Acta Mathematicae Applicatae Sinica, English Series
Vol. 22, No. 2 (2006) 243–256

Acta Mathematicae Applicatae Sinica, English Series

Travelling Wave Solutions in Delayed Reaction Diffusion Systems with Partial Monotonicity

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Abstract This paper deals with the existence of travelling wave fronts of delayed reaction diffusion systems with partial quasi-monotonicity. We propose a concept of "desirable pair of upper-lower solutions", through which a subset can be constructed. We then apply the Schauder's fixed point theorem to some appropriate operator in this subset to obtain the existence of the travelling wave fronts.

Keywords Travelling wave fronts, upper-lower solution, partial monotonicity, Schauder's fixed point theorem
2000 MR Subject Classification 34K10, 35B20, 35K57

1 Introduction

It is known that in certain situations traveing wave solutions play a key role in characterizing the behavior of general solutions of reaction diffusion systems with given initial conditions. For reaction diffusion systems without time delay, existence of travelling wave solutions have been extensively and intensively studied, see Gardner[3], Fife[2], Britton[1], Murray[9], Volpert et al.[17] and so on. However, little is known on travelling wave solutions for reaction diffusion systems with time delay. Schaaf[11], in a pioneer work, systematically studied a *scalar* reaction diffusion equation with a *single* discrete delay by using the phase-plane technique, the maximum principle for parabolic functional differential equations and general theory for ordinary functional differential equations. Recently Schaaf's work has drawn much attention and initiated the study of travelling wave solutions to delayed reaction diffusion systems. Zou and Wu[21] considered *systems* with quasimonotonicity and a single delay, and established existence of travelling wavefronts by first truncating the unbounded domain and then passing to infinity. Smith and Zhao[14] also considered a delayed reaction diffusion equation with quasi-monotonicity and studied the global asymptotic stability, Lyapunov stability, and uniqueness of travelling wave solutions by the elementary super- and subsolution comparison and squeezing methods. By establishing some spectral properties for the variation equations for delayed systems, Huang[7], obtained monotone heteraclinic orbits, corresponding to travelling wave fronts in the setting of delayed reaction diffusion systems.

Recently, Wu and Zou[19] further studied general reaction diffusion systems with general finite delays, where both quasimonotone and a type of weakened quasimonotone (we will call it

Manuscript received June 5, 2003. Revised July 21, 2005.
Supported by the National Natural Science Foundation of China (No.19971032), and the second author is supported by Natural Science Foundation of Canada and by a Petro Canada Young Innovator Award.

exponetially quasimonotone) reaction terms were considered. More precisely, Wu and Zou[19] considered systems of the form

$$\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t) + f(u_t(x)), \qquad t>0, \quad x \in R. \tag{1.1}$$

Here, $u \in R^n, D = diag(d_1,\ldots,d_n)$ with $d_i > 0, i = 1,\cdots,n;$ $f : C([-\tau,0]), R^n) \to R^n$ is continuous, $f(0) = 0 = f(K)$ and $f(u) \neq 0$ for $u \in (0,K)$. $u_t(x) \in C([-\tau,0], R^n)$, parameterized by $x \in R$, is defined by $u_t(x)(\theta) = u(t+\theta, x)$ for any $\theta \in [-\tau, 0]$. In the sequel, when there is no danger confusion, we will suppress the x and write $u_t(x) = u_t$. By quasi-monotonicity of (1.1), we mean the following

(QM) There exists a matrix $\beta = diag(\beta_1,\cdots,\beta_n)$ with $\beta_i \geq 0, i = 1,\cdots,n$, such that

$$f(\phi(x)) - f(\psi(x)) + \beta[\phi(x)(0) - \psi(x)(0)] \geq 0$$

for $\phi(x), \psi(x) \in X = C([-\tau,0]; R^n)$ with $0 \leq \psi(x)(s) \leq \phi(x)(s) \leq K$ for $s \in [-\tau,0]$.

Here and in the sequel, an inequality in R^n is in the sense of standard ordering in R^n, that is, in the componentwise sense. If $\beta = 0$, (QM) reduces to the usual monotonocity. By exponentially quasi-monotonicity for (1.1), we mean the following weakened quasi-monotonicity

(QM*) There exists a matrix β =diag $(\beta_1,\cdots,\beta_n)$ with $\beta_i \geq 0,\ i = 1,\cdots,n$, such that

$$f(\phi(x)) - f(\psi(x)) + \beta[\phi(x)(0) - \psi(x)(0)] \geq 0$$

for $\phi(x), \psi(x) \in X = C([-\tau,0]; R^n)$ with (i) $0 \leq \psi(x)(s) \leq \phi(x)(s) \leq K$ for $s \in [-\tau,0]$; and (ii) $e^{\beta s}[\phi(x)(s) - \psi(x)(s)]$ non-decreasing in $s \in [-\tau,0]$.

The approach in Wu and Zou[19] is a combination of the upper-lower solutions and a monotone iteration scheme for functions defined on the whole real line (thus, no truncation is needed), and therefore, has the advantage of numerical approximations. In applying the main results in Wu and Zou[19] to particular models, one faces two major chanllenges: (A) verifying (QM) or (QM*); (B) under (QM) or (QM*), constructing the required pair of upper-lower solutions starting with which the iteration generates a monotone sequence that converges to the profile of the travelling wave front. There have been some successes in these two aspects, reported in the application section of Wu and Zou[19], as well as in the recent work of Gourley[4], Huang and Zou[5], So, Wu and Zou[15] and So and Zou[16]. In the mean time, there have been more *unsuccesful* attempts for many other model systems.

Addressing (B), Ma[8], and Huang and Zou[6] employ the Schauder's fixed point theorem to relax the requirements for the upper-lower solutions, but with the cost of sacrificing the monotonicity of the iteration sequence. For (A), it is quite common that the reaction terms in model systems arising from a practical problem may not satisfy **(QM)** or **(QM*)**. One can find many such systems, and an immediate yet simple one is the following Lotka-Volterra system of competition-cooperation type

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\dfrac{\partial^2 u(x,t)}{\partial x^2} + r_1 u(x,t)[1 - a_1 u(x,t-\tau_1) - b_1 v(x,t)], \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\dfrac{\partial^2 v(x,t)}{\partial x^2} + r_2 v(x,t)[1 + a_2 u(x,t) - b_2 v(x,t-\tau_2)]. \end{cases} \tag{1.2}$$

Therefore, it is of both theoretic and practical importance and interest to find ways to establish the existence of travelling wave fronts for delayed reaction diffusion systems that do not satisfy (QM) and (QM*), and this constitutes the purpose of this paper.

In this paper we will consider systems with "partial quasi-monotonicity" (PQM) or weakened "partially exponential quasi-monotonicity" (PQM*) in the sense to be specified later. In order for the mathematical ideas not to be obscured by the complexity of a system, we only focus on delayed systems of *two* equations of the form

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = d_1\dfrac{\partial^2}{\partial x^2}u(x,t) + f_1\big(u_t(x), v_t(x)\big), \\ \dfrac{\partial}{\partial t}v(x,t) = d_2\dfrac{\partial^2}{\partial x^2}v(x,t) + f_2\big(u_t(x), v_t(x)\big). \end{cases} \tag{1.3}$$

The approach can be extended to systems of *more than two* equations without essential difficulties. Our new *partial quasi-monotonicity* conditions are motivated by (1.2) and many other systems such as diffusive competition-cooperation system, while our approach to tackle such systems is a combination of the cross iteration method which has been used to study the initial and boundary value problems for reaction diffusion systems without delay (see, e.g., Pao[10] or Ye and Li[18] and the references therein), and the Schauder's fixed point theorem technique employed in Ma[8], and Huang and Zou[6].

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. The main theorems on the existence of travelling wavefronts for System (1.3) satisfying partial quasi-monotonicity (PQM) or (PQM*) are presented in Sections 3 and 4.

2 Preliminaries

Throughout this paper, we adopt the usual notations for the standard ordering in R^2. Thus, for $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$, we denote $u \le v$ if $u_i \le v_i,\ i = 1, 2$; $u < v$ if $u \le v$ but $u \ne v$; and $u \ll v$ if $u \le v$ but $u_i \ne v_i, i = 1, 2$. If $u \le v$, we also denote $(u, v] = \{w \in R^2 : u < w \le v\}$, $[u, v) = \{w \in R^2 : u \le w < v\}$, and $[u, v] = \{w \in R^2 : u \le w \le v\}$. We use $|\cdot|$ to denote the Euclidean norm in R^2 and $||\cdot||$ to denote the supremum norm in $C([-\tau, 0], R^2)$.

A travelling wave solution of (1.3) is a solution $(u(x,t), v(x,t))$ with the special form $u(x,t) = \phi(x + ct)$, $v(x,t) = \psi(x + ct)$ where $\phi, \psi \in C^2(R,R)$, and $c > 0$ is a positive constant accounting for the wave speed. Substituting $u(x,t) = \phi(x + ct), v(x,t) = \psi(x + ct)$ and denoting the travelling wave coordinate $x + ct$ still by t, we obtain the corresponding wave equations

$$\begin{cases} d_1\phi''(t) - c\phi'(t) + f_{1c}(\phi_t, \psi_t) = 0, \\ d_2\psi''(t) - c\psi'(t) + f_{2c}(\phi_t, \psi_t) = 0 \end{cases} \tag{2.1}$$

where $f_{ic}(\phi_s, \psi_s) : X_{c\tau} = C([-c\tau, 0], R^2) \to R$ is defined by

$$f_{ic}(\phi, \psi) = f_i(\phi^c, \psi^c), \quad \phi^c(s) = \phi(cs), \qquad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0], \qquad i = 1, 2. \tag{2.2}$$

Here (ϕ, ψ) is called a profile of the travelling wave solution, and (2.2) is called the corresponding wave equation for (2.1). In this work, we are only interested in travelling wave fronts which are travelling wave solutions with the profile satisfying the following asymptotic boundary conditions:

$$\begin{aligned} &\lim_{t\to-\infty}\phi(t) = \phi_- \qquad \lim_{t\to-\infty}\psi(t) = \psi_-, \\ &\lim_{t\to+\infty}\phi(t) = \phi_+ \qquad \lim_{t\to+\infty}\psi(t) = \psi_+. \end{aligned} \tag{2.3}$$

Without loss of generality, we assume that $\phi_- = 0,\ \psi_- = 0$ and $\phi_+ = k_1,\ \psi_+ = k_2$, under which (2.3) reads

$$\begin{aligned} &\lim_{t\to-\infty}\phi(t) = 0, \qquad \lim_{t\to-\infty}\psi(t) = 0, \\ &\lim_{t\to+\infty}\phi(t) = k_1, \qquad \lim_{t\to+\infty}\psi(t) = k_2. \end{aligned} \tag{2.4}$$

Corresponding to (2.4), we make the following hypotheses:

(A1) $f(\widetilde{0}) = f(\widetilde{K}) = 0$ with $0 < K = (k_1, k_2)$, where by $\widetilde{u}$ we mean a constant function from $[-\tau, 0]$ to R^2 taking the value u for all $t \in [-\tau, 0]$.

(A2) $f_2(\phi, \psi) = \psi(0)[h(\psi) + a\phi(0)]$ where the functional $h(\phi)$ is continuous and $a > 0$.

(A3) There exist two positive constants $L_1 > 0,\ L_2 > 0$ such that

$$|f_1(\phi_1, \psi_1) - f_1(\phi_2,\ \psi_2)| \le L_1||\Phi - \Psi||,$$
$$|f_2(\phi_1, \psi_1) - f_2(\phi_2,\ \psi_2)| \le L_2||\Phi - \Psi||$$

for $\Phi = (\phi_1, \psi_1), \Psi = (\phi_2,\ \psi_2) \in C([-\tau, 0], R)$ with $0 \le \Phi(s), \Psi(s) \le K,\ s \in [-\tau, 0],\ i = 1,\ 2.$

We point out that while (A1) and (A3) are standard requirements for travelling wave fronts, (A2) is motivated by System (1.2) and various other models with Gause competitive type specices interactions.

In the next section, we will apply the Schauder fixed point theorem, which requires continuity of the operator under consideration. For this purpose, we need to introduce a topology in $C(R, R^2)$. Let $\mu > 0$ and equipped $C(R, R^2)$ with the exponential decay norm defined by

$$|\Phi|_\mu = \sup_{t \in R} e^{-\mu|t|}|\Phi(t)|_{R^2}.$$

Denote also

$$B_\mu(R, R^2) = \{\Phi \in C(R, R^2) :\ |\Phi|_\mu < \infty\}.$$

Then it is easy to show that $(B_\mu(R, R^2), |\cdot|_\mu)$ is a Banach space.

In order to obtain a subset of $C(R, R^2)$ in which the Schauder's fixed point theorem can be applied, we introduce the concept of *disirable pair of upper-lower solutions* for (2.1).

Definition 2.1. *A pair of continuous function $\overline{\rho}(t) = (\overline{\phi}(t), \overline{\psi}(t))$ and $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\psi}(t))$ for $t \in R$ is called a desirable pair of upper-lower solutions of (2.1) if $\overline{\rho}'$, $\overline{\rho}''$, $\underline{\rho}'$ and $\underline{\rho}''$ exist almost everywhere (a.e.) in R and they are essentially bounded on R, and there hold*

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + f_{1c}(\overline{\phi}_t, \underline{\psi}_t) \le 0, \quad a.e. \quad in \quad R. \tag{2.5}$$
$$d_2\overline{\psi}''(t) - c\overline{\psi}'(t) + f_{2c}(\overline{\phi}_t, \overline{\psi}_t) \le 0, \quad a.e. \quad in \quad R. \tag{2.6}$$

and

$$d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + f_{1c}(\underline{\phi}_t, \overline{\psi}_t) \ge 0, \quad a.e. \quad in \quad R. \tag{2.7}$$
$$d_2\underline{\psi}''(t) - c\underline{\psi}'(t) + f_{2c}(\underline{\phi}_t, \underline{\psi}_t) \ge 0, \quad a.e. \quad in \quad R. \tag{2.8}$$

Note that, unlike the standard upper solutions and lower solutions defined in [19], f_{1c} is evaluated in a cross iteration scheme given in (2.5) and (2.7), which possess some suitable monotonicity (see [18]).

3 Partially Quasimonotone Case

Although many model systems do not satisfy (QM) or QM*, they may be quasi-monotone with respect to some particular component(s). System (1.2) provide a prototype of such systems, by which we are motivated to propose the following partial quasi-monotonicity condition.

(PQM) There exist two positive constants $\beta_1 > 0,\ \beta_2 > 0$ such that

$$\begin{aligned}
&f_{1c}(\phi_1, \psi_1) - f_{1c}(\phi_2, \psi_1) + \beta_1[\phi_1(0) - \phi_2(0)] \geq 0,\\
&f_{1c}(\phi_1, \psi_1) - f_{1c}(\phi_1, \psi_2) \leq 0,\\
&f_{2c}(\phi_1, \psi_1) - f_{2c}(\phi_2, \psi_2) + \beta_2[\psi_1(0) - \psi_2(0)] \geq 0.
\end{aligned}$$

where $\phi_1, \phi_2, \psi_1, \psi_2 \in C([-\tau, 0], R)$ with $0 \leq \phi_2(s) \leq \phi_1(s) \leq k_1,\ 0 \leq \psi_2(s) \leq \psi_1(s) \leq k_2,\ s \in [-\tau, 0]$.

Note that for convenience, (PQM) (also (PQM*) in Section 4 is expressed in terms of $f_c = (f_{1c}, f_{2c})$ in $C([-\tau, 0], R)$, but this may be stated equivalently as $f = (f_1, f_2)$ in $C([-\tau, 0], R)$, as that in (QM) and (QM*) in Section 1.

In this section, we explore the existence of solutions of (2.1)–(2.4) with $f_{1c}(\phi_t, \psi_t)$ and $f_{1c}(\phi_t, \psi_t))$ satisfying (PQM). In what follows, we assume that a desired pair of upper-lower solutions $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ of (2.1) are given so that

(P1) $(0, 0) \leq (\underline{\phi}(t), \underline{\psi}(t)) \leq (\overline{\phi}(t), \overline{\psi}(t)) \leq (k_1, k_2), \quad t \in R.$

(P2) $\lim\limits_{t \to -\infty} (\overline{\phi}(t), \overline{\psi}(t)) = (0, 0), \quad \lim\limits_{t \to \infty} (\overline{\phi}(t), \overline{\psi}(t)) = (k_1, k_2),$

(P3) $\sup\limits_{s \leq t} (\underline{\phi}(s), \underline{\psi}(s)) \leq (\overline{\phi}(t), \overline{\psi}(t))$ for all $t \in R$.

We point out that if either $(\overline{\phi}(t), \overline{\psi}(t))$ or $(\underline{\phi}(s), \underline{\psi}(s))$ is non-decreasing, then (P3) is implied by (P1).

For the constants $\beta_1 > 0$ and $\beta_2 > 0$ in (PQM), define $H :\ C(R, R^2) \to C(R, R^2)$ by

$$H_1(\phi,\ \psi)(t) = f_{1c}(\phi_t, \psi_t) + \beta_1 \phi(t), \qquad \phi, \psi \in C(R, R), \tag{3.1}$$

$$H_2(\phi,\ \psi)(t) = f_{2c}(\phi_t, \psi_t) + \beta_2 \psi(t), \qquad \phi, \psi \in C(R, R). \tag{3.2}$$

The operator H_1 and H_2 enjoy the following nice properties:

Lemma 3.1. *Assume that (A1) and (PQM) hold. Then*

$$H_1(\phi_2, \psi_1)(t) \leq H_1(\phi_1, \psi_1)(t), \qquad H_1(\phi_1, \psi_1)(t) \leq H_1(\phi_1, \psi_2)(t)$$

for $t \in R$ *and* $\phi_i,\ \psi_i \in C(R, R),\ i = 1, 2,$ *with* $0 \leq \phi_2(t) \leq \phi_1(t) \leq k_1, \quad 0 \leq \psi_2(t) \leq \psi_1(t) \leq k_2$.

Proof. By (PQM), direct calculation shows that

$$\begin{aligned}
&H_1(\phi_1, \psi_1)(t) - H_1(\phi_2, \psi_1)(t) = f_{1c}(\phi_{1t}, \psi_{1t}) - f_{1c}(\phi_{2t}, \psi_{1t}) + \beta_1[\phi_1(t) - \phi_2(t)] \geq 0,\\
&H_1(\phi_1, \psi_1)(t) - H_1(\phi_1, \psi_2)(t) = f_{1c}(\phi_{1t}, \psi_{1t}) - f_{1c}(\phi_{1t}, \psi_{2t}) \leq 0.
\end{aligned}$$

This completes the proof.

Lemma 3.2[19]. *Assume that (A1) and (PQM) hold. Then for any* $(0, 0) \leq (\phi, \psi) \leq (k_1, k_2)$ *which* $\phi(t)$ *and* $\psi(t)$ *are nondecreasing in* t*, we have*

(i) $H_2(\phi, \psi)(t) \geq 0, \quad t \in R,$

(ii) $H_2(\phi, \psi)(t)$ *is nondecreasing for* $t \in R,$

(iii) $H_2(\phi_2, \psi_2)(t) \leq H_2(\phi_1, \psi_1)(t)$ *for* $t \in R$ *and* $\psi_i, \psi_i \in C(R, R),\ i = 1, 2,$ *with* $0 \leq \phi_2(t) \leq \phi_1(t) \leq k_1, \quad 0 \leq \psi_2(t) \leq \psi_1(t) \leq k_2.$

In terms of H_1 and H_2, (2.1) can be rewritten as

$$\begin{cases} d_1 \phi''(t) - c\phi'(t) - \beta_1 \phi(t) + H_1(\phi, \psi)(t) = 0, \\ d_2 \psi''(t) - c\psi'(t) - \beta_2 \psi(t) + H_2(\phi, \psi)(t) = 0, \end{cases} \qquad t \in R. \tag{3.3}$$

Define

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1},$$
$$\lambda_3 = \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, \qquad \lambda_4 = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}.$$

One easily sees that $\lambda_1 < 0,\ \ \lambda_2 > 0,\ \ \lambda_3 < 0$ and $\lambda_4 > 0$. Let

$$C_K(R, R^2) = \{(\phi, \psi) \in C(R, R^2) : (0, 0) \le (\phi, \psi) \le (k_1, k_2)\},$$

and define $F = (F_1, F_2) : C_K(R, R^2) \to C(R, R^2)$ by

$$F_1(\phi,\psi)(t) = \frac{1}{d_1(\lambda_2 - \lambda_1)}\Big[\int_{-\infty}^{t} e^{\lambda_1(t-s)} H_1(\phi,\psi)(s)ds + \int_{t}^{\infty} e^{\lambda_2(t-s)} H_1(\phi,\psi)(s)ds\Big],$$
$$F_2(\phi,\psi)(t) = \frac{1}{d_2(\lambda_4 - \lambda_3)}\Big[\int_{-\infty}^{t} e^{\lambda_3(t-s)} H_2(\phi,\psi)(s)ds + \int_{t}^{\infty} e^{\lambda_4(t-s)} H_2(\phi,\psi)(s)ds\Big],$$

for $(\phi, \psi) \in C_K(R, R^2)$. It is easy to show that $F :\ C_K(R, R^2) \to C(R, R^2)$ is well defined, and for any $\phi, \psi \in C_K(R, R)$, $F_1(\phi, \psi)$ $F_2(\phi, \psi)$ satisfy

$$d_1 F_1''(\phi,\psi) - cF_1'(\phi,\psi) - \beta_1 F_1(\phi,\psi) + H_1(\phi,\psi) = 0, \tag{3.4}$$
$$d_2 F_2''(\phi,\psi) - cF_2'(\phi,\psi) - \beta_2 F_2(\phi,\psi) + H_2(\phi,\psi) = 0. \tag{3.5}$$

Thus, if $F(\phi, \psi) = (F_1(\phi, \psi), F_2(\phi, \psi)) = (\phi, \psi)$, i.e., (ϕ, ψ) is a fixed point of F, then (3.4), (3.5) reduce to (3.3), meaning that (3.3) has a solution (ϕ, ψ). If this solution further satisfies the boundary condition (2.4), then it gives a travelling wavefront.

Corresponding to Lemmas 3.1, 3.2, we have the following lemma for F, which is a direct consequence of Lemmas 3.1, 3.2.

Lemma 3.3. *Assume that (A1) and (PQM) hold. Then for any* $(0, 0) \le (\phi, \psi) \le (k_1, k_2)$, *we have*

(i) $F_2(\phi, \psi)(t)$ *is nondecreasing for* $t \in R$*;*

(ii) $F_1(\phi_2, \psi_1) \le F_1(\phi_1, \psi_1)$, $\ F_1(\phi_1, \psi_1) \le F_1(\phi_1, \psi_2)$, $\ F_2(\phi_2, \psi_2)(t) \le F_2(\phi_1, \psi_1)(t)$ *for* $t \in R$ *and* $\phi_i, \psi_i \in C(R, R)$, $i = 1, 2$, *with* $0 \le \phi_2(t) \le \phi_1(t) \le k_1 \quad 0 \le \psi_2(t) \le \psi_1(t) \le k_2$.

Proof. We just prove (i). To prove (i), let $t \in R$ and $s > 0$ be given, we have

$$\begin{aligned}
&F_2(\phi,\psi)(t+s) - F_2(\phi,\psi)(t)\\
&= \frac{1}{d_2(\lambda_4 - \lambda_3)}\Big[\int_{-\infty}^{t+s} e^{\lambda_3(t+s-\theta)} H_2(\phi,\psi)(\theta)d\theta + \int_{t+s}^{\infty} e^{\lambda_4(t+s-\theta)} H_2(\phi,\psi)(\theta)d\theta\Big]\\
&\quad - \frac{1}{d_2(\lambda_4 - \lambda_3)}\Big[\int_{-\infty}^{t} e^{\lambda_3(t-\theta)} H_2(\phi,\psi)(\theta)d\theta + \int_{t}^{\infty} e^{\lambda_4(t-\theta)} H_2(\phi,\psi)(\theta)d\theta\Big]\\
&= \frac{1}{d_2(\lambda_4 - \lambda_3)}\Big[\int_{-\infty}^{t} e^{\lambda_3(t-\theta)}\big(H_2(\phi,\psi)(s+\theta) - H_2(\phi,\psi)(\theta)\big)d\theta\\
&\quad + \int_{t}^{\infty} e^{\lambda_4(t-\theta)}\big(H_2(\phi,\psi)(s+\theta) - H_2(\phi,\psi)(\theta)\big)d\theta\Big].
\end{aligned}$$

It follows from Lemma 3.2 (ii) that $H_2(\phi, \psi)(s + \theta) - H_2(\phi, \psi)(\theta) > 0$, which implies that $F_2(\phi, \psi)(t)$ is nondecreasing in $t \in R$.

Now, we define the set

$$\Gamma\big((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi})\big) := \left\{ (\phi,\psi)\in C(R,R^2) \begin{array}{ll} (i) & \psi(t) \text{ is nondecreasing in } R, \\ (ii) & \underline{\phi}(t)\le\phi(t)\le\overline{\phi}(t), \text{and } \underline{\psi}(\mathrm{t})\le\psi(\mathrm{t})\le\overline{\psi}(\mathrm{t}) \end{array} \right\}$$

It is easy to see that $\Gamma((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$ is non-empty. In fact, let $\phi_0(t) = \sup\limits_{s\le t}\underline{\phi}(s),\ \ \psi_0(t) = \sup\limits_{s\le t}\underline{\psi}(s)$, then (P3) implies $(\phi_0(t),\ \psi_0(t)) \in \Gamma((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$. Moreover, it is obvious that $\Gamma((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$ is convex, closed and bounded.

From now on, we choose the parameter $\mu>0$ for the exponential decay norm to be such that $\mu<\min\{-\lambda_1,\ \lambda_2,\ -\lambda_3,\ \lambda_4\}$. We next verify the continuity of F.

Lemma 3.4. *Assume (A3) holds, then $F=(F_1,F_2)$ is continuous with respective to the norm $|\cdot|_\mu$ in $B_\mu(R,R^2)$.*

Proof. We prove Lemma 3.4 by two steps. The first step is to prove that $H=(H_1,H_2): B_\mu(R,R^2)\to B_\mu(R,R^2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^2)$.

For any fixed $\varepsilon>0$, let $\delta<\frac{\varepsilon}{L_1e^{\mu c\tau}+\beta_1}$. Then for $\Phi=(\phi_1,\psi_1),\ \Psi=(\phi_2,\psi_2)\in B_\mu(R,R^2)$ with

$$|\Phi-\Psi|_\mu = \underset{t\in R}{|}\Phi(t)-\Psi(t)|e^{-\mu|t|}<\delta, \sup$$

we have

$$\begin{aligned}
&|H_1(\psi_1,\psi_1)(t)-H_1(\phi,\psi)(t)|e^{-\mu|t|}\\
&\le |f_1(\phi_{1t},\psi_{1t})-f_1(\phi_{2t},\psi_{2t})|e^{-\mu|t|}+\beta_1|\phi_1-\phi_2|_\mu\\
&\le L_1||\Phi_t-\Psi_t||_{X_{c\tau}}e^{-\mu|t|}+\beta_1|\phi_1-\phi_2|_\mu\\
&= L_1\sup_{s\in[-c\tau,0]}|\Phi(s+t)-\Psi(s+t)|e^{-\mu|t|}+\beta_1|\Phi-\Psi|_\mu\\
&\le L_1\sup_{s\in[-c\tau,0]}|\Phi(s+t)-\Psi(s+t)|e^{-\mu|t+s|}\sup_{s\in[-c\tau,0]}e^{\mu|t+s|}e^{-\mu|t|}+\beta_1|\Phi-\Psi|_\mu\\
&\le L_1|\Phi-\Psi|_\mu|e^{-\mu|t|}e^{\mu|t|}e^{\mu c\tau}+\beta_1|\Phi-\Psi|_\mu\\
&\le L_1e^{\mu c\tau}|\Phi-\Psi|_\mu+\beta_1|\Phi-\Psi|_\mu\\
&\le (L_1e^{\mu c\tau}+\beta_1)|\Phi-\Psi|_\mu\le\varepsilon.
\end{aligned}$$

which implies that $H_1: B_\mu(R,R^2)\to B_\mu(R,R^2)$ is continuous.

Similarly, it can be shown that $H_2: B_\mu(R,R^2)\to B_\mu(R,R^2)$ is continuous. Thus, we obtain that $H=(H_1,H_2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^2)$.

Next, we prove the continuity of $F=(F_1,F_2)$. If $t\ge0$,

$$\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\\
&\le\frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_2-\lambda_1}{(\mu-\lambda_1)(\lambda_2-\mu)}+\frac{2\mu}{\lambda_1^2-\mu^2}e^{(\lambda_1-\mu)t}\Big]|H_1(\phi_1,\psi_1)-H_1(\phi_2,\psi_2)|_\mu\\
&\le\frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_2-\lambda_1}{(\mu-\lambda_1)(\lambda_2-\mu)}+\frac{2\mu}{\lambda_1^2-\mu^2}\Big]|H_1(\Phi)(t)-H_1(\Psi)(t)|_\mu.
\end{aligned}$$

If $t<0$, we have

$$\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\\
&\le\frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_2-\lambda_1}{-(\mu+\lambda_1)(\lambda_2+\mu)}+\frac{2\mu}{\lambda_1^2-\mu^2}e^{(\lambda_2+\mu)t}\Big]|H_1(\phi_1,\psi_1)-H_1(\phi_2,\psi_2)|_\mu\\
&\le\frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_2-\lambda_1}{-(\mu+\lambda_1)(\lambda_2+\mu)}+\frac{2\mu}{\lambda_1^2-\mu^2}\Big]|H_1(\Phi)-H_1(\Psi)|_\mu.
\end{aligned}$$

Therefore, the continuity of F_1 follows from that of H_1.

The proof of continuity of F_2 with respect to the norm $|\cdot|_\mu$ in $B_\mu(R, R^2)$ is similar to that of F_1. The proof is completed.

Lemma 3.5. *Assume (A1) and (PQM) hold. Then* $F(\Gamma((\underline{\phi}, \underline{\psi}),\ (\overline{\phi}, \overline{\psi})) \subset \Gamma((\underline{\phi}, \underline{\psi}),\ (\overline{\phi}, \overline{\psi}))$.

Proof. For any (ϕ, ψ) with $(\underline{\phi}, \underline{\psi}) \le (\phi, \psi) \le (\overline{\phi}, \overline{\psi})$, we first claim that

$$F_1(\underline{\phi}, \overline{\psi}) \le F_1(\phi, \psi) \le F_1(\overline{\phi}, \underline{\psi}), \qquad F_2(\underline{\phi}, \underline{\psi}) \le F_2(\phi, \psi) \le F_2(\overline{\phi}, \overline{\psi}). \tag{3.6}$$

In fact, since $\underline{\phi} \le \phi \le \overline{\phi}$, $\underline{\psi} \le \psi \le \overline{\psi}$, by (PQM), it follows

$$\begin{aligned} &H_1(\overline{\phi}, \underline{\psi})(t) - H_1(\phi, \psi)(t) = f_1(\overline{\phi}_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t) + \beta_1(\overline{\phi}(t) - \phi(t)) \\ &= f_1(\overline{\phi}_t, \underline{\psi}_t) - f_1(\phi_t, \underline{\psi}_t) + \beta_1(\overline{\phi}(t) - \phi(t)) + f_1(\phi_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t) \\ &\ge f_1(\phi_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t) \ge 0, \end{aligned}$$

which implies that $H_1(\overline{\phi}, \underline{\psi})(t) \ge H_1(\phi, \psi)(t)$. Similarly, $H_1(\underline{\phi}, \overline{\psi}) \le H_1(\phi, \psi)$. Hence, we obtain

$$H_1(\underline{\phi}, \overline{\psi}) \le H_1(\phi, \psi) \le H_1(\overline{\phi}, \underline{\psi}). \tag{3.7}$$

By a similar argument, we can also get

$$H_2(\underline{\phi}, \underline{\psi}) \le H_2(\phi, \psi) \le H_2(\overline{\phi}, \overline{\psi}). \tag{3.8}$$

From (3.7), it follows that

$$\begin{aligned} F_1(\overline{\phi}, \underline{\psi})(t) - F_1(\phi, \psi)(t) = &\frac{1}{d_1(\lambda_2 - \lambda_1)} \Big[\int_{-\infty}^{t} e^{\lambda_1(t-s)} \big[H_1(\overline{\phi}, \underline{\psi}) - H_1(\phi, \psi)\big] ds \\ &+ \int_t^{\infty} e^{\lambda_2(t-s)} \big[H_1(\overline{\phi}, \underline{\psi}) - H_1(\phi, \psi)\big] ds \Big] \ge 0 \end{aligned}$$

and

$$\begin{aligned} F_1(\phi, \psi)(t) - F_1(\underline{\phi}, \overline{\psi})(t) = &\frac{1}{d_1(\lambda_2 - \lambda_1)} \Big[\int_{-\infty}^{t} e^{\lambda_1(t-s)} \big[F_1(\phi, \psi) - F_1(\underline{\phi}, \overline{\psi})\big] ds \\ &+ \int_t^{\infty} e^{\lambda_2(t-s)} \big[F_1(\phi, \psi) - F_1(\underline{\phi}, \overline{\psi})\big] ds \Big] \ge 0 \end{aligned}$$

This establishes the first part of (3.6). Repeating the above argument but using (3.8) instead, we arrive at the second part of (3.6).

Next, we prove $F_1(\overline{\phi}, \underline{\psi}) \le \overline{\phi}$ and $F_1(\underline{\phi}, \overline{\psi}) \ge \underline{\phi}$. By the definition of the desirable pair of upper-lower solutions, we know

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) - \beta_1\overline{\phi}(t) + H_1(\overline{\phi}, \underline{\psi})(t) \le 0. \tag{3.9}$$

Choosing $(\phi, \psi) = (\overline{\phi}, \underline{\psi})$ in (3.4) and denoting $\overline{\phi}_1(t) = F_1(\overline{\phi}, \underline{\psi})(t)$, we have

$$d_1\overline{\phi}_1''(t) - c\overline{\phi}_1'(t) - \beta_1\overline{\phi}_1(t) + H_1(\overline{\phi}, \underline{\psi})(t) = 0. \tag{3.10}$$

Setting $w(t) = \overline{\phi}_1(t) - \overline{\phi}(t)$ and combining (3.9) and (3.10) gives

$$d_1 w(t)'' - cw'(t) - \beta_1 w(t) \ge 0 \tag{3.11}$$

Repeating the proof of Lemma 3.3 in Wu and Zou[19] shows that $w(t) \leq 0$, which implies that $F_1(\overline{\phi}, \underline{\psi}) \leq \overline{\phi}$.

By a similar argument, we can prove that $F_1(\underline{\phi}, \overline{\psi}) \geq \underline{\phi}$, $F_2(\overline{\phi}, \underline{\psi}) \leq \overline{\psi}$ and $F_2(\underline{\phi}, \overline{\psi}) \geq \underline{\psi}$. This completes the proof.

Lemma 3.6. *Assume (PQM) holds, then* $F : \Gamma((\underline{\phi}, \underline{\psi}),\ (\overline{\phi}, \overline{\psi})) \to \Gamma((\underline{\phi}, \underline{\psi}),\ (\overline{\phi}, \overline{\psi}))$ *is compact.*

Proof. We first establish an estimate for F'. For any $(\phi, \psi) \in \Gamma((\underline{\phi}, \underline{\psi}),\ (\overline{\phi}, \overline{\psi}))$,

$$F_1'(\phi,\psi)(t) = \frac{\lambda_1 e^{\lambda_1 t}}{d_1(\lambda_2 - \lambda_1)} \int_{-\infty}^{t} e^{-\lambda_1 s} H_1(\phi,\psi)(s) ds + \frac{\lambda_2 e^{\lambda_2 t}}{d_1(\lambda_2 - \lambda_1)} \int_{t}^{\infty} e^{-\lambda_2 s} H_1(\phi,\psi)(s) ds$$

Thus,

$$\begin{aligned}
||F_1'(\phi,\psi)(t)||_\mu &= \sup_{t\in R}\Big[e^{-\mu|t|}\frac{\lambda_1 e^{\lambda_1 t}}{d_1(\lambda_2-\lambda_1)}\int_{-\infty}^{t} e^{-\lambda_1 s}H_1(\phi,\psi)(s)ds \\
&\quad + e^{-\mu|t|}\frac{\lambda_2 e^{\lambda_2 t}}{d_1(\lambda_2-\lambda_1)}\int_{t}^{\infty} e^{-\lambda_2 s}H_1(\phi,\psi)(s)ds\Big] \\
&\leq \frac{|\lambda_1|}{d_1(\lambda_2-\lambda_1)}\sup_{t\in R} e^{\lambda_1 t-\mu|t|}\int_{-\infty}^{t} e^{-\lambda_1 s}e^{\mu|s|}e^{-\mu|s|}H_1(\phi,\psi)(s)ds \\
&\quad + \frac{\lambda_2}{d_1(\lambda_2-\lambda_1)}\sup_{t\in R} e^{\lambda_2 t-\mu|t|}\int_{t}^{\infty} e^{-\lambda_2 s}e^{\mu|s|}e^{-\mu|s|}H_1(\phi,\psi)(s)ds \\
&\leq \frac{|\lambda_1|}{d_1(\lambda_2-\lambda_1)}||H_1(\phi,\psi)||_\mu \sup_{t\in R} e^{\lambda_1 t-\mu|t|}\int_{-\infty}^{t} e^{-\lambda_1 s}e^{\mu|s|}ds \\
&\quad + \frac{\lambda_2}{d_1(\lambda_2-\lambda_1)}||H_1(\phi,\psi)||_\mu \sup_{t\in R} e^{\lambda_2 t-\mu|t|}\int_{t}^{\infty} e^{-\lambda_2 s}e^{\mu|s|}ds.
\end{aligned}$$

If $t > 0$, then

$$\begin{aligned}
&||F_1'(\phi,\psi)(t)||_\mu \\
&\leq \frac{\lambda_1}{d_1(\mu+\lambda_1)(\lambda_2-\lambda_1)}||H_1(\phi,\psi)||_\mu + \frac{\lambda_2}{d_1(\lambda_2-\mu)(\lambda_2-\lambda_1)}||H_1(\phi,\psi)||_\mu \\
&\leq \frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_1}{\mu+\lambda_1} + \frac{\lambda_2}{\lambda_2-\mu}\Big]||H_1(\phi,\psi)||_\mu.
\end{aligned}$$

If $t < 0$, then

$$\begin{aligned}
||F_1'(\phi,\psi)(t)||_\mu \leq & \frac{-\lambda_1}{d_1(\lambda_2-\lambda_1)}\frac{1}{|\mu+\lambda_1|}||H_1(\phi,\psi)||_\mu \\
& + \frac{\lambda_2}{d_1(\lambda_2-\lambda_1)}\Big[\Big|\frac{1}{\lambda-2-\mu} - \frac{1}{\lambda_2+\mu}\Big| + \frac{1}{\mu+\lambda_2}\Big]||H_1(\phi,\psi)||_\mu \\
\leq & \frac{1}{d_1(\lambda_2-\lambda_1)}\Big[\frac{\lambda_1}{\mu+\lambda_1} + \frac{\lambda_2}{\lambda_2-\mu}\Big]||H_1(\phi,\psi)||_\mu.
\end{aligned}$$

By the opposite monotonicity of $H_1(\phi, \psi)$ w.r.t. ϕ and ψ respectively (see Lemma 3.1) and the fact that $(0, 0) \leq (\phi, \psi) \leq (k_1, k_2)$, we know that $||H_1(\phi, \psi)||_\mu$ is bounded by a positive number. Therefore, there exists a constant M so that $||F_1'(\phi, \psi)(t)||_\mu \leq M$.

For $F_2(\phi, \psi)$, direct calculation shows that

$$F_2'(\phi,\psi)(t) = \frac{\lambda_3 e^{\lambda_3 t}}{d_2(\lambda_4 - \lambda_3)} \int_{-\infty}^{t} e^{-\lambda_3 \theta} H_2(\phi,\psi)(\theta) d\theta + \frac{\lambda_4 e^{\lambda_4 t}}{d_2(\lambda_4 - \lambda_3)} \int_{t}^{+\infty} e^{-\lambda_4 \theta} H_2(\phi,\psi)(\theta) d\theta.$$

It follows from Lemma 3.3 (ii)that $F'(\phi,\psi)(t) \geq 0$. By Lemma 3.2 (i) and the fact that $\lambda_3 < 0,\ \lambda_4 > 0$, we then have

$$\begin{aligned}0 \leq F_2'(\phi,\psi)(t) \leq & \frac{\lambda_4 e^{\lambda_4 t}}{d_2(\lambda_4-\lambda_3)} \int_t^{+\infty} e^{-\lambda_4\theta} H_2(\phi,\psi)(\theta)d\theta \\ \leq & \frac{\lambda_4 e^{\lambda_4 t}}{d_2(\lambda_4-\lambda_3)} H_2(\overline{\phi},\overline{\psi})(t)) \int_t^{+\infty} e^{-\lambda_4\theta} d\theta \\ \leq & \frac{1}{d_2(\lambda_4-\lambda_3)} H_2(\overline{\phi},\overline{\psi})(t).\end{aligned}$$

Hence, (P1) implies that $||F_1'(\phi,\psi)(t)||_\mu$ is also bounded by some positive constant. The above estimate for F' shows that $F(\Gamma((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi}))$ is equicontinuous. It is also easily seen that $F(\Gamma((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi})$ is uniformly bounded.

Next, we define

$$F^n(\phi,\psi)(t) = \begin{cases} F(\phi,\psi)(t), & t \in [-n,n]; \\ F(\phi,\psi)(n), & t \in (n,+\infty); \\ F(\phi,\psi)(-n), & t \in (-\infty,-n). \end{cases}$$

Then for each $n \geq 1$, $F^n(\Gamma((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi})))$ is also equicontinuous and uniformly bounded on $\Gamma((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi}))$. Now, in the interval $[-n,n]$, Ascoli-Arzela Theorem can be applied to F^n, implying that F_n is compact. On the other hand, $F^n \to F$ in $B_\mu(R,R^2)$ as $n \to \infty$, since

$$\begin{aligned}&\sup_{t\in R}|F^n(\phi,\psi)(t) - F(\phi,\psi)(t)|e^{-\mu|t|} \\ &= \sup_{t\in(-\infty,-n)\bigcup(n,\infty)} |F^n(\phi,\psi)(t) - F(\phi,\psi)(t)|e^{-\mu|t|} \\ &\leq 2Ke^{-\mu n} \to 0, \quad n \to \infty.\end{aligned}$$

Now, by Proposition 2.12 in [20], we know that $F : \Gamma((\underline{\phi},\ \underline{\psi}),(\overline{\phi},\overline{\psi})) \to \Gamma((\underline{\phi},\ \underline{\psi}),(\overline{\phi},\overline{\psi}))$ is also compact. The proof is completed.

Now, we are in the position to state and prove the following main theorem.

Theorem 3.1. *Assume that (A1)–(A3) and (PQM) hold. Suppose there is a desirable pair of upper-lower solutions* $(\overline{\phi},\overline{\psi})$ *and*$(\underline{\phi},\ \underline{\psi})$ *for (2.1) satisfying (P1)–(P3) and*

(P4) $\sup_{t\in R}\underline{\psi}(t) > 0$, $\underline{\phi}(t)$ *is non-decreasing with* $\sup_{t\in R}\underline{\phi}(t) > 0$, *and*

$$f(\widetilde{u},\widetilde{v}) = (f_1(\widetilde{u},\widetilde{v}), f_2(\widetilde{u},\widetilde{v})) \neq (0,0) \quad \textit{for any } (u,v) \in [\sup_{t\in R}\underline{\phi}(t), k_1) \times [\sup_{t\in R}\underline{\psi}(t), k_2).$$

Then, (2.1)–(2.4) has a solution, with the second component $\psi(t)$ *nondecreasing in* $t \in R$. *That is, System (1.3) has a travelling wave front.*

Proof. Combining Lemmas 3.1–3.6 with the Schauder's fixed point theorem, we know that there exits a fixed point $(\phi^*(t),\psi^*(t))$ of F in $\Gamma((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi}))$, which gives a solution of (2.1). It remains to show that this fixed point satisfies the asymptotic boundary condition (2.4).

First of all, by (P2) and the fact that $0 \leq (\underline{\phi},\underline{\psi})(t) \leq (\phi^*(t),\psi^*(t)) \leq (\overline{\phi},\overline{\psi})(t) \leq (k_1,k_2)$, we know that

$$\lim_{t\to-\infty}(\phi^*,\psi^*)(t) = (0,0).$$

Secondly, $(\phi^*(t), \psi^*(t)) \in \Gamma((\underline{\phi}, \underline{\psi}), (\overline{\phi}, \overline{\psi}))$ implies that $\psi^*(t)$ is monotone nondecreasing in $t \in R$, and hence, $\lim_{t\to\infty} \psi^*(t)$ exists and satisfies $k_2^* := \lim_{t\to\infty} \psi^*(t) = \sup_{t\in R} \psi^*(t) \geq \sup_{t\in R} \underline{\psi}(t) > 0$. Now, employing the Hopital's rule to $\psi^*(t) = F_2(\phi^*(t), \psi^*(t))$, we have

$$\begin{aligned}
&\lim_{t\to\infty} \psi^*(t) = \lim_{t\to\infty} F_2(\phi^*(t), \psi^*(t)) \\
&= \lim_{t\to\infty} \frac{1}{d_2(\lambda_4 - \lambda_3)} \Big[\int_{-\infty}^{t} e^{\lambda_3(t-s)} H_2(\phi^*, \psi^*)(s) ds + \int_{t}^{\infty} e^{\lambda_4(t-s)} H_2(\phi^*, \psi^*)(s) ds \Big] \\
&= \lim_{t\to\infty} \frac{1}{d_2(\lambda_4 - \lambda_3)} \Big[\frac{H_2(\phi^*, \psi^*)(t)}{-\lambda_3} + \frac{H_2(\phi^*, \psi^*)(t)}{\lambda_4} \Big] \\
&= \lim_{t\to\infty} \Big[\frac{f_{2c}(\phi_t^*, \psi_t^*)}{\beta_2} + \psi^*(t) \Big] = \lim_{t\to\infty} \frac{f_{2c}(\phi_t^*, \psi_t^*)}{\beta_2} + \lim_{t\to\infty} \psi^*(t),
\end{aligned}$$

which implies that

$$\lim_{t\to\infty} f_{2c}(\phi_t^*, \psi_t^*) = 0.$$

By $f_{2c}(\phi_t^*, \psi_t^*) = \psi^*(t)[h_c(\psi_t^*) + a\phi^*(t))$, we know that

$$\phi^*(t) = \frac{1}{a} \Big[\frac{f_{2c}(\phi_t^*, \psi_t^*)}{\psi^*(t)} - h_c(\psi_t^*) \Big].$$

This shows that $k_1^* := \lim_{t\to\infty} \phi^*(t)$ also exists. By Proposition 2.1 in [19], we must have $(f_{1c}(\widetilde{k}_1^*, \widetilde{k}_2^*), f_{2c}(\widetilde{k}_1^*, \widetilde{k}_2^*) = (0, 0))$. Note (P4) implies that

$$\begin{aligned}
&0 < \sup_{t\in R} \underline{\psi}(t) \leq k_2^* \leq k_2, \\
&0 < \sup_{t\in R} \underline{\phi}(t) = \lim_{t\to\infty} \underline{\phi}(t) \leq \lim_{t\to\infty} \phi^*(t) = k_1^* \leq k_1.
\end{aligned}$$

Again by (P4), we conclude that $k_1^* = k_1$ and $k_2^* = k_2$. Therefore, the fixed point does satisfy the boundary condition (2.4), giving a travelling wavefront of (1.3). The proof is completed.

Remark 3.1. From the proof of Theorem 3.1, we see that the second component $\psi^*(t)$ is nondecreasing, while the first component $\phi^*(t)$ may not possess the monotonicity. Such a difference between the components of the travelling wavefront is due to the fact that we only assumed partial quasi-monotonicity (PQM) for the nonlinear reaction term. The same remark also applies to Theorem 4.1 in the next section.

4 Partially Exponential Non-quasimonotone Case

It is well known that a negative delayed term usually destroy the standard monotonicity. A simple example is the delayed logistic reaction term. Even for the partial quasi-monotonicity (PQM), one faces the same situation. On the other hand, for systems with such negative delayed terms, alternative ordering may help an otherwise non-monotone system gain some other type monotonicity related to the new ordering. Smith, and Thieme[12,13] provide succesful examples in this direction, where an *exponotential ordering* are adopted. Employing this exponential ordering idea, Wu and Zou[19], and Huang and Zou[6] are able to establish the existence of travelling wavefronts for delayed reaction diffusion systems with a weakened quasi-monotonicity (may be called exponential quasi-monotonicity). In this section, we will use the the same idea to weaken the partially quasi-monotonicity used in Section 3 and replace (PQM) by the following weaker one:

(PQM*) There exist two positive constants $\beta_1 > 0,\ \beta_2 > 0$ such that

$$\begin{aligned} &f_{1c}(\phi_1,\psi_1) - f_{1c}(\phi_2,\psi_1) + \beta_1[\phi_1(0) - \phi_2(0)] \geq 0, \\ &f_{1c}(\phi_1,\psi_1) - f_{1c}(\phi_1,\psi_2) \leq 0, \\ &f_{2c}(\phi_1,\psi_1) - f_{2c}(\phi_2,\psi_2) + \beta_2[\psi(0) - \psi_1(0)] \geq 0, \end{aligned}$$

where $\phi_1,\phi_2,\psi_1,\psi_2 \in C([-\tau,0],R)$ with (i) $0 \leq \phi_2(s) \leq \phi_1(s) \leq k_1,\ 0 \leq \psi_2(s) \leq \psi_1(s) \leq k_2,\ s \in [-\tau,0]$. (ii) $e^{\beta_1 s}[\phi_1(s) - \phi_2(s)]$ and $e^{\beta_2 s}[\psi_1(s) - \psi_2(s)]$ are nondecreasing in $s \in [-\tau,0]$.

There will be a cost for this weakening of requirements on the non-linear reaction terms in that more restrictions are imposed on the desired pair of upper-lower solutions. More precisely, we will replace (P3) (used to guarantee that $\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$ is non-empty) with the following assumption.

(P3*) The set

$$\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi})) = \left\{ (\phi,\psi) \in C(R,R^2) \quad \begin{array}{l} (i) \quad \psi \text{ is nondecreasing in } R, \\ (ii) \quad (\underline{\phi},\underline{\psi})(t) \leq (\phi,\psi)(t) \leq (\overline{\phi},\overline{\psi})(t); \\ (iii) \quad e^{\beta_1 t}[\overline{\phi}(t) - \phi(t)], \quad e^{\beta_2 t}[\overline{\psi}(t) - \psi(t)], \\ \qquad e^{\beta_1 t}[\phi(t) - \underline{\phi}(t)] \text{ and } \quad e^{\beta_2 t}[\psi(t) - \underline{\psi}(t)] \\ \qquad \text{are nondecreasing in } t \in R; \\ (iv) \quad e^{\beta_2 t}[\psi(t+s) - \psi(t)] \text{ is nondecreasing} \\ \qquad \text{in } t \in R \text{ for every } s > 0. \end{array} \right\}$$

is non-empty.

We will adopt the same setting as that for H and F in Section 3. Thus, parallel to Lemmas 3.1–3.3, we can obtain the following Lemmas 4.1–4.3 by similar argument.

Lemma 4.1. *Assume that (A1) and (PQM*) hold. Then*

$$H_1(\phi_2,\psi_1)(t) \leq H_1(\phi_1,\psi_1)(t), \qquad H_1(\phi_1,\psi_1)(t) \leq H_1(\phi_1,\psi_2)(t)$$

for $t \in R,\ \phi_i,\ \psi_i \in C(R,R),\ i = 1,2,$ *with (i)* $0 \leq \phi_2(t) \leq \phi_1(t) \leq k_1,\ \ 0 \leq \psi_2(t) \leq \psi_1(t) \leq k_2$*; (ii)* $e^{\beta_1 t}[\phi_1(t) - \phi_2(t)]$ *and* $e^{\beta_2 t}[\psi_1(t) - \psi_2(t)]$ *are nondecreasing in* t.

Lemma 4.2. *Assume that (A1) and (PQM*) hold. Then for any* $(\phi,\psi) \in \Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$, *we have*

(i) $H_2(\phi,\psi)(t) \geq 0, \quad t \in R.$

(ii) $H_2(\phi,\psi)(t)$ *is nondecreasing for* $t \in R$

(iii) $H_2(\phi_1,\psi_1)(t) \leq H_2(\phi,\psi)(t)$ *for* $t \in R,\ \phi_i,\ \psi_i \in C(R,R),\ i = 1,2,$ *with (i)* $0 \leq \phi_2(t) \leq \phi_1(t) \leq k_1, \quad 0 \leq \psi_2(t) \leq \psi_1(t) \leq k_2$*; (ii)* $e^{\beta_1 t}[\phi_1(t) - \phi_2(t)]$ *and* $e^{\beta_2 t}[\psi_1(t) - \psi_2(t)]$ *are nondecreasing in* t.

Lemma 4.3. *Assume that (A1) and (PQM*) hold. Then for any* $(\phi,\psi) \in \Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$, *we have*

(i) $F_2(\phi,\psi)(t)$ *is nondecreasing for* $t \in R$*;*

(ii) $F_2(\phi_1,\psi_1)(t) \leq F_2(\phi,\psi)(t)$ *for* $t \in R,\ \phi_i,\ \psi_i \in C(R,R),\ i = 1,2,$ *with (i)* $0 \leq \phi_2(t) \leq \phi_1(t) \leq k_1, \quad 0 \leq \psi_2(t) \leq \psi_1(t) \leq k_2$*; (ii)* $e^{\beta_1 t}[\phi_1(t) - \phi_2(t)]$ *and* $e^{\beta_2 t}[\psi_1(t) - \psi_2(t)]$ *are nondecreasing in* t.

It is obvious that if (P3*) holds, then the set $\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$ is also a closed, bounded and covex subset of $B_\mu(R,R^2)$. Also, the continuity of F does not depend on (PQM), and thus remains true. In order to apply the Schauder's fixed point theorem, we rerquire that F map Γ^* into Γ^* and be compact.

Lemma 4.4. *Assume that (PQM*) holds. Then* $F(\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))) \subset \Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$.

Proof. Let $(\phi,\psi) \in \Gamma^*((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi}))$. By a similar argument to that of Lemma 3.5, we can verify that $F(\phi,\psi) = (F_1(\phi,\psi),F_2(\phi,\psi))$ satisfies the first and second conditions of $\Gamma^*((\underline{\phi},\underline{\psi}),(\overline{\phi},\overline{\psi}))$. It is easily seen that if (PQM*) holds, we can always choose β_1 and β_2 sufficiently large such that $c \geq 1-\min\{\beta_1 d_1,\beta_2 d_2\}$, and hence $\beta_1+\lambda_1>0$, $\beta_1+\lambda_2>0$, $\beta_2+\lambda_3>0$ and $\beta_2+\lambda_4>0$. Now, by Lemma 4.2 and some simple calculations, we obtain

$$\begin{aligned}
&\frac{d}{dt}e^{\beta_2 t}\big[(F_2(\phi,\psi)(t+s)-F_2(\phi,\psi)(t))\big]\\
&=(\beta_2+\lambda_3)e^{(\beta_2+\lambda_3)t}\int_{-\infty}^{t}\frac{e^{-\lambda_3\theta}}{d_2(\lambda_4-\lambda_3)}\big[H_2(\phi,\psi)(\theta+s)-H_2(\phi,\psi)(\theta)\big]d\theta\\
&\quad+(\beta_2+\lambda_4)e^{(\beta_2+\lambda_4)t}\int_{t}^{\infty}\frac{e^{-\lambda_4\theta}}{d_2(\lambda_4-\lambda_3)}\big[H_2(\phi,\psi)(\theta+s)-H_2(\phi,\psi)(\theta)\big]d\theta\\
&\geq 0,
\end{aligned}$$

which verifies Condition (iv) of Γ^* for $F(\phi,\psi)$. For Condition (iii) of Γ^*, we proceed as follows. By the definition of $(\overline{\phi},\overline{\psi})$ and $(\underline{\psi},\underline{\psi})$, we have

$$d_1\overline{\phi}''(t)-c\overline{\phi}'(t)-\beta_1\overline{\phi}(t)+H_1(\overline{\phi},\underline{\psi})(t)\leq 0, \tag{4.1}$$
$$d_2\overline{\psi}''(t)-c\overline{\psi}'(t)-\beta_a\overline{\psi}(t)+H_2(\overline{\phi},\overline{\psi})(t)\leq 0, \tag{4.2}$$
$$d_1\underline{\phi}''(t)-c\underline{\phi}'(t)-\beta_1\underline{\phi}(t)+H_1(\underline{\phi},\overline{\psi})(t)\geq 0, \tag{4.3}$$
$$d_2\underline{\psi}''(t)-c\underline{\psi}'(t)-\beta_a\underline{\psi}(t)+H_2(\underline{\phi},\underline{\psi})(t)\geq 0. \tag{4.4}$$

Substracting (3.4) from (4.1), (3.5) from (4.2) and tetting $w_1=\overline{\phi}-F_1(\phi,\psi)$ and $w_2=\overline{\psi}-F_2(\phi,\psi)$, by Lemmas 4.1-4.2, we see that

$$\begin{aligned}
d_1w_1''(t)-cw_1'(t)-\beta_1w_1(t)\leq 0,\\
d_1w_1''(t)-cw_1'(t)-\beta_1w_1(t)\leq 0.
\end{aligned}$$

Now, using the same auguments as that in the proof of Lemma 4.3 in Wu and Zou[19], we can obtain

$$\frac{d}{dt}\big[e^{\beta_1 t}w_1(t)\big]\geq 0,\qquad \frac{d}{dt}\big[e^{\beta_1 t}w_1(t)\big]\geq 0. \tag{4.5}$$

Similarly, using (3.4), (3.5), (4.3) and (4.4), we can obtain

$$\frac{d}{dt}\big[e^{\beta_2 t}u_1(t)\big]\geq 0,\qquad \frac{d}{dt}\big[e^{\beta_2 t}u_2(t)\big]\geq 0, \tag{4.6}$$

where $u_1=F_1(\phi,\psi)-\underline{\phi}$ and $u_2=F_2(\phi,\psi)-\underline{\psi}$. This shows that $F(\phi,\psi)$ also satisfies Condition (iii) of Γ^*. Therefore $F(\phi,\psi)\in\Gamma^*$, and the proof is completed.

Lemma 4.5. *If (PQM*) holds, then* $F:\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))\to\Gamma^*((\underline{\phi},\underline{\psi}),\ (\overline{\phi},\overline{\psi}))$ *is compact.*

Proof. The proof is similar to that of Lemma 3.6, and is omitted here.

Finally, combining lemmas 4.3–4.5 with the Schauder's fixed pointed theorem and following that in the proof of Theorem 3.1, with the Schauder's fixed point theorem and following the same arguments as in the proof of Theorem 3.1, we can establish the following result akin to Theorem 3.1.

Theorem 4.1. *Assume that (A1)–(A3) and (PQM*) hold. Suppose there is a desirable pair of upper-lower solutions* $(\overline{\phi},\overline{\psi})$ *and* $(\underline{\phi},\underline{\psi})$ *satisfying (P1), (P2), (P3*) and (P4). Then,* *(2.1)–(2.4) has a solution, with the second component $\psi(t)$ nondecreasing in $t \in R$. That is, System (1.3) has a travelling wave front.*

Acknowledgment. The authors would like to thank the referees for their valuable comments which have led to an improvement of the presentation of the paper.

References

[1] Britton, N.F. Reaction diffusion equations and their application to biology. Academic Press, New York, 1986
[2] Fife, P.C. Mathematical aspects of reaction and diffusion systems. Lecture Notes in Biomathematics, Vol. 28, Springer-Verlag, Berlin, New York, 1979
[3] Gardner, R. Review on travelling wave solutions of Parabolic Systems by A. I. Volpert, V. A. Volpert and V. A. Volpert. *Bull. Amer. Math. Soc.*, 32: 446–452 (1995)
[4] Gourley, S.A. Wave front solutions of a diffusive delay model for population of Daphnia magna. *Comp. Math. Appl.*, 42: 1421–1430 (2001)
[5] Huang, J., Zou, X. Travelling wavefronts in diffusive and cooperative lotka-volterra system with delays. *J. Math. Anal. Appl.*, 271: 455–466 (2002)
[6] Huang, J., Zou, X. Existence of travelling wavefronts of delayed reaction diffusion systems without monotonicity. *Disc. Conti. Dynam. Syst.* (series A), 9: 925–936 (2003)
[7] Huang, W. Monotonicity of heteroclinic orbits and spectral properties of variational equations for delay differential equations. *J. Differential Equations*, 162: 91–139 (2000)
[8] Ma, S. Travelling wavefronts for delayed reaction-diffusion systems via a fixed point theorem. *J. Differential Equations*, 171: 294–314 (2001)
[9] Murray J. D. Murray. *Mathematical Biology*, Springer-Verlag, New York, 1989
[10] Pao, C.V. Nonlinear parabolic and elliptic equations. New York, Plenum Press, 1992
[11] Schaaf, K. Asymptotic behavior and travelling wave solutions for parabolic functional differential equations. *Trans. Amer. Math. Soc.*, 302: 587–615 (1987)
[12] Smith, H.L., Thieme, H.R. Monotone semiflows in scalar non-quasimonotone functional differential equations. *J. Math. Anal. Appl.*, 150: 289–306 (1990)
[13] Smith, H.L., Thieme, H.R. Strongly order preserving semiflows generated by functional differential equations. *J. Differential Equations*, 93: 322–363 (1991)
[14] Smith, H.L., Zhao, X.Q. Global asymptotic stability of travelling waves in delayed reaction-diffusion equations. *SIAM J. Math. Anal.*, 31: 514–534 (2000)
[15] So, J. W.H., Wu, J., Zou, X. A reaction diffusion model for a single species with age structure –I. Travelling wave fronts on unbounded domains. *Proc. Royal Soc. London, Ser. A*, 457: 1841-1854, 2001
[16] So, J. W.-H., Zou, X. Travelling waves for the diffusive Nicholson's blowflies equation. *Appl. Math. Compt.*, 122: 385–392 (2001)
[17] Volpert, A.I., Volpert, V.A., Volpert, V.A. Travelling wave solutions of parabolic Systems. Translations of mathematical monographs Vol. 140, Amer. math. Soc., Providence, 1994
[18] Ye, Q., Li, Y. Introduction of reaction diffusion equations. Academy Press, BeiJing, 1985
[19] Wu, J., Zou, X. Travelling wave fronts of reaction diffusion systems with delay. *J Dynam. Diff. Eqns.*, 13(3): 651–687 (2001)
[20] Zeidler, E. Nonlinear functional analysis and its applications, I, Fixed-point Theorems. Springer-Verlag, New York, New York, 1986
[21] Zou, X., Wu, J. Existence of travelling wavefronts in delayed reaction-diffusion system via monotone iteration method. *Proc. Amer. Math. Soc.*, 125: 2589–2598 (1997)