$. Then

Claim 2.1. *If* $(p/\delta) > 1$, *then* $f_c(\hat{N}_0) = f_c(\hat{N}_e) = 0$, *and* $f_c(\hat{K}) \neq 0$ *for any* $K \in (N_0, N_e)$, *where* $\hat{K}$ *denotes the constant function taking the value* K *on* $[-c\tau, 0]$.

Claim 2.2. *If* $1 < (p/\delta) \leqslant e$, *then* f_c *satisfies the following quasi-monotonicity condition*:

For all $\beta \geqslant \delta$, *we have*

$$f_c(\phi_1) - f_c(\phi_2) + \beta[\phi_1(0) - \phi_2(0)] \geqslant 0$$

for all $\phi_1, \phi_2 \in C([-c\tau, 0], \mathbb{R})$ *with* $N_0 \leqslant \phi_2(s) \leqslant \phi_1(s) \leqslant N_e$ *for all* $s \in [-c\tau, 0]$.

Proof.

$$f_c(\phi_1) - f_c(\phi_2) = -\delta[\phi_1(0) - \phi_2(0)] + p\big[\phi_1(-c\tau)\mathrm{e}^{-a\phi_1(-c\tau)} - \phi_2(-c\tau)\mathrm{e}^{-a\phi_1(-c\tau)}\big].$$

Consider the function $h(y) = y\mathrm{e}^{-ay}$. Then

$$h'(y) = \mathrm{e}^{-ay}[1 - ay] = \begin{cases} > 0 & \text{for } y < 1/a, \\ > 0 & \text{for } y > 1/a. \end{cases}$$

So, $h(y)$ is increasing on $[0, 1/a]$. Now, since $p/\delta \leqslant e$,

$$0 \leqslant \phi_2(s) \leqslant \phi_1(s) \leqslant N_e = \frac{1}{a} \ln \frac{p}{\delta} \leqslant \frac{1}{a}.$$

Thus,

$$\phi_1(-c\tau)\mathrm{e}^{-a\phi(-c\tau)} - \phi_2(-c\tau)\mathrm{e}^{-a\phi(-c\tau)} > 0$$

and therefore

$$f_c(\phi) - f_c(\phi_2) + \delta[\phi_1(0) - \phi_2(0)] \geqslant 0. \quad \square$$

Define the profile set

$$\Gamma = \phi \in C(\mathbb{R}, \mathbb{R}),$$

(i) ϕ is non-decreasing in $\mathbb{R}$,
(ii) $\lim_{t \to -\infty} \phi(t) = N_0$ and $\lim_{t \to \infty} = N_e$.

Definition. ϕ is called an *upper* (resp., *lower*) *solution* of (2) if $\phi \in C(\mathbb{R}, \mathbb{R})$ is differentiable a.e. and it satisfies

$$c\phi'(t) \geqslant \phi''(t) - \delta\phi(t) + p\phi(t - c\tau)\mathrm{e}^{-a\phi(t-c\tau)}, \quad \text{a.e. in } \mathbb{R},$$

resp.,

$$c\phi'(t) \leqslant \phi''(t) - \delta\phi(t) + p\phi(t - c\tau)\mathrm{e}^{-a\phi(t-c\tau)}, \quad \text{a.e. in } \mathbb{R}.$$

Define

$$\Delta_c(\lambda) = \lambda^2 - c\lambda - \delta + p\mathrm{e}^{-\lambda c\tau} = p\mathrm{e}^{-\lambda c\tau} - [c\lambda + \delta - \lambda^2].$$

Claim 2.3. *There exists $c^* > 0$ such that for $c > c^*$, $\Delta_c(\lambda) = 0$ has two positive real roots, $0 < \lambda_1 < \lambda_2$ and*

$$\Delta_c(\lambda) = \begin{cases} > 0 & \text{for } \lambda > \lambda_2, \\ < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2), \\ > 0 & \text{for } \lambda < \lambda_1. \end{cases}$$

Proof. The curve $\lambda \mapsto p\mathrm{e}^{-\lambda c\tau}$ is an exponentially decaying curve which is concave up, whereas the curve $\lambda \mapsto c\lambda + \delta - \lambda^2$ is a parabolic curve which is concave down. The result follows from graphing these two curves and making use of the facts $\Delta_c(0) > 0$, $\Delta_c(\infty) = \infty$, $(\partial^2 \Delta_c(\lambda)/\partial\lambda^2) > 0$ and $(\partial\Delta_c(\lambda)/\partial c) < 0$, for $\lambda > 0$. $\square$

Remark. Clearly, $c^* < 2\sqrt{p-\delta}$. Also c^* depends on τ. Indeed, c^* is obtained by solving for c and λ the following equations:

$$pe^{-\lambda c\tau} = c\lambda + \delta - \lambda^2, \qquad pc\tau e^{-\lambda c\tau} = 2\lambda - c.$$

Claim 2.4. *Assume $c > c^*$. Then $\overline{\phi}(t) = \min\{N_e, e^{\lambda_1 t}\}$ is an upper solution of* (2) *and $\phi \in \Gamma$.*

Proof. $\overline{\phi} \in \Gamma$ is obvious. Let t_0 be such that $e^{\lambda_1 t_0} = N_e$.
(i) For $t \geqslant t_0$, $\overline{\phi}(t) = N_e$, $\overline{\phi}'(t) = 0 = \overline{\phi}''(t)$, $\overline{\phi}(t - c\tau) \leqslant N_e$. Thus,

$$\begin{aligned}&\overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)} \\ &\quad = -\delta N_e + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)} \leqslant -\delta N_e + pN_e e^{-aN_e} \\ &\quad = N_e[-\delta + pe^{-aN_e}] = 0,\end{aligned}$$

since ye^{-ay} is increasing for $y < 1/a$ and $\overline{\phi}(t-c\tau) \leqslant N_e = 1/a\ln(p/\delta) \leqslant (1/a)$.

(ii) For $t < t_0$, $\overline{\phi}(t) = e^{\lambda_1 t}$ and $\overline{\phi}(t-c\tau) = e^{\lambda_1(t-c\tau)}$. Thus,

$$\begin{aligned}&\overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) + p\overline{\phi}(t-c\tau)e^{-a\overline{\phi}(t-c\tau)} \\ &\quad \leqslant \overline{\phi}''(t) - c\overline{\phi}'(t) - \delta\overline{\phi}(t) + p\overline{\phi}(t-c\tau) \\ &\quad = e^{\lambda_1 t}[\lambda_1^2 - c\lambda_1 - \delta + pe^{-\lambda_1 c\tau}] = e^{\lambda_1 t}\Delta_c(\lambda_1) = 0.\end{aligned}$$

Therefore, $\overline{\phi}$ is an upper solution of (2). □

Now let $c > c^*$ and $0 < \lambda_1 < \lambda_2$ be as in Claim 2.3 Choose $\epsilon > 0$ such that $\epsilon < \lambda_1 < \lambda_1 + \epsilon < \lambda_2$. Define $\underline{\phi}(t) = \max\{0, (1 - Me^{\epsilon t})e^{\lambda_1 t}\}$, where the constant $M > 1$ is to be determined.

Claim 2.5. *For sufficiently large M, $\underline{\phi}(t)$ is a lower solution of* (2).

Proof. Let $t_1 = (1/\epsilon)\ln(1/M)$. Then $t_1 < 0$ for $M > 1$ and

$$\underline{\phi}(t) = \begin{cases} 0 & \text{for } t > t_1, \\ (1 - Me^{\epsilon t})e^{\lambda_1 t} & \text{for } t < t_1. \end{cases}$$

(i) For $t > t_1$, $\underline{\phi}(t) = 0$ and $\underline{\phi}(t - c\tau) = 0$. So,

$$\underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) + p\underline{\phi}(t-c\tau)e^{-a\underline{\phi}(t-c\tau)} = p\underline{\phi}(t-c\tau)e^{-a\underline{\phi}(t-c\tau)} = 0.$$

(ii) For $t < t_1$,

$$\begin{aligned}&\underline{\phi}(t) = [1 - Me^{\epsilon t}]e^{\lambda_1 t}, \\ &\underline{\phi}'(t) = [\lambda_1 - M(\epsilon + \lambda_1]e^{\epsilon t})e^{\lambda_1 t},\end{aligned}$$

$$\underline{\phi}''(t) = [\lambda_1^2 - M(\lambda_1+\epsilon)^2 e^{\epsilon t}]e^{\lambda_1 t} \quad \text{and}$$
$$\underline{\phi}(t-c\tau) = [1 - Me^{\epsilon(t-c\tau)}]e^{\lambda_1(t-c\tau)}.$$

Note that $e^y \geqslant 1+y$ for all $y \in \mathbb{R}$. Therefore

$$\begin{aligned}
&\underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) + p\underline{\phi}(t-c\tau)e^{-a\underline{\phi}(t-c\tau)} \\
&\quad \geqslant \underline{\phi}''(t) - c\underline{\phi}'(t) - \delta\underline{\phi}(t) + p\underline{\phi}(t-c\tau)[1 - a\underline{\phi}(t-c\tau)] \\
&\quad = [\lambda_1^2 - M(\lambda_1+\epsilon)^2 e^{\epsilon t}]e^{\lambda_1 t} - c[\lambda_1 - M(\lambda_1+\epsilon)e^{\epsilon t}]e^{\lambda_1 t} - \delta[1 - Me^{\epsilon t}]e^{\lambda_1 t} \\
&\qquad + p[1 - Me^{\epsilon(t-c\tau)}]e^{\lambda_1(t-c\tau)}[1 - a(1 - Me^{\epsilon(t-c\tau)})e^{\lambda_1(t-c\tau)}] \\
&\quad = e^{\lambda_1 t}\Big\{[\lambda_1^2 - c\lambda_1 - \delta] - Me^{\epsilon t}[(\lambda_1+\epsilon)^2 - c(\lambda_1+\epsilon) - \delta] \\
&\qquad\qquad + pe^{-\lambda_1 c\tau}[1 - Me^{\epsilon t}e^{-\epsilon c\tau}] - pae^{-2\lambda_1 c\tau}e^{\lambda_1 t}[1 - Me^{\epsilon(t-c\tau)}]^2\Big\} \\
&\quad = e^{\lambda_1 t}\Big\{[\lambda_1^2 - c\lambda_1 - \delta + pe^{-\lambda_1 c\tau}] - Me^{\epsilon t}[(\lambda_1+\epsilon)^2 - c(\lambda_1+\epsilon) - \delta \\
&\qquad\qquad + pe^{-(\lambda_1+\epsilon)c\tau}] - pae^{-2\lambda_1 c\tau}e^{\lambda_1 t}[1 - Me^{\epsilon(t-c\tau)}]^2\Big\} \\
&\quad = e^{\lambda_1 t}\Big\{\Delta_c(\lambda_1) - Me^{\epsilon t}\Delta_c(\lambda_1+\epsilon) - pae^{-2\lambda_1 c\tau}e^{\lambda_1 t}[1 - Me^{\epsilon(t-c\tau)}]^2\Big\} \\
&\quad = e^{\lambda_1 t}\Big\{-Me^{\epsilon t}\Delta_c(\lambda_1+\epsilon) - pae^{-2\lambda_1 c\tau}e^{\lambda_1 t}[1 - Me^{\epsilon(t-c\tau)}]^2\Big\} \\
&\quad \geqslant e^{\lambda_1 t}\Big\{-Me^{\epsilon t}\Delta_c(\lambda_1+\epsilon) - pae^{-2\lambda_1 c\tau}e^{\epsilon t}[1 - Me^{\epsilon(t-c\tau)}]^2\Big\} \\
&\quad = e^{(\lambda_1+\epsilon)t}\Big\{-M\Delta_c(\lambda_1+\epsilon) - pae^{-2\lambda_2 c\tau}[1 - M^{\epsilon(t-c\tau)}]^2\Big\}
\end{aligned}$$

since $t < t_1 < 0$ and $\epsilon < \lambda_1$.

Now

$$1 - Me^{\epsilon(t-c\tau)} = 1 - Me^{\epsilon t}e^{-\epsilon c\tau} < 1 + Me^{\epsilon t_1}e^{-\epsilon c\tau} = 1 + e^{-\epsilon c\tau}$$

and

$$1 - Me^{\epsilon(t-c\tau)} > 1 - Me^{\epsilon t_1}e^{-\epsilon c\tau} = 1 - e^{-\epsilon c\tau} > 0.$$

Therefore, $[1 - Me^{\epsilon(t-c\tau)}]^2 < (1 + e^{-\epsilon c\tau})^2$ and thus

$$\begin{aligned}
&\underline{\phi}''(t) - c\underline{\phi}(t) - \delta\underline{\phi} + p\underline{\phi}(t-c\tau)e^{-a\underline{\phi}(t-c\tau)} \\
&\quad \geqslant e^{(\lambda_1+\epsilon)t}\Big\{-M\Delta_c(\lambda_1+\epsilon) - pae^{-2\lambda_1 c\tau}(1 + e^{-\epsilon c\tau})^2\Big\} \\
&\quad \geqslant e^{(\lambda_1+\epsilon)t}[-\Delta_c(\lambda_1+\epsilon)]\left\{M - \frac{pae^{-2\lambda_1 c\tau}(1 + e^{-\epsilon c\tau})^2}{-\Delta_c(\lambda_1+\epsilon)}\right\}.
\end{aligned}$$

Due to the choice of λ_1 and ϵ, $\Delta_c(\lambda_1+\epsilon)<0$. Thus the right hand side of the above inequality is non–negative if we choose

$$M > \frac{pa\mathrm{e}^{-2\lambda_1 c\tau}(1+\mathrm{e}^{-\epsilon c\tau})^2}{-\Delta_c(\lambda_1+\epsilon)}.$$

Thus $\underline{\phi}$ is a lower solution of (2).

Clearly, $N_0 \leqslant \underline{\phi} \leqslant \overline{\phi} \leqslant N_e$. By Theorem A in Section 1, we complete the proof of the main theorem.

3. Numerical simulations

By the results in Section 3 in [7], the wave profile ϕ can be obtained by the convergence of the following iteration:

$$\phi_m(t) = \frac{1}{\lambda_2-\lambda_1}\left[\int_{-\infty}^{t} \mathrm{e}^{\lambda_1(t-s)}H(\phi_{m-1})(s)\,\mathrm{d}s + \int_{t}^{\infty} \mathrm{e}^{\lambda_2(t-s)}H(\phi_{m-1})(s)\,\mathrm{d}s\right], \quad \phi_0(t)=\overline{\phi}(t),$$

where $t\in\mathbb{R},\ m=1,2,\ldots,$ and $H:\ C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is defined by

$$H(\phi)(t) = p\phi(t-c\tau)\mathrm{e}^{-a\phi(t-c\tau)}, \quad \phi\in C(\mathbb{R},\mathbb{R}), \quad t\in\mathbb{R}.$$

Now we take some particular values for the parameters values, $p=2$, $\delta=1$, $a=1$, $\tau=1$ and $c=2$. Then $N_e=0.6931471806$, $\lambda_1=0.1954954948$, $\lambda_2=2.408480501$ and $t_0=-1.874789600$. The graphs of $\overline{\phi}$ and ϕ_1 (i.e., after one iteration) are shown in Figs. 1 and 2, respectively.

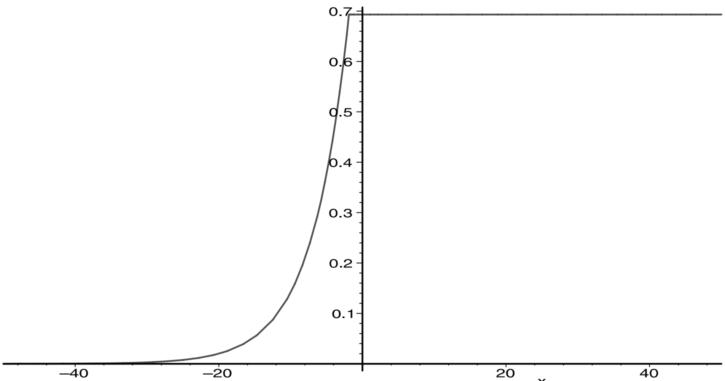

Fig. 1. Graph of $\overline{\phi}$.

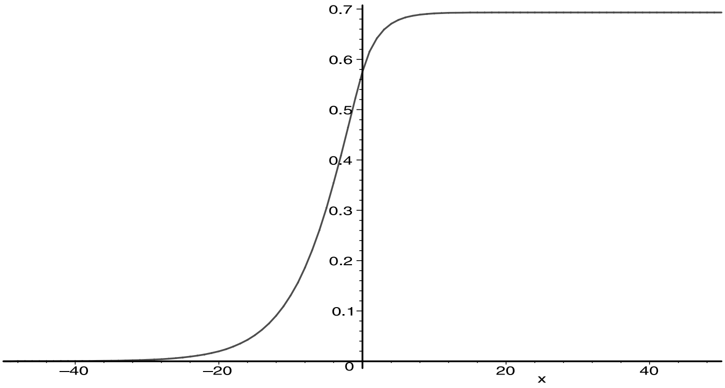

Fig. 2. Graph of ϕ_1.

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