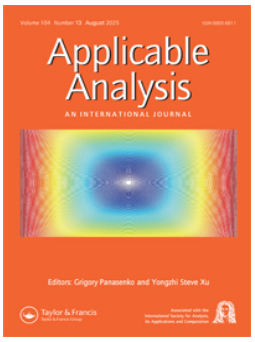


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# On steady states of a predator-prey model with predator-taxis and spatial heterogeneity*

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# On steady states of a predator-prey model with predator-taxis and spatial heterogeneity*

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**ABSTRACT**
In order to better accommodate the actual ecological environment, a more practical predator-prey model with fear effect, predator-taxis and degeneracy in spatially heterogeneous environment is proposed and analyzed in this paper. We are mainly concerned with positive steady state solutions and we investigate not only the individual effect of degeneracy but also the combined effects of predator-taxis and degeneracy on the such solutions of the models. Our results not only demonstrate some similarities but also reveal differences between the models with and without predator-taxis that partially reflects the fear effect.

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## 1. Introduction

Traditional predator-prey models focus on the direct effect on the prey by catching and consuming. However, some ecologists' field researches indicate that there are indirect effects mainly in the form of fear, which may also impact the results of predator-prey interactions. Such fear effects are mainly due to prey's anti-predation responses, including the changes in their foraging times and locations, avoidance (like habitat choice and distribution), vigilance, alarm calls, grouping and even defences against predators [1, 2]. These strategies not only diminish direct mortality from predation temporally (benefit), but will also reduce lifetime fitness at the same time (cost). Furthermore, some experimental evidences show that indirect effects in predator-prey interactions are common and can even be very significant. For example, Schmitz et al. [3] found that such indirect effects may cause same level of grasshopper mortality as direct predation by spider. Nelson et al. [4] surgically shortened the mouthparts of damsel bugs to prevent them from eating, but the bugs still interfered with the prey aphids, and it turned out that this can still reduce aphid population by 30%. Zanette et al. [5] conducted a field experiment on song sparrows in which they protected the birds from direct predation and played the calls and sounds of

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their predators to impose fear on the birds, and such a fear effect resulted in a 40% reduction in the number of song sparrows's offspring produced per year.

As far as modelling of fear effect is concerned, Brown et al. [6] firstly modelled the ecology of fear by conjoin the Rosenzweig-MacArthur model with foraging theory, where fear was represented by the level of vigilance. In recent years, many different types of mathematical models with fear (indirect) effects have been proposed and analyzed. For example, an ODE model is proposed and analyzed in [7] where the fear effect is reflected by a reduction to the reproduction rate; an *age-structured DDE model* is proposed in [8] where the cost of fear and adaptive avoidance of predators are considered; a DDE model with both cost and benefit of anti-predation response as well as a digestion delay is analyzed in [9]; a food web ODE model with anti-predation responses is investigated in [10]. Some models with fear effects have also incorporated spatial effects, see e.g. the reaction-diffusion model in [11] and the patch model in [12].

In this paper, we will further explore the fear effects in the presence of some spatial factors. Let $u(x,t)$ and $v(x,t)$ be the populations of prey and predator at time $t$ and location $x$, respectively. For the movements of prey and predators, we consider a scenario that (i) the predators move randomly, with the flux $J_v = -d_v \nabla v$; (ii) due to fear, there is a predator-taxis effect in the prey's movement, meaning that in addition to the random diffusion, the prey also tends to move from locations with higher predator density to places with lower predator density. Feature (ii) can be represented by the flux for the prey given by

$$J_u = -d_u \nabla u - \alpha\beta(u) u \nabla v,$$

where the first part accounts for the random diffusion while the second part reflects the predator-repulsion effect. Then a general predator-prey model with the above scenario for the movement reads

$$\begin{cases} u_t = \nabla \cdot (d_u \nabla u + \alpha\beta(u) u \nabla v) + h(u,v), \\ v_t = d_u \Delta v + g(u,v). \end{cases} \tag{1}$$

Here the parameter $\alpha$ measures the strength of the predator-repulsion *partially reflects the fear level*, and $\beta(u)$ is assumed, following [13, 14] to have the volume filling effect described by

$$\beta(u) = \begin{cases} 1 - \dfrac{u}{M}, & 0 \le u \le M, \\ 0, & u > M, \end{cases} \tag{2}$$

where $M$ measures the maximum number of prey that a unit volume can accommodate. If the number of prey exceeds the volume $M$, then prey can no longer squeeze into nearby space, and therefore the tendency of directed movement goes to 0.

The fear effect can also be reflected in the reaction terms. For example, in the ODE model in [7], the authors considered the cost of fear in production by introducing a factor $f(k,v)$ to reduce the birth rate from $r_0 u$ to $f(k,v) r_0 u$, leading to the following forms of $h$

$$h(u,v) = r_0 f(k,v) u - du - au^2 - p(u,v)v.$$

Here $r_0$, $d$ and $a$ represent the birth rate, natural death rate and intra-specific interaction coefficient of prey, respectively,

$$f(k,v) = \frac{1}{1+kv} \tag{3}$$

with $k$ measuring the level of prey's anti-predation response caused by fear, $p(u,v)$ is the functional response. In [9], in addition to cost, the authors also considered the benefit of the prey's anti-predation response by adopting the following forms for the two reaction terms:

$$\begin{cases} h(u,v) = r_0 f(k,v)u - du - au^2 - \rho(k)p(u,v)v, & x \in \Omega, \quad t > 0, \\ g(u,v) = -m_1 v - m_2 v^2 + c\rho(k)p(u,v)v, & x \in \Omega, \quad t > 0. \end{cases} \tag{4}$$

Here the $k$ dependent factor $\rho(k) \in (0,1)$ reflects the benefit because it reduces the predation rate, $c$ measures the conversion efficiency of biomass from prey to predator, $m_1$ is the intrinsic growth rate of the predator with $m_1 > 0$ accounting for a specialist predator and $m_1 < 0$ explaining the scenario of generalist predator, and $m_2 \geq 0$ accounts for the intra-specific competition of the predator. In the real world, the predator's intra-specific pressure depends on the conditions of the habitat (environment) which are generally heterogeneous. This suggests us to consider $m_2$ as location dependent: $m_2 = m_2(x)$. Such location dependent parameter may vanish in some locations, presenting a degenerate situation. As is shown in [15], when a location dependent parameter vanishes on a nonempty proper subdomain of $\Omega$ and is positive in the rest of $\Omega$, some qualitative changes will occur to the behavior of model. Adopting the location dependence of $m_2$ and denoting $m = -m_1$ and $s(x) = m_2(x)$, (1)–(4) lead to the following system of reaction diffusion equations:

$$\begin{cases} u_t - d_u \Delta u - \alpha \nabla \cdot (\beta(u) u \nabla v) \\ \quad = r_0 f(k,v)u - du - au^2 - \rho(k)p(u,v)v, & x \in \Omega, \quad t > 0. \\ v_t - d_v \Delta v = mv - s(x)v^2 + c\rho(k)p(u,v)v, \end{cases} \tag{5}$$

For the function response function and the benefit level, we will adopt the following forms:

$$p(u,v) = \frac{pu}{1 + q_1 u + q_2 v}, \quad \rho(k) = \frac{1}{1 + c_1 k},$$

where $p$ is the maximal predation rate of predator, $q_1$ and $q_2$ stand for the interference effects of prey and predator, respectively, and $c_1$ is the decreasing rate of reproduction with respect to the fear level $k$. Moreover, except for $m$ which can take negative values, all other constants are positive, and in the light of [14], we always assume that $M > (r_0 - d)/a$.

For the spatial domain, we consider a scenario that the two species live in an isolated bounded domain $\Omega \in R^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega \in C^{2+\epsilon}$ with $\epsilon \in (0,1)$. Then, the isolation (no-flux) boundary condition is presented as

$$J_u \cdot \mathbf{n} = d_u \frac{\partial u}{\partial \mathbf{n}} + \alpha \beta(u) u \frac{\partial v}{\partial \mathbf{n}} = 0, \quad J_v \cdot \mathbf{n} = d_v \frac{\partial v}{\partial \mathbf{n}} = 0,$$

which is equivalent to the following homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega, \tag{6}$$

where $\mathbf{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$.

We remark that for the special cases of $\rho(k) = 0$, $m = -m_1 < 0$ and $s(x) = m_2$ being non-positive constant, (5)–(6) reduces to the model discussed in Wang-Zou [8]. In [8], in addition to the global existence, uniqueness, positiveness and boundedness of solution of the model with given positive initial functions, the authors also explored the existence and stability of co-existence steady state, as well as the pattern formations for four types of the functional response function $p(u,v)$: (i) $p(u,v) = pu$; (ii) $p(u,v) = pu/(1+qu)$; (iii) $p(u,v) = pu/(1+q_1u+q_2v)$; (iv) $p(u,v) = pu/(u+qv)$. A particular interesting finding is that the impact of the predator-repulsion level on pattern formation actually depends on the type of $p(u,v)$.

In this paper, we are mainly interested in the pattern formation of the model (5)–(6), which is related to the system governing the steady state solutions, given by

$$\begin{cases} -d_u\Delta u - \alpha\nabla\cdot(\beta(u)u\nabla v) = \dfrac{r_0}{1+kv}u - du - au^2 - \dfrac{p\rho(k)uv}{1+q_1u+q_2v}, & x\in\Omega, \\ -d_v\Delta v = mv - s(x)v^2 + \dfrac{cp\rho(k)uv}{1+q_1u+q_2v}, & x\in\Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{7}$$

For the location dependent parameter $s(x)$, we assume it has a degeneracy in $\Omega$, that is, $s(x)$ satisfies

$$s(x)\equiv 0,\quad x\in\overline{\Omega}_0,\quad s(x)>0,\quad x\in\overline{\Omega}\backslash\overline{\Omega}_0 \tag{8}$$

with $\overline{\Omega}_0$ being a smooth subdomain in $\Omega$. Biologically, (8) means that $\Omega_0$ is a favourable subregion for predator since the intra-specific pressure of predator vanishes in $\Omega_0$.

We point out that in recent decades, many researchers have investigated individual effect of prey's fear against predators (see, e.g. [7–12, 16]), the combined effects of random diffusion and predator-taxis [17–21], the combined effects of random diffusion and degeneracy [22–24], and the combined effects of random diffusion, prey's fear and degeneracy [25] on the dynamics for predator-prey models, respectively. However, to the authors knowledge, the joint impacts of fear induced cost and benefit, random diffusion, predator-taxis, and degeneracy in spatial factors on the dynamics for predator-prey models have not been studied yet. Such joint impact constitutes the aim of this study. Moreover, to highlight the role of the degeneracy in conjunction with the predator-taxis level, we also explore, in more details, the special case without predator-taxis, that is, the following steady state model

$$\begin{cases} -d_u\Delta u = \dfrac{r_0}{1+kv}u - du - au^2 - \dfrac{p\rho(k)uv}{1+q_1u+q_2v}, & x\in\Omega, \\ -d_v\Delta v = mv - s(x)v^2 + \dfrac{cp\rho(k)uv}{1+q_1u+q_2v}, & x\in\Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0, & x\in\partial\Omega. \end{cases} \tag{9}$$

More specifically, we would like to seek answers to the following three meaningful questions:

- What common properties would the solutions to (7) and (9) share?
- Is/are there any essential difference(s) between the solutions to (7) and (9)?

- How do the parameter $k$ and the location dependence of $s(x)$ with the vanishing property (8) affect the solution behavior of (7) and (9)?

To address these questions, we explore positive solutions of (7) and (9) respectively in Sections 2 and 3, including existence, uniqueness/multiplicity, bifurcation structure, and behaviors when the prey's growth rate varies, the impact of the degeneracy and predator-taxis effect. The similarity and differences between (7) and (9) discussed in the respective remarks in Section 3 after the major results. In Section 4, we present some numerical results to demonstrate the obtained analytic results, which visually illustrate the impact of fear effect and the spatial heterogeneity function $s(x)$ on the positive solutions and their patterns. Our results provide more new insights on predator-prey interactions in the presence of spatial factors and fear effect.

## 2. On solutions of model (7)

To proceed, we first give some notations that will be used in the rest of the paper. Suppose that $T$ is an differential operator, $\phi$ is a Hölder continuous function in $\Omega$, and $d > 0$. Let $\lambda_1^D(\pm d, T+\phi, \Omega)$ and $\lambda_1^N(\pm d, T+\phi, \Omega)$ denote the first eigenvalues of $\mp d\Delta + T + \phi$ over $\Omega$ under Dirichlete and Neumann boundary conditions respectively. When $T = \phi = 0$, we will omit $T$ and $\phi$ to write $\lambda_1^D(\pm d, \Omega) = \lambda_1^D(\pm d, 0, \Omega)$ and $\lambda_1^N(\pm d, \Omega) = \lambda_1^N(\pm d, 0, \Omega)$. When $T = 0$ and $\phi \neq 0$, we will omit $T$ to write $\lambda_1^D(\pm d, \phi, \Omega) = \lambda_1^D(\pm d, 0+\phi, \Omega)$ and $\lambda_1^N(\pm d, \phi, \Omega) = \lambda_1^N(\pm d, 0+\phi, \Omega)$. Moreover, we denote the usual norms in $L^p(\Omega)$ with $p \geq 1$ and $C(\overline{\Omega})$ as $\|\phi\|_p = (\int_\Omega |\phi(x)|^p \mathrm{d}x)^{1/p}$ and $\|\phi\|_\infty = \max_{\overline{\Omega}} |\phi(x)|$, respectively.

It can be seen from Theorem 2.2(1) in [22] that the problem

$$-d_v \Delta v = mv - s(x)v^2, \quad x \in \Omega, \qquad \partial_{\mathbf{n}} v = 0, \quad x \in \partial\Omega \tag{10}$$

has a unique positive solution $v_m^{d_v}(x)$ when $m < \lambda_1^D(d_v, \Omega_0)$ and it has no positive solution when $m \geq \lambda_1^D(d_v, \Omega_0)$ where $s(x)$ satisfies (8). This means that model (7) has a semi-trivial solution $(0, v_m^{d_v})$ as $0 < m < \lambda_1^D(d_v, \Omega_0)$. Besides, model (7) has another semi-trivial solution $((r_0 - d)/a, 0)$ as $r_0 > d$, and trivial solution $(0, 0)$ for any parameter values.

With the above preparation, we can easily identify that

$$m < \lambda_1^D(d_v, \Omega_0) \tag{11}$$

is actually also a necessary condition for (7) and also (9) to have positive solutions, as stated in the following lemma.

**Lemma 2.1:** *Equation* (7) (*resp. Equation* (9) *as well*) *can have positive solution*(*s*) *only when* (11) *holds.*

***Proof:*** For the sake of contradiction, assume that there is such a positive solution $(u, v)$ to Equation (7) (resp. Equation (9)) for some $m \geq \lambda_1^D(d_v, \Omega_0)$. Then $v$ is a super solution for (10) and a subsolution of (10) can be $\varepsilon\phi$ where $\phi$ is a positive eigenfunction corresponding to $\lambda_1^D(d_v, \Omega_0)$. This implies that there is a positive solution to (10) while $m \geq \lambda_1^D(d_v, \Omega_0)$, a contradiction. ■

Due to this lemma, in the rest of this paper, we always assume (11) holds. In the rest of this section, we discuss the local bifurcation structures, the stability of local bifurcation solutions, and the global bifurcation structures of model (7). Then, we investigate existence of positive solutions and the asymptotic behavior of positive solutions for (7).

### 2.1. Local bifurcation structures and stability of local bifurcation solutions

In this subsection, by taking $r_0$ as a bifurcation parameter and using the local bifurcation theory proposed by Crandall and Rabinowitz [26], we explore the local bifurcation structures of positive solutions for model (7) emitting from the semi-trivial solutions $((r_0 - d)/a, 0)$ and $(0, v_m^{d_v})$ respectively, together with the stability of local bifurcation solutions. In order to accomplish these goals, it is necessary to present an *a priori* estimate of positive solutions for (7), which is shown in below.

**Theorem 2.1:** *Suppose that $a$, $M$ and $d$ are fixed positive constants and assume that $m < \lambda_1^D(d_v, \Omega_0)$. Then there exists a positive constant $W = W(M)$ such that any positive solution $(u, v)$ of model (7) with $d < r_0 < aM + d$ satisfies $\|u\|_\infty + \|v\|_\infty \le W$.*

***Proof:*** Assume that the conclusion is not true. Then there exists a sequence $\{r_0^n\}_{n=1}^{\infty}$ with $d < r_0^n < aM + d$ such that model (7) with $r_0 = r_0^n$ has a positive solution $(u_n, v_n)$ satisfying

$$\|u_n\|_\infty + \|v_n\|_\infty \to \infty \quad \text{as } n \to \infty. \tag{12}$$

For each $n$, we define the differential operator $L_n : C^2(\overline{\Omega}) \to C^0(\overline{\Omega})$ and function $\mathfrak{f}_n$ as

$$L_n u = -d_u \Delta u - \alpha \left(1 - \frac{2u_n(x)}{M}\right) \nabla v_n(x) \nabla u$$

and

$$\mathfrak{f}_n(x, u) = \alpha \Delta v_n(x) u \left(1 - \frac{u}{M}\right) + \frac{r_0^n}{1 + kv_n(x)} g_n(u) - du - au^2 \\ - \frac{p\rho(k) v_n(x)}{1 + q_1 u_n(x) + q_2 v_n(x)} u$$

respectively. Here $g_n(u)$ for $u \in [0, \infty)$ is such a function that satisfies (i) $g_n(u) \ge 0$ for $u \ge 0$ and $g_n(0) > 0$; (ii) $g_n(u_n) = u_n$; and (iii) $g_n(M) = 1 + kv_n$. There are many such functions, for example, one can set $g(0) = 1$ and use polynomial interpolation to obtain a quadratic function $g_n(u)$ satisfying the above properties. Then, for such a $g_n(u)$ defining $f_n(x, u)$, $u_n$ obviously solves $L_n u = f_n(u)$ with the boundary condition $\partial_{\mathbf{n}} u = 0$. Moreover, we can easily verify that

$$\mathfrak{f}_n(x, M) = 0 + \frac{r_0^n}{1 + kv_n(x)} g_n(M) - dM - aM^2 - \frac{p\rho(k) v_n(x)}{1 + q_1 u_n(x) + q_2 v_n(x)} M \\ \le (r_0^n - d - aM)M < 0 = L_n M, \quad x \in \Omega$$

and $\partial_{\mathbf{n}} M = 0$, meaning that $M$ is an upper solution of $L_n u = f_n(u)$ with $\partial_{\mathbf{n}} u = 0$. Also, it is obvious that $u = 0$ is a lower solution of $L_n u = f_n(u)$:

$$f_n(x,0) = \frac{r_0^n}{1+kv_n} g(0) > 0 = L_n 0, \quad x \in \Omega.$$

Therefore, by a sup-sub solution argument for elliptic equations, we know that $0 \le u_n \le M$ for all $x \in \overline{\Omega}$. Indeed, because $f_n(x,0) > 0$, the iteration starting from this lower solution $u = 0$ will generate an increasing sequence of *positive* functions which will converge to a *positive solution* $\hat{u}$ of $L_n u = f_n(x,u)$ satisfying $0 < \hat{u} \le M$. On the other hand, it is easy to see that $f_n(x,u)$ satisfies the lower-sided Lipschitz condition, and hence, $L_n u = f_n(x,u)$ can only have one positive solution, implying that $0 \le u_n = \hat{u}_n \le M$ for all $x \in \overline{\Omega}$. The proved boundedness of $u_n$, together with (12) leads to $\|v_n\|_\infty \to \infty$ as $n \to \infty$.

Set $\hat{v}_n = v_n/\|v_n\|_\infty$. From the second equation of (7), one sees that $\hat{v}_n$ satisfies

$$-d_v \Delta \hat{v}_n = \hat{v}_n \left( m - s(x)\|v_n\|_\infty \hat{v}_n + \frac{cp\rho(k)u_n}{1+q_1u_n+q_2v_n} \right), \quad x \in \Omega, \quad \partial_{\mathbf{n}} \hat{v}_n = 0, \quad x \in \partial\Omega. \tag{13}$$

Then $-d_v \Delta \hat{v}_n \le (m + cp\rho(k)/q_1)\hat{v}_n$, and then

$$d_v \int_\Omega |\nabla \hat{v}_n|^2 \,\mathrm{d}x + \int_\Omega \hat{v}_n^2 \,\mathrm{d}x \le \left( m + \frac{cp\rho(k)}{q_1} + 1 \right) \int_\Omega \hat{v}_n^2 \,\mathrm{d}x \le \left( m + \frac{cp\rho(k)}{q_1} + 1 \right) |\Omega|,$$

which implies that $\{\hat{v}_n\}_{n=1}^\infty$ is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence of $\{\hat{v}_n\}_{n=1}^\infty$, still denoted by itself, such that $\hat{v}_n$ converges to some $\hat{v}$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\|\hat{v}_n\|_\infty = 1$, we can set $\hat{v}_n \to \hat{v}$ in $L^p(\Omega)$ for any $p > 1$. Clearly, $0 \le \hat{v} \le 1$ in $\overline{\Omega}$. Further, by applying the technique of Proposition 4.1 in [22], we conclude $\hat{v} \not\equiv 0$ in $\overline{\Omega}$.

Next, we will deduce $\hat{v} \equiv 0$ in $\overline{\Omega}$ below to contradict the above. Firstly, we clarify that $\hat{v} = 0$ almost everywhere in $\Omega \backslash \overline{\Omega}_0$. Multiplying both sides of (13) by $\hat{v}_n$ and integrating over $\Omega$, one has

$$\|v_n\|_\infty \int_\Omega s(x)\hat{v}_n^3 \,\mathrm{d}x = \int_\Omega \left( \hat{v}_n^2 (m + \frac{cp\rho(k)u_n}{1+q_1u_n+q_2v_n}) - d_v|\nabla \hat{v}_n|^2 \right) \mathrm{d}x. \tag{14}$$

It is evident that the right hand side of (14) is uniformly bounded, then

$$0 = \lim_{n\to\infty} \int_\Omega s(x)\hat{v}_n^3 \,\mathrm{d}x = \int_\Omega s(x)\hat{v}^3 \,\mathrm{d}x = \int_{\Omega \backslash \overline{\Omega}_0} s(x)\hat{v}^3 \,\mathrm{d}x.$$

Since $s(x) > 0$ in $\Omega \backslash \overline{\Omega}_0$, $\hat{v} = 0$ almost everywhere in $\Omega \backslash \overline{\Omega}_0$. Now on $\Omega_0$, we have $\hat{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ since $\partial\Omega_0$ is smooth enough. In the following, we prove $\hat{v} \equiv 0$ in $\overline{\Omega}_0$. Multiplying both sides of (13) by $\varphi \in C_0^\infty(\Omega_0)$ and integrating over $\Omega_0$, we obtain

$$d_v \int_{\Omega_0} \nabla \hat{v}_n \nabla \varphi \, dx = m \int_{\Omega_0} \hat{v}_n \varphi \, dx + \int_{\Omega_0} \frac{cp\rho(k) u_n \hat{v}_n}{1 + q_1 u_n + q_2 v_n} \varphi \, dx.$$

On account of

$$\left| \int_{\Omega_0} \frac{cp\rho(k) u_n \hat{v}_n}{1 + q_1 u_n + q_2 v_n} \varphi \, dx \right| = \left| \int_{\Omega_0} \frac{cp\rho(k) u_n \hat{v}_n}{1 + q_1 u_n + q_2 \hat{v}_n \|v_n\|_\infty} \varphi \, dx \right|$$
$$\leq \frac{cp\rho(k) \|u_n\|_\infty}{q_2 \|v_n\|_\infty} \|\varphi\|_{L^1(\Omega_0)} \to 0$$

as $n \to \infty$, $\hat{v}|_{\Omega_0} \geq 0$ is a weak solution of

$$-d_v \Delta \hat{v} = m \hat{v}, \quad x \in \Omega_0.$$

Since we have shown that $\hat{v} = 0$ almost everywhere in $\Omega \backslash \overline{\Omega}_0$, $\hat{v}$ satisfies the homogeneous Dirichlete boundary condition on $\partial \Omega_0$. Now by the maximum principle, we conclude that $\hat{v} \equiv 0$ in $\overline{\Omega}_0$. Combining the above, we have shown that $\hat{v} \equiv 0$ in $\overline{\Omega}$, contradicting $\hat{v} \not\equiv 0$ in $\overline{\Omega}$. This contradiction completes the proof of theorem. ■

Now, we study the local bifurcation phenomenon of model (7). Firstly, consider the function

$$\varsigma(r_0) = \lambda_1^N \left( -d_u, \alpha \nabla v_m^{d_v} \nabla + \alpha \Delta v_m^{d_v} + \frac{r_0}{1 + k v_m^{d_v}} - \frac{p\rho(k) v_m^{d_v}}{1 + q_2 v_m^{d_v}}, \Omega \right)$$

with $0 < m < \lambda_1^D(d_v, \Omega_0)$ and $r_0 \in (d, \infty)$. Since $1/(1 + k v_m^{d_v}) > 0$, $\varsigma(r_0)$ is a continuous and strictly increasing function with respect to $r_0$. By using the argument similar to that of given in [27], Theorem 2.1, the standard regularity theory of elliptic equations, the embedding theorems and the assumption $\partial\Omega \in C^{2+\epsilon}, 0 < \epsilon < 1$, we get that there exists a positive constant $M^\circ$ such that for any positive solution $(u, v) \in C^2(\Omega) \times C^2(\Omega)$, there holds $\max_{\overline{\Omega}}\{\|\nabla u\|_{C^1}, \|\nabla v\|_{C^1}, \|\Delta v\|_{C^1}\} \leq M^\circ$. Thus, we can assume that there exists a $\alpha$ such that $\varsigma(d) < d$, then we can derive that no matter $\varsigma(d) < 0$ or $\varsigma(d) = 0$ or $\varsigma(d) > 0$, there is always a unique $r_0^{**} \in (d, \infty)$ such that

$$\varsigma(r_0^{**}) = \lambda_1^N \left( -d_u, \alpha \nabla v_m^{d_v} \nabla + \alpha \Delta v_m^{d_v} + \frac{r_0^{**}}{1 + k v_m^{d_v}} - \frac{p\rho(k) v_m^{d_v}}{1 + q_2 v_m^{d_v}}, \Omega \right) = d. \tag{15}$$

Figure 1 draws the curves of $\varsigma$ with respect to $r_0$ when $\varsigma(d) < 0$, $\varsigma(d) = 0$ and $\varsigma(d) > 0$, respectively. From this figure, the existence of $r_0^{**}$ is verified.

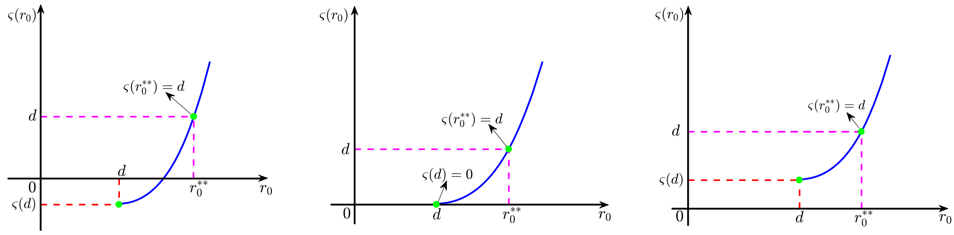
For $p > N$, let $X = W_{\mathbf{n}}^{2,p}(\Omega) \times W_{\mathbf{n}}^{2,p}(\Omega)$, $Y = L^p(\Omega) \times L^p(\Omega)$, where

$$W_{\mathbf{n}}^{2,p}(\Omega) = \{u \in W^{2,p}(\Omega) : \partial_{\mathbf{n}} u = 0, \ x \in \partial\Omega\}.$$

The two semi-trivial solution curves, denoted by $\Theta_u$ and $\Theta_v$ respectively, are given by

$$\Theta_u = \{(r_0, u, v) = (r_0, (r_0 - d)/a, 0) : r_0 > d\}, \quad \Theta_v = \{(r_0, u, v) = (r_0, 0, v_m^{d_v}) : r_0 > 0\}.$$

Next, we study the local bifurcation branches of positive solutions for model (7) bifurcating from $\Theta_u$ and $\Theta_v$, respectively. The results read as follows.




(a) $\varsigma(d) < 0$ (b) $\varsigma(d) = 0$ (c) $\varsigma(d) > 0$

**Figure 1.** Curves of $\varsigma$ with respect to $r_0$, where $\varsigma(d) < 0$ in (a), $\varsigma(d) = 0$ in (b) and $\varsigma(d) > 0$ in (c).

**Theorem 2.2:** *Let $r_0^* = d - am/(mq_1 + cp\rho(k))$ and $r_0^{**}$ be defined in (15). Then the followings hold.*

(i) *If $-cp\rho(k)/q_1 < m < 0$, then a branch of positive solutions for model (7) bifurcates from $\Theta_u$ if and only if $r_0 = r_0^*$; Moreover, all positive solutions of (7) near $(r_0^*, (r_0^* - d)/a, 0) \in \mathbb{R} \times X$ are on a smooth curve $\Theta_1$, which is given by*

$$\Theta_1 = \{(r_0(\vartheta), u(\vartheta), v(\vartheta)) = (r_0^* + \vartheta r_0'(0) + o(|\vartheta|),\ (r_0^* - d)/a + \vartheta\phi_* + o(|\vartheta|), \vartheta\psi_* + o(|\vartheta|)) : 0 < \vartheta < \vartheta_1\},$$

*where $\vartheta_1 > 0$ is a certain number, $\phi_*$ and $\psi_*$ are definite functions in $W_{\mathbf{n}}^{2,p}(\Omega)$, and $r_0'(0)$ is defined by*

$$r_0'(0) = \frac{\int_\Omega \left(s(x) + \frac{cpq_2\rho(k)(r_0^*-d)}{a(1+q_1(r_0^*-d)/a)^2}\right)\psi_*^3\,dx - \int_\Omega \frac{cp\rho(k)\phi_*\psi_*^2}{(1+q_1(r_0^*-d)/a)^2}\,dx}{\int_\Omega \frac{cp\rho(k)\psi_*^2}{a(1+q_1(r_0^*-d)/a)^2}\,dx}.$$

*Additionally, the bifurcation of $\Theta_1$ at $(r_0^*, (r_0^* - d)/a, 0)$ is supercritical if $q_2 > \tilde{q}$ and subcritical if $q_2 < \tilde{q}$, where $\tilde{q}$ is given by*

$$\tilde{q} = \frac{\int_\Omega \frac{cp\rho(k)\phi_*\psi_*^2}{(1+q_1(r_0^*-d)/a)^2}\,dx - \int_\Omega s(x)\psi_*^3\,dx}{\int_\Omega \frac{cp\rho(k)(r_0^*-d)}{a(1+q_1(r_0^*-d)/a)^2}\psi_*^3\,dx}.$$

(ii) *If $0 < m < \lambda_1^D(d_v, \Omega_0)$, then a branch of positive solutions for model (7) bifurcates from $\Theta_v$ if and only if $r_0 = r_0^{**}$. Moreover, all positive solutions of (7) near $(r_0^{**}, 0, v_m^{d_v}) \in \mathbb{R} \times X$ are on a smooth curve $\Theta_2$, which is given by*

$$\Theta_2 = \{(r_0(\vartheta), u(\vartheta), v(\vartheta)) = (r_0^{**} + \vartheta\hat{r}_0'(0) + o(|\vartheta|),\ \vartheta\phi_{**} + o(|\vartheta|), v_m^{d_v} + \vartheta\psi_{**} + o(|\vartheta|)) : 0 < \vartheta < \vartheta_2\},$$

*where $\vartheta_2 > 0$ is a certain number, $\phi_{**}$ and $\psi_{**}$ are definite functions in $W^{2,p}_{\mathbf{n}}(\Omega)$, and $\hat{r}'_0(0)$ is defined by*

$$\hat{r}'_0(0) = \int_\Omega \left( \left( \frac{2\alpha}{M} \nabla v_m^{d_v} \nabla \phi_{**} - \alpha \Delta \psi_{**} + \frac{k r_0^{**}}{(1 + k v_m^{d_v})^2} \psi_{**} + \frac{p\rho(k)\psi_{**}}{(1 + q_2 v_m^{d_v})^2} \right) \phi_{**}^2 + \left( \frac{\alpha}{M} \Delta v_m^{d_v} + a - \frac{p q_1 \rho(k) v_m^{d_v}}{(1 + q_2 v_m^{d_v})^2} \right) \phi_{**}^3 - \alpha \phi_{**} \nabla \phi_{**} \nabla \psi_{**} \right) dx \Big/ \int_\Omega \frac{\phi_{**}^2}{1 + k v_m^{d_v}} dx.$$

*Additionally, the bifurcation of $\Theta_2$ at $(r_0^{**}, 0, v_m^{d_v})$ is supercritical if $a > \tilde{a}$ and subcritical if $a < \tilde{a}$, where $\tilde{a}$ is given by*

$$\tilde{a} = \int_\Omega \left( \left( \frac{p q_1 \rho(k) v_m^{d_v}}{(1 + q_2 v_m^{d_v})^2} - \frac{\alpha}{M} \Delta v_m^{d_v} \right) \phi_{**}^3 + \alpha \phi_{**} \nabla \phi_{**} \nabla \psi_{**} - \left( \frac{2\alpha}{M} \nabla v_m^{d_v} \nabla \phi_{**} - \alpha \Delta \psi_{**} + \frac{k r_0^{**}}{(1 + k v_m^{d_v})^2} \psi_{**} + \frac{p\rho(k)\psi_{**}}{(1 + q_2 v_m^{d_v})^2} \right) \phi_{**}^2 \right) dx \Big/ \int_\Omega \phi_{**}^3 \, dx.$$

***Proof:*** We only give the detailed proof of (ii) since (i) can be proved similarly. Let $w = v - v_m^{d_v}$ and define $\mathcal{L}_1 : \mathbb{R} \times X \to Y$ as $\mathcal{L}_1(r_0, u, w) = (f_1, f_2)^T$ with

$$f_1 = d_u \Delta u + \alpha \left(1 - \frac{2u}{M}\right) \nabla u \nabla (w + v_m^{d_v}) + \alpha u \left(1 - \frac{u}{M}\right) \Delta (w + v_m^{d_v}) + \left( \frac{r_0}{1 + k(w + v_m^{d_v})} - d - au - \frac{p\rho(k)(w + v_m^{d_v})}{1 + q_1 u + q_2 (w + v_m^{d_v})} \right) u$$

and

$$f_2 = w + \left( m - s(x)(w + v_m^{d_v}) + \frac{c p \rho(k) u}{1 + q_1 u + q_2 (w + v_m^{d_v})} \right) (w + v_m^{d_v}).$$

Clearly, $\mathcal{L}_1(r_0, u, w) = 0$ if and only if $(u, w + v_m^{d_v})$ is a solution of model (7). Hence, to find the positive solutions branch of (7) near $(r_0, u, v) = (r_0, 0, v_m^{d_v})$, we only need to find the positive solutions branch of $\mathcal{L}_1(r_0, u, w) = 0$ near $(r_0, u, w) = (r_0, 0, 0)$.

Denote by $\mathcal{L}_{1(u,w)}(r_0, u, w)$ the Fréchet derivative of $\mathcal{L}_1$ with respect to $(u, w)$. Then some straightforward calculations yield

$$\mathcal{L}_{1(u,w)}(r_0, u, w)(\phi, \psi)^T = \begin{pmatrix} \mathfrak{B}_1(\phi, \psi) + \mathfrak{B}_2(\phi, \psi) \\ d_v \Delta \psi + \mathfrak{B}_3(\phi, \psi) \end{pmatrix},$$

where

$$
\begin{aligned}
\mathfrak{B}_1(\phi,\psi) &= \frac{r_0\phi}{1+k(w+v_m^{d_v})} - d\phi - 2au\phi - p\rho(k)\frac{(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^2}\phi \\
&\quad - \frac{kr_0u\psi}{(1+k(w+v_m^{d_v}))^2} - \frac{p\rho(k)u(1+q_1u)}{(1+q_1u+q_2(w+v_m^{d_v}))^2}\psi, \\
\mathfrak{B}_2(\phi,\psi) &= d_u\Delta\phi - \frac{2\alpha\phi}{M}\nabla u\nabla(w+v_m^{d_v}) + \alpha\left(1-\frac{2u}{M}\right)\nabla(w+v_m^{d_v})\nabla\phi \\
&\quad + \alpha\left(1-\frac{2u}{M}\right)\phi\Delta(w+v_m^{d_v}) + \alpha\left(1-\frac{2u}{M}\right)\nabla u\nabla\psi + \alpha u\left(1-\frac{u}{M}\right)\Delta\psi, \\
\mathfrak{B}_3(\phi,\psi) &= \frac{cp\rho(k)(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^2}\phi \\
&\quad + \left(m - 2s(x)(w+v_m^{d_v}) + \frac{cp\rho(k)u(1+q_1u)}{(1+q_1u+q_2(w+v_m^{d_v}))^2}\right)\psi
\end{aligned}
$$

with $\phi,\psi \in W_{\mathbf{n}}^{2,p}(\Omega)$.

By the Krein-Rutman theorem, we obtain that the second equation of $\mathcal{L}_{1(u,w)}(r_0,0,0)(\phi,\psi)^T = (0,0)^T$ has a solution $\psi > 0$ if and only if $r_0 = r_0^{**}$. Next, we verify that $(r_0^{**},0,v_m^{d_v})$ is the only bifurcation point of positive solutions for model (7) bifurcating from $\Theta_u$.

Set $\Phi = (\phi,\psi)^T$. Then $\mathcal{L}_{1(u,w)}(r_0,u,w)(\phi,\psi)^T$ can be rewritten as

$$
\mathcal{L}_{1(u,w)}(r_0,u,w)(\phi,\psi)^T = \mathbf{A}_0(\alpha,\Phi)\Delta\Phi + \mathbf{A}_1(\alpha,\Phi)\nabla\Phi + \mathbf{A}_2(\alpha,\Phi),
$$

where

$$
\begin{aligned}
\mathbf{A}_0(\alpha,\Phi) &= \begin{pmatrix} d_u & \alpha\left(1-\frac{aw+r_0-d}{aM}\right)\left(w+\frac{r_0-d}{a}\right) \\ 0 & d_v \end{pmatrix}, \\
\mathbf{A}_1(\alpha,\Phi) &= \begin{pmatrix} \alpha\left(1-\frac{2(aw+r_0-d)}{aM}\right)\nabla v & \alpha\left(1-\frac{2(aw+r_0-d)}{aM}\right)\nabla w \\ 0 & 0 \end{pmatrix}, \\
\mathbf{A}_2(\alpha,\Phi) &= \begin{pmatrix} -\frac{2\alpha\phi}{M}\nabla w\nabla v + \alpha\left(1-\frac{2(aw+r_0-d)}{aM}\right)\phi\Delta v + \mathfrak{B}_1(\phi,\psi) \\ \mathfrak{B}_3(\phi,\psi) \end{pmatrix}.
\end{aligned}
$$

Obviously, $\mathrm{tr}\mathbf{A}_0(\alpha,\Phi) > 0$ and $\det\mathbf{A}_0(\alpha,\Phi) > 0$ for any $\Phi \in X$, this implies that the operator $\mathcal{L}_{1(u,w)}(r_0,u,w)$ is elliptic. Furthermore, by case 2 with $N = 1$ of remark 2.5.5 in Shi and Wang [28], one can also obtain that $\mathcal{L}_{1(u,w)}(r_0,u,w)$ is strongly elliptic and satisfies the Agmon's condition for angles $\theta \in [-\pi/2,\pi/2]$. Therefore, in view of Theorem 3.3 and Remark 3.4 in [28], $\mathcal{L}_{1(u,w)}(r_0,u,w)(\phi,\psi)^T$ is the Fredholm operator with zero index.

Next, we claim $\dim(\ker(\mathcal{L}_{1(u,w)}(r_0^{**},0,0))) = \mathrm{codim}(\mathrm{R}(\mathcal{L}_{1(u,w)}(r_0^{**},0,0))) = 1$. Actually, one can easy to get $\mathrm{Ker}(\mathcal{L}_{1(u,w)}(r_0^{**},\ 0,0)) = \mathrm{span}\{(\phi_{**},\psi_{**})\}$ with $\phi_{**} > 0$ and

$(\phi_{**}, \psi_{**})$ satisfying

$$\begin{cases} d_u \Delta \phi_{**} + \alpha \nabla v_m^{d_v} \nabla \phi_{**} + \alpha \phi_{**} \Delta v_m^{d_v} + \dfrac{r_0^{**}}{1+kv_m^{d_v}} \phi_{**} - d\phi_{**} \\ -\dfrac{p\rho(k) v_m^{d_v}}{1+q_2 v_m^{d_v}} \phi_{**} = 0, & x \in \Omega, \\ d_v \Delta \psi_{**} + m\psi_{**} - 2s(x) v_m^{d_v} \psi_{**} + \dfrac{cp\rho(k) v_m^{d_v}}{1+q_2 v_m^{d_v}} \phi_{**} = 0, & x \in \Omega, \\ \partial_{\mathbf{n}} \phi_{**} = \partial_{\mathbf{n}} \psi_{**} = 0, & x \in \partial\Omega. \end{cases} \tag{16}$$

From (10), one sees that $(d_v \Delta + m - 2s(x) v_m^{d_v})^{-1}$ is a negative operator, thus, the second equation of (16) means

$$\psi_{**} = -(d_v \Delta + m - 2s(x) v_m^{d_v})^{-1} (cp\rho(k) v_m^{d_v}/(1+q_2 v_m^{d_v}) \phi_{**}) > 0.$$

So, $\dim(\ker(\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0))) = 1$. Further, since $\mathcal{L}_{1(u,w)}(r_0, u, w)(\phi, \psi)^T$ is the Fredholm operator with zero index, $\mathrm{codim}(\mathrm{R}(\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0))) = 1$.

On the other side, by some calculations, we have

$$\mathcal{L}_{1r_0(u,w)}(r_0, u, w)(\phi, \psi)^T = \left( \frac{\phi}{1+k(w+v_m^{d_v})} - \frac{ku\psi}{(1+k(w+v_m^{d_v}))^2}, 0 \right)^T.$$

Then

$$\mathcal{L}_{1r_0(u,w)}(r_0^{**}, 0, 0)(\phi_{**}, \psi_{**})^T = \left( \frac{\phi_{**}}{1+k(w+v_m^{d_v})} - \frac{ku\psi_{**}}{(1+k(w+v_m^{d_v}))^2}, 0 \right)^T.$$

We now prove

$$\mathcal{L}_{1r_0(u,w)}(r_0^{**}, 0, 0)(\phi_{**}, \psi_{**})^T \notin \mathrm{R}(\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0)).$$

Assume $\mathcal{L}_{1r_0(u,w)}(r_0^{**}, 0, 0)(\phi_{**}, \psi_{**})^T \in \mathrm{R}(\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0))$. Then there exists a nontrivial solution $(u, w)$ such that $\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0)(u, w)^T = \mathcal{L}_{1r_0(u,w)}(r_0^{**}, 0, 0)(\phi_{**}, \psi_{**})^T$, that is,

$$\begin{cases} d_u \Delta u + \alpha \nabla v_m^{d_v} \nabla u + \alpha u \Delta v_m^{d_v} + \dfrac{r_0^{**}}{1+kv_m^{d_v}} u - du - \dfrac{p\rho(k) v_m^{d_v}}{1+q_2 v_m^{d_v}} u = \\ \dfrac{\phi_{**}}{1+k(w+v_m^{d_v})} - \dfrac{ku\psi_{**}}{(1+k(w+v_m^{d_v}))^2}, & x \in \Omega, \\ d_v \Delta w + mw - 2s(x) v_m^{d_v} w + \dfrac{cp\rho(k) v_m^{d_v}}{1+q_2 v_m^{d_v}} u = 0, & x \in \Omega, \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0, & x \in \partial\Omega. \end{cases} \tag{17}$$

From (16) and the definition of $r_0^{**}$, we know that the determinant of the coefficient matrix on the left-hand of (17) is zero. Hence, (17) is impossible, and therefore,

$\mathcal{L}_{1r_0(u,w)}(r_0^{**},0,0)(\phi_{**},\psi_{**})^T \notin \mathrm{R}(\mathcal{L}_{1(u,w)}(r_0^{**},0,0))$. Then from the results in [26], we conclude that $(r_0^{**},0,v_m^{d_v})$ is the only bifurcation point of positive solutions for model (7) bifurcating from $\Theta_v$, and the positive solutions set of model (7) near $(r_0^{**},0,v_m^{d_v})$ is on a smooth curve

$$\begin{aligned}\Theta_2 = \{(r_0(\vartheta),u(\vartheta),v(\vartheta)) = (r_0^{**}+\vartheta r_0'(0)+o(|\vartheta|),\vartheta\phi_{**}+o(|\vartheta|),v_m^{d_v}+\vartheta\psi_{**}\\ +o(|\vartheta|)) : 0<\vartheta<\vartheta_2.\}.\end{aligned}$$

The remaining derives $\hat{r}_0'(0)$. Set $\hat{\mathcal{L}}_1(r_0,u,w) := \mathcal{L}_{1(u,w)}(r_0,u,w)(\phi,\psi)^T$. Then $\hat{\mathcal{L}}_{1(u,w)}(r_0,u,w)(\phi,\psi)^T = (C,D)^T$ with

$$\begin{aligned}
C = &-\frac{4\alpha\phi}{M}\nabla(w+v_m^{d_v})\nabla\phi - \frac{2\alpha\phi^2}{M}\Delta(w+v_m^{d_v}) - 2a\phi^2\\
&+\frac{2pq_1\rho(k)(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi^2\\
&-\frac{4\alpha\phi}{M}\nabla u\nabla\psi + 2\alpha\left(1-\frac{2u}{M}\right)\nabla\phi\nabla\psi + 2\alpha\left(1-\frac{2u}{M}\right)\phi\Delta\psi\\
&-\frac{2kr_0}{(1+k(w+v_m^{d_v}))^2}\phi\psi\\
&-p\rho(k)\frac{(1+2q_1u)(1+q_1u+q_2(w+v_m^{d_v}))-2q_1u(1+q_1u)}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi\psi\\
&-p\rho(k)\frac{(1+2q_2(w+v_m^{d_v}))(1+q_1u+q_2(w+v_m^{d_v})) - 2q_2(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi\psi\\
&+\frac{2k^2r_0}{(1+k(w+v_m^{d_v}))^3}u\psi^2 + 2pq_2\rho(k)\frac{u(1+q_1u)}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\psi^2,\\
D = &-\frac{2cpq_1\rho(k)(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi^2 - 2s(x)\psi^2\\
&-2cpq_2\rho(k)\frac{u(1+q_2u)}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\psi^2\\
&+cp\rho(k)\frac{(1+2q_1u)(1+q_1u+q_2(w+v_m^{d_v}))-2q_1u(1+q_1u)}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi\psi\\
&+cp\rho(k)\frac{(1+2q_2(w+v_m^{d_v}))(1+q_1u+q_2(w+v_m^{d_v})) - 2q_2(w+v_m^{d_v})(1+q_2(w+v_m^{d_v}))}{(1+q_1u+q_2(w+v_m^{d_v}))^3}\phi\psi.
\end{aligned}$$

It follows from [29] that $\hat{r}_0'(0)$ is given by

$$\hat{r}_0'(0) = -\frac{\langle \hat{\mathcal{L}}_{1(u,w)}(r_0^{**},0,0)(\phi_{**},\psi_{**})^T, l_1\rangle}{2\langle \mathcal{L}_{1r_0(u,w)}(r_0^{**},0,0)(\phi_{**},\psi_{**})^T, l_1\rangle}$$
$$= \int_\Omega \left(\left(\frac{2\alpha}{M}\nabla v_m^{d_v}\nabla\phi_{**} - \alpha\Delta\psi_{**} + \frac{kr_0^{**}}{(1+kv_m^{d_v})^2}\psi_{**} + \frac{p\rho(k)\psi_{**}}{(1+q_2v_m^{d_v})^2}\right)\phi_{**}^2\right.$$
$$\left. + \left(\frac{\alpha}{M}\Delta v_m^{d_v} + a - \frac{pq_1\rho(k)v_m^{d_v}}{(1+q_2v_m^{d_v})^2}\right)\phi_{**}^3 - \alpha\phi_{**}\nabla\phi_{**}\nabla\psi_{**}\right)\mathrm{d}x \Big/ \int_\Omega \frac{\phi_{**}^2}{1+kv_m^{d_v}}\mathrm{d}x$$

with $l_1$ being a linear functional in $Y$ defined as $\langle (f,g)^T, l_1\rangle = \int_\Omega f(x)\phi_{**}\mathrm{d}x$. Additionally, because $\hat{r}_0'(0) > 0$ if $a > \tilde{a}$ and $\hat{r}_0'(0) < 0$ if $a < \tilde{a}$, the bifurcation of $\Theta_2$ at $(r_0^{**}, 0, v_m^{d_v})$ is supercritical if $a > \tilde{a}$ and subcritical if $a < \tilde{a}$. The proof is completed. ■

Notice that by the Sobolev embedding theorems, we can verify that if the space $X = W_{\mathbf{n}}^{2,p}(\Omega) \times W_{\mathbf{n}}^{2,p}(\Omega)$ is replaced by $\mathcal{C} = C_{\mathbf{n}}^1(\Omega) \times C_{\mathbf{n}}^1(\Omega)$ with $C_{\mathbf{n}}^1(\Omega) = \{u \in C^1(\Omega) : u(x) = 0 \text{ for } x \in \partial\Omega\}$, then the conclusions in Theorem 2.2 also hold. Next, the stability of local bifurcation solutions of model (7) obtained in Theorem 2.2 is investigated.

**Theorem 2.3:** *The following statements hold.*

(i) *If* $-cp\rho(k)/q_1 < m < 0$, *then there exists a small positive number* $\vartheta_1$ *such that the positive solution* $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ *of model* (7) *bifurcating from* $(r_0^*, (r_0^* - d)/a, 0)$ *is non-degenerate for* $\vartheta \in (0, \vartheta_1)$. *Moreover,* $(u(\vartheta), v(\vartheta))$ *is locally asymptotically stable if* $q_2 > \tilde{q}$ *and unstable if* $q_2 < \tilde{q}$.

(ii) *If* $0 < m < \lambda_1^D(d_v, \Omega_0)$, *then there exists a small positive number* $\vartheta_2$ *such that the positive solution* $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ *of model* (7) *bifurcating from* $(r_0^{**}, 0, v_m^{d_v})$ *is non-degenerate for* $\vartheta \in (0, \vartheta_2)$. *Moreover, if* $\lim_{\vartheta\to 0^+} \nabla v(\vartheta)/\vartheta \neq \infty$, *then* $(u(\vartheta), v(\vartheta))$ *is locally asymptotically stable if* $a > \tilde{a}$ *and unstable if* $a < \tilde{a}$.

***Proof:*** We only provide detailed proof of (ii) since (i) can be proved similarly. Firstly, according to the proof of Theorem 2.2(ii), one sees that there exists $\vartheta_2 > 0$ such that (7) admits the positive solution $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ emitting from $(r_0^{**}, 0, v_m^{d_v})$ with $\vartheta \in (0, \vartheta_2)$. Linearizing (7) at $(u, v) = (u(\vartheta), v(\vartheta))$ with $r_0 = r_0(\vartheta)$, we have

$$\mathcal{X}(\vartheta)\begin{pmatrix}\phi(\vartheta)\\ \psi(\vartheta)\end{pmatrix} = \beta(\vartheta)\begin{pmatrix}\phi(\vartheta)\\ \psi(\vartheta)\end{pmatrix} \quad \text{with } \mathcal{X}(\vartheta) = \begin{pmatrix}\mathcal{X}_1(\vartheta) & \mathcal{X}_2(\vartheta)\\ \mathcal{X}_3(\vartheta) & \mathcal{X}_4(\vartheta)\end{pmatrix}, \tag{18}$$

$$\mathcal{X}_1(\vartheta) = -d_u\Delta - \alpha\left(-\frac{2}{M}\nabla u(\vartheta)\nabla v(\vartheta) + \left(1 - \frac{2u(\vartheta)}{M}\right)\nabla v(\vartheta)\nabla\right.$$
$$\left. + \left(1 - \frac{2u(\vartheta)}{M}\right)\Delta v(\vartheta)\right)$$
$$- \left(\frac{r_0(\vartheta)}{1+kv(\vartheta)} - d - 2au(\vartheta) - \frac{p\rho(k)v(\vartheta)(1+q_2v(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\right),$$

$$\mathcal{X}_2(\vartheta) = -\alpha\left(\left(1-\frac{u(\vartheta)}{M}\right)u(\vartheta)\Delta + \left(1-\frac{2u(\varsigma)}{M}\right)\nabla u(\vartheta)\nabla\right)$$
$$+\left(\frac{kr_0(\vartheta)u(\vartheta)}{(1+kv(\vartheta))^2} + \frac{p\rho(k)u(\vartheta)(1+q_1u(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\right),$$
$$\mathcal{X}_3(\vartheta) = -\frac{cp\rho(k)v(\vartheta)(1+q_2v(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}, \quad \mathcal{X}_4(\vartheta) = -d_v\Delta - m + 2s(x)v(\vartheta)$$
$$-\frac{cp\rho(k)u(\vartheta)(1+q_1u(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}.$$

Owing to $(r_0(\vartheta), u(\vartheta), v(\vartheta)) \to (r_0^{**}, 0, v_m^{d_v})$ as $\vartheta \to 0^+$, we get $\mathcal{X}(\vartheta) \to \mathcal{X}_0$ as $\vartheta \to 0^+$, where $\mathcal{X}_0$ is denoted as

$$\mathcal{X}_0 = \begin{pmatrix} -d_u\Delta - \alpha\nabla v_m^{d_v}\nabla - \alpha\Delta v_m^{d_v} - \dfrac{r_0^{**}}{1+kv_m^{d_v}} + d + \dfrac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} & 0 \\ -\dfrac{cp\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} & -d_v\Delta - m + 2s(x)v_m^{d_v} \end{pmatrix}.$$

It is easy to see that 0 is the least eigenvalue of $\mathcal{X}_0$, and the corresponding positive eigenfunction is $(\phi_{**}, \psi_{**})$, which is defined in the proof of Theorem 2.2(ii). Furthermore, the real parts of all the other eigenvalues of $\mathcal{X}_0$ are positive and bounded away from 0. Thus, the perturbation theory of linear operators [30] means that $\mathcal{X}(\vartheta)$ has a unique eigenvalue $\beta(\vartheta)$ with eigenfunction $(\varphi(\vartheta), \psi(\vartheta))$ when $\vartheta > 0$ is small enough, where $\beta(\vartheta)$ and $(\phi(\vartheta), \psi(\vartheta))$ satisfy $\beta(\vartheta) \to 0$ and $(\phi(\vartheta), \psi(\vartheta)) \to (\phi_{**}, \psi_{**})$ as $\vartheta \to 0^+$. Additionally, all other eigenvalues of $\mathcal{X}(\vartheta)$ have positive real parts, which are apart from 0. Therefore, there exists a small positive number $\vartheta_2$ such that the positive solution $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ of model (7) bifurcating from $(r_0^{**}, 0, v_m^{d_v})$ is non-degenerate for $\vartheta \in (0, \vartheta_2)$.

Now, we investigate the stability of positive solution $(u(\vartheta), v(\vartheta))$. From (18), one sees that $\phi(\vartheta)$ satisfies

$$\beta(\vartheta)\phi(\vartheta) = -d_u\Delta\phi(\vartheta) - \alpha\left(-\frac{2}{M}\nabla u(\vartheta)\nabla v(\vartheta) + \left(1-\frac{2u(\vartheta)}{M}\right)\nabla v(\vartheta)\nabla\right.$$
$$\left.+\left(1-\frac{2u(\vartheta)}{M}\right)\Delta v(\vartheta)\right)\phi(\vartheta)$$
$$-\left(\frac{r_0(\vartheta)}{1+kv(\vartheta)} - d - 2au(\vartheta) - \frac{p\rho(k)v(\vartheta)(1+q_2v(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\right)\phi(\vartheta)$$
$$-\alpha\left(\left(1-\frac{u(\vartheta)}{M}\right)u(\vartheta)\Delta + \left(1-\frac{2u(\vartheta)}{M}\right)\nabla u(\vartheta)\nabla\right)\psi(\vartheta)$$

$$+\left(\frac{kr_0(\vartheta)u(\vartheta)}{(1+kv(\vartheta))^2}+\frac{p\rho(k)u(\vartheta)(1+q_1u(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\right)\psi(\vartheta). \tag{19}$$

Multiplying both sides of (19) by $u(\vartheta)$ and integrating over $\boldsymbol{\Omega}$, and then applying the equation corresponding to $u(\vartheta)$, we obtain

$$\begin{aligned}
&\beta(\vartheta)\int_\Omega \phi(\vartheta)u(\vartheta)\,\mathrm{d}x\\
&=\alpha\int_\Omega\Bigg(\left(1-\frac{2u(\vartheta)}{M}\right)\phi(\vartheta)\nabla u(\vartheta)\nabla v(\vartheta)+u(\vartheta)\left(1-\frac{u(\vartheta)}{M}\right)\\
&\times\phi(\vartheta)\Delta v(\vartheta)+\frac{2}{M}\phi(\vartheta)u(\vartheta)\nabla u(\vartheta)\nabla v(\vartheta)\\
&-\left(1-\frac{2u(\vartheta)}{M}\right)u(\vartheta)\nabla v(\vartheta)\nabla\phi(\vartheta)-\left(1-\frac{2u(\vartheta)}{M}\right)u(\vartheta)\phi(\vartheta)\Delta v(\vartheta)\\
&-\left(1-\frac{u(\vartheta)}{M}\right)u^2(\vartheta)\Delta\psi(\vartheta)-\left(1-\frac{2u(\vartheta)}{M}\right)u(\vartheta)\nabla u(\vartheta)\nabla\psi(\vartheta)\Bigg)\,\mathrm{d}x\\
&+\int_\Omega\Bigg(au^2(\vartheta)\phi(\vartheta)-\frac{pq_1\rho(k)u^2(\vartheta)v(\vartheta)\phi(\vartheta)}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\\
&+\left(\frac{kr_0(\vartheta)}{(1+kv(\vartheta))^2}+\frac{p\rho(k)(1+q_1u(\vartheta))}{(1+q_1u(\vartheta)+q_2v(\vartheta))^2}\right)u^2(\vartheta)\psi(\vartheta)\Bigg)\,\mathrm{d}x.
\end{aligned} \tag{20}$$

Dividing both sides of (20) by $\vartheta^2$ and letting $\vartheta\to 0$, we get

$$\begin{aligned}
\lim_{\vartheta\to0^+}\frac{\beta(\vartheta)}{\vartheta}=&\Bigg(\alpha\lim_{\vartheta\to0^+}\int_\Omega\frac{\nabla v(\vartheta)(\phi(\vartheta)\nabla u(\vartheta)-u(\vartheta)\nabla\phi(\vartheta))}{\vartheta^2}\mathrm{d}x+\int_\Omega\Bigg(\Bigg(\frac{\alpha\Delta v_m^{d_v}}{M}+a\\
&-\frac{p\rho(k)q_1v_m^{d_v}}{(1+q_2v_m^{d_v})^2}\Bigg)\phi_{**}^3+\frac{2\alpha}{M}\phi_{**}^2\nabla\phi_{**}\nabla v_m^{d_v}-\alpha\phi_{**}(\phi_{**}\Delta\psi_{**}+\nabla\phi_{**}\nabla\psi_{**})\\
&+\left(\frac{r_0^{**}k}{(1+kv_m^{d_v})^2}+\frac{p\rho(k)}{(1+q_2v_m^{d_v})^2}\right)\phi_{**}^2\psi_{**}\Bigg)\mathrm{d}x\Bigg)\Big/\int_\Omega\phi_{**}^2\,\mathrm{d}x.
\end{aligned} \tag{21}$$

Since $\lim_{\vartheta\to0^+}\nabla v(\vartheta)/\vartheta\neq\infty$,

$$\begin{aligned}
&\lim_{\vartheta\to0^+}\int_\Omega\frac{\nabla v(\vartheta)(\phi(\vartheta)\nabla u(\vartheta)-u(\vartheta)\nabla\phi(\vartheta))}{\vartheta^2}\mathrm{d}x\\
&=\lim_{\vartheta\to0^+}\int_\Omega\frac{\nabla v(\vartheta)}{\vartheta}\mathrm{d}x\cdot\lim_{\vartheta\to0^+}\int_\Omega\frac{\phi(\vartheta)\nabla u(\vartheta)-u(\vartheta)\nabla\phi(\vartheta)}{\vartheta}\mathrm{d}x\\
&=\lim_{\vartheta\to0^+}\int_\Omega\frac{\nabla v(\vartheta)}{\vartheta}\mathrm{d}x\cdot\int_\Omega(\phi_{**}\nabla\phi_{**}-\phi_{**}\nabla\phi_{**})\mathrm{d}x\\
&=0.
\end{aligned}$$

Combining (21), Theorem 2.2(ii) and the stability theory of bifurcation solution, we conclude that the conclusion in Theorem 2.3(ii) follows. ■

### 2.2. Global bifurcation structures

This subsection focuses on the global bifurcation structures of positive solutions for model (7).

Combining Theorems 2.1–2.2 and the global bifurcation theory presented in [31], we next investigate the global bifurcation structures of positive solutions to model (7) emitting from $\Theta_u$ and $\Theta_v$, respectively.

**Theorem 2.4:** *The following statements hold.*

(i) *If* $-cp\rho(k)/q_1 < m < 0$, *then an unbounded continuum* $\Theta_*$ *of positive solution for model* (7) *bifurcates from* $\Theta_u$ *at* $(r_0, u, v) = (r_0^*, (r_0^* - d)/a, 0)$ *and* $\mathrm{Proj}_{r_0}\Theta_* \supset (r_0^*, \infty)$.

(ii) *If* $0 < m < \lambda_1^D(d_v, \Omega_0)$, *then an unbounded continuum* $\Theta_{**}$ *of positive solution for model* (7) *bifurcates from* $\Theta_v$ *at* $(r_0, u, v) = (r_0^{**}, 0, v_m^{d_v})$ *and* $\mathrm{Proj}_{r_0}\Theta_{**} \supset (r_0^{**}, \infty)$.

***Proof:*** We only prove (ii) since (i) can be verified similarly. Firstly, by virtue of Theorem 2.2(ii), there is a smooth curve $\Theta_2$ of positive solutions for model (7) bifurcating from $(r_0, u, v) = (r_0^{**}, 0, v_m^{d_v}) \in \Theta_v$. Let $\Theta_{**} \subset \mathbb{R} \times \mathcal{C}$ be the maximal connected set of positive solutions of (7) satisfying

$$\Theta_2 \subset \Theta_{**} \subset \{(r_0, u, v)\backslash(r_0^{**}, 0, v_m^{d_v}) : (r, u, v) \text{ is a solution of model (7)}\}.$$

Then Theorem 1.2 in [31] shows that $\Theta_{**}$ must satisfy one of the following alternatives

(1) $\Theta_{**}$ is unbounded in $\mathbb{R}^+ \times \mathcal{C}$;
(2) $\Theta_{**}$ contains a point $(\tilde{r}_0, 0, v_m^{d_v})$ with $\tilde{r}_0 \neq r_0^{**}$;
(3) $\Theta_{**}$ contains a point $(r_0, u, v) \in \mathbb{R}^+ \times \{\mathcal{Y}\backslash\{(0, v_m^{d_v})\}\}$, where $\mathcal{Y}$ is a closed subspace of $\mathcal{C}$ such that $\mathcal{C} = \ker(\mathcal{L}_{1(u,w)}(r_0^{**}, 0, 0)) \oplus \mathcal{Y}$.

Denote $\mathfrak{H} = \{u \in C^1_{\mathbf{n}}(\Omega) : u > 0 \text{ in } \overline{\Omega}\}$. Next, we derive that if $(r_0, u, v) \in \Theta_{**}\backslash\{(r_0^{**}, 0, v_m^{d_v})\}$, then $(r_0, u, v) \in \mathbb{R}^+ \times \mathfrak{H} \times \mathfrak{H}$, that is, $u > 0$ and $v > 0$ in $\overline{\Omega}$. Once this has been shown, then both alternatives (2) and (3) can be excluded. Suppose on the contrary, that $\Theta_{**}$ contains a point $(r_0, u, v) \neq (r_0^{**}, 0, v_m^{d_v})$ which lies outside of $\mathbb{R}^+ \times \mathfrak{H} \times \mathfrak{H}$. Then there exist a sequence $\{(r_0^n, u_n, v_n)\}_{n=1}^{\infty} \subset \Theta_{**} \cap (\mathbb{R}^+ \times \mathfrak{H} \times \mathfrak{H})$ and $(\check{r}_0, \check{u}, \check{v}) \in \Theta_{**} \cap (\mathbb{R}^+ \times \partial(\mathfrak{H} \times \mathfrak{H}))$ such that

$$\lim_{n\to\infty} (r_0^n, u_n, v_n) = (\check{r}_0, \check{u}, \check{v}) \in \Theta_{**} \cap (\mathbb{R}^+ \times \partial(\mathfrak{H} \times \mathfrak{H})) \quad \text{in } \mathbb{R}^+ \times \mathcal{C},$$

where $(\check{u}, \check{v})$ is a non-negative solution of model (7) with $r_0 = \check{r}_0$. Similar to the previous analysis, we conclude that there exists a positive constant $M^\star$ such that

$\max_{\overline{\Omega}}\{\|\nabla\check{u}\|_{C^1},\|\nabla\check{v}\|_{C^1},\|\Delta\check{v}\|_{C^1}\}\le M^\star$. Then by the first equation of (7), we have

$$\begin{aligned}
&-d_u\Delta\check{u}+P\check{u}-\alpha\nabla\cdot(\beta(\check{u})\check{u}\nabla\check{v})+\check{u}\left(-\frac{\check{r}_0}{1+k\check{v}}+d+a\check{u}+\frac{p\rho(k)\check{v}}{1+q_1\check{u}+q_2\check{v}}\right)\\
&=-d_u\Delta\check{u}-\alpha\nabla\check{v}\nabla\check{u}+\left(P+\frac{2\alpha}{M}\nabla\check{u}\nabla\check{v}-\alpha\left(1-\frac{\check{u}}{M}\right)\Delta\check{v}-\frac{\check{r}_0}{1+k\check{v}}\right.\\
&\quad\left.+d+a\check{u}+\frac{p\rho(k)\check{v}}{1+q_1\check{u}+q_2\check{v}}\right)\check{u}\\
&=P\check{u}\ge 0
\end{aligned}$$

with $P$ being a sufficiently large positive constant so that

$$P+\frac{2\alpha}{M}\nabla\check{u}\nabla\check{v}-\alpha\left(1-\frac{\check{u}}{M}\right)\Delta\check{v}-\frac{\check{r}_0}{1+k\check{v}}+d+a\check{u}+\frac{p\rho(k)\check{v}}{1+q_1\check{u}+q_2\check{v}}$$

is positive and bounded for all $x\in\Omega$. Then the strong maximum principle asserts that one of the followings must hold:

(a) $\check{u}\equiv 0,\check{v}\equiv 0,x\in\overline{\Omega}$; (b) $\check{u}>0,\check{v}\equiv 0,x\in\overline{\Omega}$; (c) $\check{u}\equiv 0,\check{v}>0,x\in\overline{\Omega}$.

If case (a) is correct, then $(\check{r}_0,\check{u},\check{v})$ lies on the trivial solution branch $\Theta_0=\{(r_0,u,v)=(r_0,0,0):r_0>0\}$. It holds that the only nontrivial and nonnegative solution of (7) closing to $\Theta_0$ lies on the semi-trivial solution branch $\Theta_u$. Hence, there cannot exist a sequence in $\Theta_{**}\cap(\mathbb{R}^+\times\mathfrak{H}\times\mathfrak{H})$ converging to $(\check{r}_0,0,0)$. This implies that case (a) cannot occur.

If case (b) is true, then as $n\to\infty$, $\check{u}$ satisfies

$$-d_u\Delta\check{u}=\check{u}(\check{r}_0-d-a\check{u}),\quad x\in\Omega,\quad \partial_{\mathbf{n}}\check{u}=0,\quad x\in\partial\Omega.$$

Clearly, $\check{u}=(\check{r}_0-d)/a$. This derives $(\check{r}_0,\check{u},\check{v})\in\Theta_u$, thus $(\check{r}_0,\check{u},\check{v})$ is a bifurcation point of (7) on $\Theta_u$ bifurcating from the nontrivial and nonnegative solution. Let $v$ be the principal eigenfunction of $-d_v\Delta-(m+cp\rho(k)(\check{r}_0-d)/(a+q_1(\check{r}_0-d)))$ corresponding to the principal eigenvalue 0. Then $v$ satisfies

$$-d_v\Delta v-\left(m+\frac{cp\rho(k)(\check{r}_0-d)}{a+q_1(\check{r}_0-d)}\right)v=0,\quad x\in\Omega,\quad \partial_{\mathbf{n}}v=0,\quad x\in\partial\Omega,$$

and then $\check{r}_0=r_0^*$. Further, owing to $m>0$, one deduces $r_0^*<d$, this is impossible. Thus, case (b) is not true.

Let case (c) hold. Then as $n\to\infty$, $\check{v}$ satisfies

$$-d_v\Delta\check{v}=\check{v}(m-s(x)\check{v}),\quad x\in\Omega,\quad \partial_{\mathbf{n}}\check{v}=0,\quad x\in\partial\Omega.$$

Since $0<m<\lambda_1^D(d_v,\Omega_0)$, $\check{v}=v_m^{d_v}$. So $(\check{r}_0,\check{u},\check{v})\in\Theta_v$, and thus $(\check{r}_0,\check{u},\check{v})$ is a bifurcation point of (7) on $\Theta_v$ bifurcating from the nontrivial and nonnegative solution. If $u$ is the principal eigenfunction of $-d_u\Delta-(\alpha\nabla v_m^{d_v}\nabla+\alpha\Delta v_m^{d_v}+\check{r}_0/(1+kv_m^{d_v})-d-p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}))$ corresponding to the principal eigenvalue 0, then $u$ satisfies

$$\begin{aligned}
&-d_u\Delta u-\left(\alpha\nabla v_m^{d_v}\nabla+\alpha\Delta v_m^{d_v}+\frac{\check{r}_0}{1+kv_m^{d_v}}-d-\frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}\right)\\
&\times u=0,\quad x\in\Omega,\quad \partial_{\mathbf{n}}u=0,\quad x\in\partial\Omega.
\end{aligned}$$

It holds $\check{r}_0=r_0^{**}$, contradicting the definition of $\Theta_{**}$. So, case (c) cannot occur.

Summarizing the analyses above, we conclude that if $(r_0, u, v) \in \Theta_{**}\backslash\{(r_0^{**}, 0, v_m^{d_v})\}$, then $(r_0, u, v) \in \mathbb{R}^+ \times \mathfrak{H} \times \mathfrak{H}$. This means that $\Theta_{**}$ is unbounded in $\mathbb{R}^+ \times \mathcal{C}$.

On the other side, from Theorem 2.1, we conclude that when $0 < m < \lambda_1^D(d_v, \Omega_0)$, $(u, v)$ is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$ for bounded $r_0$, and the elliptic regularity theory indicates that $(u, v)$ is uniformly bounded in $\mathcal{C}$ for bounded $r_0$. Then $\Theta_{**}$ becomes unbounded in $\mathbb{R}^+ \times \mathcal{C}$ through $r_0$. The proof is finished. ■

### 2.3. Existence of positive solutions

This subsection deals with the existence of positive solutions for model (7).

**Theorem 2.5:** *Let $r_0^{**}$ be defined in* (15). *Then, the following statements hold.*

(i) *For $-cp\rho(k)/q_1 < m \le 0$, if $r_0 > r_0^*$, then model* (7) *has at least one positive solution.*
(ii) *For $0 < m < \lambda_1^D(d_v, \Omega_0)$, if $r_0 > r_0^{**}$, then model* (7) *has at least one positive solution.*

***Proof:*** Actually, in the light of the conclusions in Theorem 2.4, we just need to prove that under the case of $m = 0$, model (7) has at least one positive solution if $r_0 > r_0^*$.

Let $m = 0$ and define $r_0^* \triangleq r_0^*(m)$. Assume on the contrary, that as $m = 0$, model (7) has no positive solution if $r_0 > r_0^*$. Then we can find a sequence $\{(m_n, u_n, v_n)\}_{n=1}^\infty$ such that $(u_n, v_n)$ is a positive solution of (7) with $m = m_n < 0$ and $\lim_{n\to\infty} m_n = 0$. Owing to $r_0^*(m_n) \to d$ as $n \to \infty$, we have $r_0 > r_0^*(m_n)$ as $n \to \infty$. Then from Theorem 2.1 and the standard regularity theory of elliptic equations, there exists a subsequence of $\{(u_n, v_n)\}_{n=1}^\infty$, also denoted by itself, such that as $n \to \infty$, $(u_n, v_n) \to (u, v)$ in $\mathcal{C}$. Passing to the limit in the equations for $u_n$ and $v_n$, one sees that $(u, v)$ is a nonnegative solution of (7) with $m = 0$. Since we have assumed that model (7) has no positive solution when $m = 0$, either $u \equiv 0$ or $v \equiv 0$.

If $u \equiv 0$, it holds from $m = 0$ that $v = 0$ and

$$\lim_{n\to\infty}\left(\frac{r_0}{1+kv_n} - d - au_n - \frac{p\rho(k)v_n}{1+q_1u_n+q_2v_n}\right) = r_0 - d > 0.$$

So, as $n$ is large enough, $\int_\Omega u_n(r_0/(1+kv_n) - d - au_n - p\rho(k)v_n/(1+q_1u_n+q_2v_n))\,\mathrm{d}x > 0$, which is impossible since

$$\begin{aligned}
&\int_\Omega u_n\left(\frac{r_0}{1+kv_n} - d - au_n - \frac{p\rho(k)v_n}{1+q_1u_n+q_2v_n}\right)\mathrm{d}x \\
&= \int_\Omega (-d_u\Delta u_n - \alpha\nabla\cdot(\beta(u_n)u_n\nabla v_n))\,\mathrm{d}x \\
&= -\alpha\int_\Omega\left(1 - \frac{2u_n}{M}\right)\nabla u_n\nabla v_n\,\mathrm{d}x + \alpha\int_\Omega \nabla\left(u_n - \frac{u_n^2}{M}\right)\nabla v_n\,\mathrm{d}x \\
&= 0,
\end{aligned}$$

thus $u > 0$.

If $v \equiv 0$, then $u = (r_0 - d)/a$ and

$$\lim_{n\to\infty}\left(m_n - s(x)v_n + \frac{cp\rho(k)u_n}{1+q_1u_n+q_2v_n}\right) = \frac{cp\rho(k)(r_0-d)}{a+q_1(r_0-d)} > 0.$$

It derives that as $n$ is large enough, $\int_\Omega v_n(m_n - s(x)v_n + cp\rho(k)u_n/(1+q_1u_n+q_2v_n))\,dx > 0$, which is also impossible, so $v > 0$.

Therefore, under the case of $m = 0$, model (7) has at least one positive solution if $r_0 > r_0^*$. The proof follows. ■

## *2.4. Asymptotic behavior of positive solutions as $r_0 \to \infty$*

This subsection refers to the asymptotic behavior of positive solutions for model (7) on the bifurcation continuum as $r_0 \to \infty$.

**Theorem 2.6:** *Assume that $\tilde{C}$ is a fixed positive constant, $\{(M_n, r_0^n)\}_{n=1}^{\infty}$ is a sequence satisfying $(M_n, r_0^n) \to (\infty,\infty)$ as $n \to \infty$ and $M_n/r_0^n < \tilde{C}$, and $(u_n, v_n)$ is an arbitrary positive solution of model (7) corresponding to $(M, r_0) = (M_n, r_0^n)$. Then the following statements hold.*

(i) $\lim_{n\to\infty} \frac{u_n}{r_0^n} = l_1 \in [0,\tilde{C}]$.

(ii) *If $l_1$ in (i) is positive and $-cp\rho(k)/q_1 < m < \lambda_1^D(d_v,\Omega_0) - cp\rho(k)/q_1$, then subject to a subsequence, $\lim_{n\to\infty} v_n = v_{\bar{m}}^{d_v}$ in $\overline{\Omega}$, where $\bar{m} = m + cp\rho(k)/q_1$ and $v_{\bar{m}}^{d_v}$ is the unique positive solution of the following problem*

$$-d_v\Delta v = \left(m + \frac{cp\rho(k)}{q_1}\right)v - s(x)v^2,\quad x\in\Omega,\qquad \partial_{\mathbf{n}}v = 0,\quad x\in\partial\Omega.$$

(iii) *If $l_1$ in (i) is positive and $\lambda_1^D(d_v,\Omega_0) - cp\rho(k)/q_1 \le m < \lambda_1^D(d_v,\Omega_0)$, then*

**(A1)** *$\|v_n\|_\infty \to \infty$ as $n \to \infty$;*

**(A2)** *subject to a subsequence, $\|v_n\|_\infty/r_0^n \to l_2 \in [0,\infty)$, $\tilde{v}_n = v_n/\|v_n\|_\infty \to \tilde{v}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p > 1$ as $n \to \infty$, where $\tilde{v}$ is a nonnegative function satisfying $\tilde{v} = 0$ for $x \in \Omega\backslash\overline{\Omega}_0$, and $\tilde{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ is a positive weak solution of*

$$-d_v\Delta\tilde{v} = \tilde{v}\left(m + \frac{cpl_1\rho(k)}{q_1l_1+q_2l_2\tilde{v}}\right),\quad x\in\Omega_0,\qquad \tilde{v} = 0,\quad x\in\partial\Omega_0. \tag{22}$$

***Proof:*** (i) Set $w_n = u_n/r_0^n$. Substitute it into the first equation of model (7) to get

$$\begin{cases} -d_u\Delta w_n - \alpha\nabla\cdot(\beta(w_nr_0^n)w_n\nabla v_n) = \dfrac{r_0^n}{1+kv_n}w_n - dw_n - aw_n^2r_0^n \\ \quad -\dfrac{p\rho(k)w_nv_n}{1+q_1w_nr_0^n+q_2v_n}, & x\in\Omega, \\ \partial_{\mathbf{n}}w_n = 0, & x\in\partial\Omega. \end{cases} \tag{23}$$

Similar to the proof of Theorem 2.1, we can obtain that $\overline{w}_n = M_n/r_0^n$ is a supersolution and $\underline{w}_n = 0$ is a subsolution of (23) for all $x \in \overline{\Omega}$. By the assumption $M_n/r_0^n < \tilde{C}$ and the super-and subsolution argument of elliptic equations, the solution $w_n$ of (23) satisfies $0 \le w_n < \tilde{C}$. It holds $\lim_{n\to\infty} w_n = l_1 \in [0, \tilde{C}]$. Therefore, $\lim_{n\to\infty} u_n/r_0^n = l_1 \in [0, \tilde{C}]$. The part (i) is proved.

(ii) We first prove that $\|v_n\|_\infty$ is uniformly bounded. Assume on the contrary, that there exists a subsequence of $\{(u_n, v_n)\}_{n=1}^\infty$, still labelled by itself, such that $\|v_n\|_\infty \to \infty$ as $n \to \infty$. Set $\tilde{v}_n = v_n/\|v_n\|_\infty$. Then by the proof of Theorem 2.1, we get that subject to a subsequence, $\tilde{v}_n \to \tilde{v}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p > 1$ as $n \to \infty$. Further, we can still obtain that $\tilde{v} \not\equiv 0$ in $\overline{\Omega}$, $\tilde{v} = 0$ almost everywhere in $\Omega\backslash\overline{\Omega}_0$, and $\tilde{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ since $\partial\Omega_0$ is smooth enough. Additionally, suppose that $z(x)$ is the weak limit of $cp\rho(k)u_n/(1 + q_1u_n + q_2v_n)$ in $H^1(\Omega)$, then $\tilde{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ is a nonnegative weak solution of

$$-d_v\Delta\tilde{v} = \tilde{v}(m + z(x)), \;\; x \in \Omega_0, \quad \tilde{v} = 0, \;\; x \in \partial\Omega_0. \tag{24}$$

Combined with the previous analysis, it can be seen that $\tilde{v} > 0$ in $\Omega_0$. Thus, (24) means $m = \lambda_1^D(d_v, -z(x), \Omega_0) \ge \lambda_1^D(d_v, \Omega_0) - cp\rho(k)/q_1$, contradicting the assumption $m < \lambda_1^D(d_v, \Omega_0) - cp\rho(k)/q_1$. Therefore, $\|v_n\|_\infty$ is uniformly bounded.

It remains to show that subject to a subsequence, $\lim_{n\to\infty} v_n = v_{\bar{m}}^{d_v}$ in $\overline{\Omega}$ with $\bar{m} = m + cp\rho(k)/q_1$. Since $l_1$ in (i) is positive means $u_n \to \infty$ as $n \to \infty$, $cp\rho(k)u_n/(1 + q_1u_n + q_2v_n) \to cp\rho(k)/q_1$ uniformly as $n \to \infty$. Then the elliptic regularity theory and Sobolev embedding theorem yield that subject to a subsequence, $v_n \to v$ in $\overline{\Omega}$ as $n \to \infty$, and $v$ is a nonnegative solution of

$$-d_v\Delta v = v\left(m + \frac{cp\rho(k)}{q_1} - s(x)v\right), \;\; x \in \Omega, \quad \partial_{\mathbf{n}} v = 0, \;\; x \in \partial\Omega. \tag{25}$$

Obviously, if $v = 0$, then the second equation of model (7) shows

$$m - s(x)v_n + \frac{cp\rho(k)u_n}{1 + q_1u_n + q_2v_n} \to m + \frac{cp\rho(k)}{q_1} > 0 \quad \text{as } n \to \infty,$$

meaning $\int_\Omega v_n(m - s(x)v_n + cp\rho(k)u_n/(1 + q_1u_n + q_2v_n))\,\mathrm{d}x > 0$, which is impossible. So, $v$ is a positive solution of (25). Combining (15), Theorem 2.2(1) in [22] and the assumption $-cp\rho(k)/q_1 < m < \lambda_1^D(d_v, \Omega_0) - cp\rho(k)/q_1$, one has $v = v_{\bar{m}}^{d_v}$. The part (ii) is verified.

(iii) Similar to the analysis in the proof of (ii), one sees that in order to finish the proof of (iii), it only needs to prove that subject to a subsequence, $\|v_n\|_\infty/r_0^n \to l_2 \in [0, \infty)$ as $n \to \infty$, and $\tilde{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ is a positive weak solution of problem (22), where $\tilde{v}$ is defined in (**A**2), and it is also the same as that of given in the proof of (ii).

Suppose on the contrary, that there exists a subsequence of $\{(u_n, v_n)\}_{n=1}^\infty$, still denoted by itself, such that $\|v_n\|_\infty/r_0^n \to \infty$ as $n \to \infty$. Then by the assumption that $l_1$ in (i) is positive, (**A**1) and the proof of (ii), we can derive $z(x) = 0$ in $\Omega_0$ with $z(x)$ being the weak limit of $cp\rho(k)u_n/(1 + q_1u_n + q_2v_n)$ in $H_1(\Omega)$. Hence, $\tilde{v}|_{\Omega_0} \in H_0^1(\Omega_0)$ is a positive weak solution of (24) with $z(x) = 0$. It holds $m = \lambda_1^D(d_v, \Omega_0)$, contradicting the assumption $m < \lambda_1^D(d_v, \Omega_0)$. Thus, $\|v_n\|_\infty/r_0^n$ is bounded, and it is evident that $\lim_{n\to\infty} \|v_n\|_\infty/r_0^n = l_2 \in$

$[0,\infty)$. Moreover, since in $\Omega_0$, $\tilde{v}_n = v_n/\|v_n\|_\infty$ satisfies

$$-d_v\Delta\tilde{v}_n = \tilde{v}_n\left(m + \frac{cp\rho(k)u_n}{1+q_1u_n+q_2\tilde{v}_n\|v_n\|_\infty}\right), \quad x\in\Omega_0,$$

we see from $\lim_{n\to\infty} u_n/r_0^n = l_1$ and $\lim_{n\to\infty}\|v_n\|_\infty/r_0^n = l_2$ that $\tilde{v}|_{\Omega_0}\in H_0^1(\Omega_0)$ is a positive weak solution of problem (22). The proof is completed. ■

## 3. On solutions of model (9)

In this section, we study the dynamics of model (9), including the *a priori* estimate, bifurcation structures, existence, uniqueness, stability and asymptotic behaviors of positive solutions.

### 3.1. An a priori estimate and bifurcation structures

This subsection aims to establish an *a priori* estimate and bifurcation structures of positive solutions for model (9). We first establish an *a priori* estimate for positive solutions as stated in the following theorem.

**Theorem 3.1:** *Suppose that a and d are fixed positive constants and assume that $m < \lambda_1^D(d_v,\Omega_0)$. Then for any given positive constant $\hat{C}$, there exists a positive constant $\hat{W} = \hat{W}(\hat{C})$ such that any positive solution $(u,v)$ of model* (9) *with $d < r_0 \le a\hat{C}+d$ satisfies $\|u\|_\infty + \|v\|_\infty \le \hat{W}$.*

***Proof:*** Assume that the conclusion is false, then there exist a positive constant $\hat{C}$ and a sequence $\{r_0^n\}_{n=1}^\infty$ with $d < r_0^n \le a\hat{C}+d$ such that model (9) with $r_0 = r_0^n$ has a positive solution $(u_n,v_n)$ satisfying $\|u_n\|_\infty + \|v_n\|_\infty \to \infty$ as $n\to\infty$. From the maximum principle, one sees $0 < u_n < (r_0^n - d)/a \le \hat{C}$. This means $\|v_n\|_\infty \to \infty$ as $n\to\infty$. However, similar to the proof of Theorem 2.1, we can obtain that this is impossible when $m < \lambda_1^D(d_v,\Omega_0)$. Thus, $\|v_n\|_\infty$ is also bounded. The proof is accomplished. ■

**Remark 3.1:** Theorem 2.1 contains the assumption $d < r_0 \le aM+d$, which corresponds to $d < r_0 \le a\hat{C}+d$ in Theorem 3.1. This difference is mainly caused by the predator-taxis. The result of introducing the predator-taxis is that $M$ in model (7) must satisfy $M > (r_0 - d)/a$, so the assumption $d < r_0 \le aM+d$ in Theorem 2.1 is necessary. On the other hand, from the maximum principle, we find that $u$ in model (9) must satisfy $u < (r_0-d)/a$. Thus, to guarantee $(r_0-d)/a > 0$ and the boundedness of $u$, we need to assume that $d < r_0 \le a\hat{C}+d$ holds in Theorem 3.1.

Assume that $r^\diamond$ satisfies $d < r^\diamond < \infty$ and

$$\varsigma(r^\diamond) = \lambda_1^N\left(-d_u, \frac{r^\diamond}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right) = d, \tag{26}$$

$r_0^*$, $\Theta_u$ and $\Theta_v$ are the same as that of defined in Section 2.1. Then it is clear that model (9) has semi-trivial solution curves $\Theta_u$ and $\Theta_v$. By using the techniques similar to that of the proof of Theorems 2.2–2.4, the local and global bifurcation structures of positive solutions for model (9) along $\Theta_u$ and $\Theta_v$, and the stability of local bifurcation solutions can be obtained, as given in the following theorems.

**Theorem 3.2 (Local bifurcation structures):** *The following statements hold.*

(i) *Let* $-c\rho(k)p/q_1 < m < 0$. *Then a branch of positive solutions for model* (9) *bifurcates from* $\Theta_u$ *if and only if* $r_0 = r_0^*$. *Specifically, all positive solutions of* (9) *near* $(r_0^*, (r_0^* - d)/a, 0) \in \mathbb{R} \times X$ *are on a smooth curve* $\tilde{\Theta}_1$, *which is given by*

$$\tilde{\Theta}_1 = \{(r_0(\vartheta), u(\vartheta), v(\vartheta)) = (r_0^* + \vartheta \bar{r}_0'(0) + o(|\vartheta|),\ (r_0^* - d)/a + \vartheta \tilde{\phi}_* + o(|\vartheta|),\ \vartheta \psi_* + o(|\vartheta|)) : 0 < \vartheta < \tilde{\vartheta}_1\},$$

*where* $\tilde{\vartheta}_1 > 0$ *is a certain number,* $\psi_*$ *is the same as that of defined in Theorem 2.2(i),*

$$\tilde{\phi}_* = (d_u \Delta + d - r_0^*)^{-1} \left( \frac{p\rho(k)(r_0^* - d)}{a + q_1(r_0^* - d)} + \frac{k r_0^*(r_0^* - d)}{a} \right) \psi_*,$$

*and*

$$\bar{r}_0'(0) = \frac{\int_\Omega \left( s(x) + \frac{c p q_2 \rho(k)(r_0^* - d)}{a(1 + q_1(r_0^* - d)/a)^2} \right) \psi_*^3 \, dx - \int_\Omega \frac{c p \rho(k) \tilde{\phi}_* \psi_*^2}{(1 + q_1(r_0^* - d)/a)^2} dx}{\int_\Omega \frac{c p \rho(k) \psi_*^2}{a(1 + q_1(r_0^* - d)/a)^2} dx}.$$

*Moreover, the bifurcation of* $\tilde{\Theta}_1$ *at* $(r_0^*, (r_0^* - d)/a, 0)$ *is supercritical if* $q_2 > \bar{q}$ *and subcritical if* $q_2 < \bar{q}$ *with* $\bar{q}$ *being given by*

$$\bar{q} = \frac{\int_\Omega \frac{c p \rho(k) \tilde{\phi}_* \psi_*^2}{(1 + q_1(r_0^* - d)/a)^2} dx - \int_\Omega s(x) \psi_*^3 dx}{\int_\Omega \frac{c p \rho(k)(r_0^* - d)}{a(1 + q_1(r_0^* - d)/a)^2} \psi_*^3 \, dx}.$$

(ii) *Let* $0 < m < \lambda_1^D(d_v, \Omega_0)$. *Then a branch of positive solutions for model* (9) *bifurcates from* $\Theta_v$ *if and only if* $r_0 = r^\diamond$. *Specifically, all positive solutions of* (9) *near* $(r^\diamond, 0, v_m^{d_v}) \in \mathbb{R} \times X$ *are on a smooth curve* $\tilde{\Theta}_2$, *which is given by*

$$\tilde{\Theta}_2 = \{(r_0(\vartheta), u(\vartheta), v(\vartheta)) = (r^\diamond + \vartheta \tilde{r}_0'(0) + o(|\vartheta|), \vartheta \widetilde{\phi}_{**} + o(|\vartheta|), v_m^{d_v} + \vartheta \widetilde{\psi}_{**} + o(|\vartheta|)) : 0 < \vartheta < \tilde{\vartheta}_2\},$$

*where* $\tilde{\vartheta}_2 > 0$ *is a certain number,* $\widetilde{\phi}_{**}$ *and* $\widetilde{\psi}_{**}$ *satisfy*

$$\begin{cases} d_u \Delta \widetilde{\phi}_{**} + \dfrac{r_0^\diamond}{1 + k v_m^{d_v}} \widetilde{\phi}_{**} - d \widetilde{\phi}_{**} - \dfrac{p\rho(k) v_m^{d_v}}{1 + q_2 v_m^{d_v}} \widetilde{\phi}_{**} = 0, & x \in \Omega, \\ d_v \Delta \widetilde{\psi}_{**} + m \widetilde{\psi}_{**} - 2s(x) v_m^{d_v} \widetilde{\psi}_{**} + \dfrac{c p \rho(k) v_m^{d_v}}{1 + q_2 v_m^{d_v}} \widetilde{\phi}_{**} = 0, & x \in \Omega, \\ \partial_{\mathbf{n}} \widetilde{\phi}_{**} = \partial_{\mathbf{n}} \widetilde{\psi}_{**} = 0, & x \in \partial\Omega, \end{cases}$$

*and $\tilde{r}_0'(0)$ is defined by*

$$\tilde{r}_0'(0) = \frac{\int_\Omega \left( (a - \frac{pq_1\rho(k)v_m^{d_v}}{(1+q_2v_m^{d_v})^2})\widetilde{\phi}_{**}^3 + (\frac{p\rho(k)}{(1+q_2v_m^{d_v})^2} + \frac{kr^\diamond}{(1+kv_m^{d_v})^2})\widetilde{\phi}_{**}^2\widetilde{\psi}_{**} \right) dx}{\int_\Omega \frac{\widetilde{\phi}_{**}^2}{1+kv_m^{d_v}} dx}.$$

*Moreover, the bifurcation of $\tilde{\Theta}_2$ at $(r^\diamond, 0, v_m^{d_v})$ is supercritical if $a > \tilde{a}_1$ and subcritical if $a < \tilde{a}_1$, where $\tilde{a}_1$ is given by*

$$\tilde{a}_1 = \frac{\int_\Omega \left( \frac{pq_1\rho(k)v_m^{d_v}}{(1+q_2v_m^{d_v})^2}\widetilde{\phi}_{**}^3 - \left( \frac{p\rho(k)}{(1+q_2v_m^{d_v})^2} + \frac{kr^\diamond}{(1+kv_m^{d_v})^2} \right)\widetilde{\phi}_{**}^2\widetilde{\psi}_{**} \right) dx}{\int_\Omega \widetilde{\phi}_{**}^3 dx}.$$

**Theorem 3.3 (Stability of bifurcation solutions):** *The following statements hold.*

(i) *Let $-cp\rho(k)/q_1 < m < 0$. Then there exists a small positive number $\tilde{\vartheta}_1$ such that the positive solution $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ of model (9) bifurcating from $(r_0^*, (r_0^* - d)/a, 0)$ is non-degenerate for $\vartheta \in (0, \tilde{\vartheta}_1)$. Moreover, $(u(\vartheta), v(\vartheta))$ is asymptotically stable if $q_2 > \bar{q}$ and unstable if $q_2 < \bar{q}$.*
(ii) *Let $0 < m < \lambda_1^D(d_v, \Omega_0)$. Then there exists a small positive number $\tilde{\vartheta}_2$ such that the positive solution $(r_0(\vartheta), u(\vartheta), v(\vartheta))$ of model (9) bifurcating from $(r^\diamond, 0, v_m^{d_v})$ is non-degenerate for $\vartheta \in (0, \tilde{\vartheta}_2)$. Moreover, $(u(\vartheta), v(\vartheta))$ is asymptotically stable if $a > \tilde{a}_1$ and unstable if $a < \tilde{a}_1$.*

**Theorem 3.4 (Global bifurcation structures):** *The following statements hold.*

(i) *Let $-cp\rho(k)/q_1 < m < 0$. Then an unbounded continuum $\tilde{\Theta}_*$ of positive solution of model (9) bifurcates from $\Theta_u$ at $(r_0^*, (r_0^* - d)/a, 0)$ and $\mathrm{Proj}_{r_0}\tilde{\Theta}_* = (r_0^*, \infty)$.*
(ii) *Let $0 < m < \lambda_1^D(d_v, \Omega_0)$. Then an unbounded continuum $\tilde{\Theta}_{**}$ of positive solution of model (9) bifurcates from $\Theta_v$ at $(r^\diamond, 0, v_m^{d_v})$ and $\mathrm{Proj}_{r_0}\tilde{\Theta}_{**} \supset (r^\diamond, \infty)$.*

**Remark 3.2:** Comparing Theorem 2.2(i) with Theorem 3.2(i), we know that the positive solution sets of models (7) and (9) near $(r_0^*, (r_0^* - d)/a, 0)$ are on the different smooth curves. This means that $\tilde{\Theta}_*$ in Theorem 3.4(i) is different from $\Theta_*$ in Theorem 2.4(i). Moreover, since $r^\diamond \neq r_0^{**}$, $\tilde{\Theta}_{**}$ in Theorem 3.4(ii) is different from $\Theta_{**}$ in Theorem 2.4(ii).

**Remark 3.3:** In the formulae $\mathrm{Proj}_{r_0}\tilde{\Theta}_* = (r_0^*, \infty)$ and $\mathrm{Proj}_{r_0}\Theta_* \supset (r_0^*, \infty)$, the former is '$=$', and the latter is '$\supset$', this difference is caused by the predator-taxis. More precisely, by the proofs of Theorems 2.1 and 3.1, one sees that $u$ in model (7) satisfies $u < M$ and in model (9) satisfies $u < (r_0 - d)/a$. Furthermore, the second equation of (7) and (9) indicates

$$m = \lambda_1^N\left(d_v, s(x)v - \frac{cp\rho(k)u}{1 + q_1u + q_2v}, \Omega\right) > \lambda_1^N\left(d_v, -\frac{cp\rho(k)u}{1 + q_1u}, \Omega\right).$$

It holds that if (7) and (9) have a positive solution, then $m$ in (7) and in (9) satisfies

$$m > \lambda_1^N\left(d_v, -\frac{cp\rho(k)M}{1+q_1M}, \Omega\right) = \frac{-cp\rho(k)M}{1+q_1M}$$

and

$$m > \lambda_1^N\left(d_v, -\frac{cp\rho(k)(r_0-d)}{a+q_1(r_0-d)}, \Omega\right) = \frac{-cp\rho(k)(r_0-d)}{a+q_1(r_0-d)},$$

respectively, where $m > -cp\rho(k)(r_0-d)/(a+q_1(r_0-d)) \Leftrightarrow r_0 > r_0^*$. Combining the proof of Theorem 2.4(i), we know that $\mathrm{Proj}_{r_0}\tilde{\Theta}_* = (r_0^*, \infty)$ and $\mathrm{Proj}_{r_0}\Theta_* \supset (r_0^*, \infty)$.

### 3.2. Existence, uniqueness and stability of positive solution

This subsection focuses on the existence, uniqueness and stability of positive solution for model (9).

**Theorem 3.5:** *The following statements hold.*

(i) *When $-cp\rho(k)/q_1 < m \le 0$, then model* (9) *has at least one positive solution if and only if $r_0 > r_0^*$.*

(ii) *Assume that $0 < m < \lambda_1^D(d_v, \Omega_0)$. If $r_0 > r^\diamond$, then model* (9) *has at least one positive solution. Let $\tilde{r}_0 = \inf\{r_0 :$ (9) has a positive solution$\}$. Then $d < \tilde{r}_0 \le r^\diamond$. Moreover, if $\tilde{r}_0 < r^\diamond$, then* (9) *has a positive solution for $r_0 = \tilde{r}_0$.*

***Proof:*** Clearly, by virtue of the conclusions in Theorem 3.4 and the proof of Theorem 2.5, we just have to prove that $\tilde{r}_0 > d$, and if $\tilde{r}_0 < r^\diamond$, $0 < m < \lambda_1^D(d_v, \Omega_0)$, then model (9) has a positive solution for $r_0 = \tilde{r}_0$.

We first prove $\tilde{r}_0 > d$ by contradiction. Let $\tilde{r}_0 = d$, then there exists a sequence $\{r_0^n\}_{n=1}^\infty$ satisfying $r_0^n > d$ and $\lim_{n\to\infty} r_0^n = d$ such that (9) with $r_0 = r_0^n$ has a positive solution $(u_n, v_n)$. By the first equation of (9), we have $r_0^n > \lambda_1^N(d_u, d + au_n + p\rho(k)v_n/(1+q_1u_n + q_2v_n), \Omega) > d$, it holds $\lim_{n\to\infty} r_0^n > d$, which contradicts to $\lim_{n\to\infty} r_0^n = d$. Hence, $\tilde{r}_0 > d$.

Now, we state that if $\tilde{r}_0 < r^\diamond$ and $0 < m < \lambda_1^D(d_v, \Omega_0)$, then model (9) has a positive solution for $r_0 = \tilde{r}_0$. Otherwise, there exists a sequence $\{(r_0^n, u_n, v_n)\}_{n=1}^\infty$ such that $(u_n, v_n)$ is a positive solution of (9) with $r_0 = r_0^n > \tilde{r}_0$ and $\lim_{n\to\infty} r_0^n = \tilde{r}_0$. Because $\tilde{r}_0 > d$, we can obtain from Theorem 3.1 and the standard regularity theory of elliptic equations that subject to a subsequence, $(u_n, v_n) \to (u, v)$ in $X$ as $n \to \infty$, where $(u, v)$ is a nonnegative solution of (9) with $r_0 = \tilde{r}_0$. Since we have assumed that model (9) has no positive solution for $r_0 = \tilde{r}_0$, either $u \equiv 0$ or $v \equiv 0$.

If $u \equiv 0$, then the equation of $v$ and $0 < m < \lambda_1^D(d_v, \Omega_0)$ derive that $v = v_m^{d_v}$. Thus, we have from the equation of $u_n$ that

$$0 = \lambda_1^N\left(d_u, \frac{r_0^n}{1+kv_n} - d - au_n - \frac{p\rho(k)v_n}{1+q_1u_n+q_2v_n}, \Omega\right)$$
$$\to \lambda_1^N\left(d_u, \frac{\tilde{r}_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right)$$

as $n \to \infty$. This indicates $\tilde{r}_0 = r^\diamond$, contradicting the assumption that $\tilde{r}_0 < r^\diamond$. So, $u > 0$.

If $v \equiv 0$, then $u = (\tilde{r}_0 - d)/a$ and

$$\lim_{n\to\infty}\left(m - s(x)v_n + \frac{cp\rho(k)u_n}{1 + q_1u_n + q_2v_n}\right) = m + \frac{cp\rho(k)(\tilde{r}_0 - d)}{a + q_1(\tilde{r}_0 - d)} > 0.$$

So, as $n$ is large enough,

$$\int_\Omega v_n\left(m - s(x)v_n + \frac{cp\rho(k)u_n}{a + q_1u_n + q_2v_n}\right)\mathrm{d}x > 0,$$

which is impossible, so $v > 0$.

Therefore, $(u, v)$ is a positive solution of (9) with $r_0 = \tilde{r}_0$ when $0 < m < \lambda_1^D(d_v, \Omega_0)$. The proof is completed. ■

**Remark 3.4:** Comparing Theorems 2.5 and 3.5, we find that the existence conditions of positive solutions for model (7) are also the existence conditions of positive solutions for model (9). This is the similarity between these two theorems. However, this similarity does not mean that the positive solutions of models (7) and (9) are in the same form, see Remark 3.2 for details. Besides that, there are some differences between Theorems 2.5 and 3.5. Specifically, we observe the followings:

(a) $r_0 > r_0^*$ is a sufficient and necessary condition for the existence of positive solutions of (9), while it is only a sufficient condition for (7), this difference is caused by the predator-taxis. The reasons are the same as described in Remark 3.3;
(b) There is a $\tilde{r}_0$ satisfying $d < \tilde{r}_0 \le r^\diamond$ such that if $\tilde{r}_0 < r^\diamond$, then (9) has a positive solution when $r_0 = \tilde{r}_0$, but there is no similar conclusion for (7), this difference is also caused by the predator-taxis. Precisely, according to the proof of Theorem 3.5, we know that the principal eigenvalue $\lambda_1^N(d_u, d + au + p\rho(k)v/(1 + q_1u + q_2v), \Omega)$ plays a very important role in determining the relationship between $\tilde{r}_0$ and $d$, and proving the existence of positive solutions for (9) at $r_0 = \tilde{r}_0$. However, for model (7), the corresponding principal eigenvalue becomes $\lambda_1^N(d_u, -\alpha(1 - 2u/M)\nabla v\nabla - \alpha(1 - u/M)\Delta v + d + au + p\rho(k)v/(1 + q_1u + q_2v), \Omega)$. Clearly, the appearance of $-\alpha(1 - 2u/M)\nabla v\nabla - \alpha(1- u/M)\Delta v$ hinders our analysis of the problem. So, whether model (7) has a similar conclusion as above remains unknown in this paper.

In what follows, on the basis of the fixed point index theory established on positive cones [32], we investigate the uniqueness and stability of positive solution for model (9). To this end, some crucial notations and preliminaries are first listed.

Let $\mathfrak{E}$ be a Banach space, $\mathfrak{M}$ be a positive cone in $\mathfrak{E}$, and $\mathfrak{D}$ be a closed subspace of $\mathfrak{M}$. Suppose that $\mathcal{A} : \overline{\mathfrak{D}} \subset \mathfrak{M} \longrightarrow \mathfrak{M}$ is a compact operator, and $(u, v) \in \mathfrak{D}$ is a fixed point of $\mathcal{A}$. If $\mathcal{A}(u, v) \neq (u, v)$ for all $(u, v) \in \partial\mathfrak{D}$, then the Leray-Schauder degree $\deg_{\mathfrak{M}}(I - \mathcal{A}, \mathfrak{D}, (u, v))$ is well-defined. Let $\mathcal{A}'(u, v)$ be the Fréchet derivative of $A$ at $(u, v)$. If $(u, v)$ is an isolated fixed point of $\mathcal{A}$ and $I - \mathcal{A}'(u, v)$ is invertible, then the fixed point index of $\mathcal{A}$ at $(u, v)$ is given by $\text{index}_{\mathfrak{M}}(\mathcal{A}, (u, v)) = \deg_{\mathfrak{M}}(I - \mathcal{A}, \mathfrak{N}(u, v), (0, 0))$, where $\mathfrak{N}(u, v)$ is a small open neighborhood of $(u, v)$ in $\mathfrak{M}$.

For $(u^\diamond, v^\diamond) \in \mathfrak{M}$, define

$$\mathfrak{M}_{(u^\diamond,v^\diamond)} = \{(u,v) \in \mathfrak{E} | (u^\diamond, v^\diamond) + \nu(u,v) \in \mathfrak{M},\ \nu > 0\}$$

and

$$S_{(u^\diamond,v^\diamond)} = \{(u,v) \in \overline{\mathfrak{M}}_{(u^\diamond,v^\diamond)} | -(u,v) \in \overline{\mathfrak{M}}_{(u^\diamond,v^\diamond)}\}.$$

Assume that $\mathfrak{E}$ has the decomposition $\mathfrak{E} = \mathfrak{E}_{(u^\diamond,v^\diamond)} \oplus \mathfrak{S}_{(u^\diamond,v^\diamond)}$ and $\mathcal{P} : \mathfrak{E} \longrightarrow \mathfrak{S}_{(u^\diamond,v^\diamond)}$ is the projection from $\mathfrak{E}$ to $\mathfrak{S}_{(u^\diamond,v^\diamond)}$, where $\mathfrak{E}_{(u^\diamond,v^\diamond)}$ is the maximum subspace of $\mathfrak{E}$ contained in $\overline{\mathfrak{M}}_{(u^\diamond,v^\diamond)}$ and $\mathfrak{S}_{(u^\diamond,v^\diamond)}$ is a closed linear subspace. Then by summarizing the ideas in [32], the index of $\mathcal{A}$ at $(u^\diamond, v^\diamond)$ can be calculated by the following lemma.

**Lemma 3.1:** *If* $(0,0)$ *is the only fixed point of* $\mathcal{A}'(u^\diamond, v^\diamond)$ *in* $\overline{\mathfrak{M}}_{(u^\diamond,v^\diamond)}$, *then there hold following results on the fixed point index* $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}, (u^\diamond, v^\diamond))$.

(i) *If* $\mathcal{P} \circ \mathcal{A}'(u^\diamond, v^\diamond)$ *has eigenvalue larger than* 1, *then* $index_{\mathfrak{M}}(\mathcal{A}, (u^\diamond, v^\diamond)) = 0$;
(ii) *If* $\mathcal{P} \circ \mathcal{A}'(u^\diamond, v^\diamond)$ *has no eigenvalue larger than* 1, *then* $index_{\mathfrak{M}}(\mathcal{A}, (u^\diamond, v^\diamond)) = index_{\mathfrak{E}_{(u^\diamond,v^\diamond)}}(\mathcal{A}'(u^\diamond, v^\diamond), (0,0)) = (-1)^\sigma$, *where* $\mathrm{index}_{\mathfrak{E}_{(u^\diamond,v^\diamond)}}(\mathcal{A}'(u^\diamond, v^\diamond), (0,0))$ *denotes the index of* $\mathcal{A}'(u^\diamond, v^\diamond)$ *at* $(0,0)$ *in* $\mathfrak{E}_{(u^\diamond,v^\diamond)}$, $\sigma$ *is the sum of algebraic multiplicities of all eigenvalues of* $\mathcal{A}'(u^\diamond, v^\diamond)$ *in* $\mathfrak{E}_{(u^\diamond,v^\diamond)}$ *which are larger than* 1.

In light of Theorem 3.1, we know that when $r_0 > d$ and $m \neq \lambda_1^D(d_v, \Omega_0)$, there exists a positive constant $W^*$ such that the positive solution $(u,v)$ of model (9) satisfies $u(x) \leq (r_0 - d)/a$ and $v(x) \leq W^*$. Set

$$\mathfrak{E} = \{(u,v) | u,v \in C^1(\overline{\Omega}), \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0,\ x \in \partial\Omega\}, \quad \mathfrak{M} = \{(u,v) \in \mathfrak{E} | u,\ v \geq 0\},$$
$$\mathfrak{D} = \{(u,v) \in \mathfrak{M} \,|\, 0 \leq u(x) < (r_0 - d)/a + \varepsilon,\ 0 \leq v(x) < W^* + \varepsilon\}.$$

For $t \in [0,1]$, define $\mathcal{A}_t : \overline{\mathfrak{D}} \to \mathfrak{M}$ by

$$\mathcal{A}_t(u,v) = \begin{pmatrix} (-d_u\Delta + \tilde{P})^{-1}\left(u\left(\dfrac{r_0}{1+tkv} - d - au - \dfrac{tp\rho(k)v}{1+q_1u+q_2v}\right) + \tilde{P}u\right) \\ (-d_v\Delta + \tilde{P})^{-1}\left(v\left(m - s(x)v + \dfrac{tcp\rho(k)u}{1+q_1u+q_2v}\right) + \tilde{P}v\right) \end{pmatrix},$$

where $\tilde{P}$ is a large positive number such that

$$\max\left\{\max_{\overline{\Omega}}\left\{\left|\frac{r_0}{1+tkv} - d - au - \frac{tp\rho(k)v}{1+q_1u+q_2v}\right|\right\},\right.$$
$$\left.\max_{\overline{\Omega}}\left\{\left|m - s(x)v + \frac{tcp\rho(k)u}{1+q_1u+q_2v}\right|\right\}\right\} < \tilde{P}.$$

Clearly, Theorem 3.1 means that $\mathcal{A}_t$ is a compact operator from $[0,1] \times \overline{\mathfrak{D}}$ to $\mathfrak{M}$, and all nonnegative solutions of model (9) lie in $\mathfrak{D}$. Moreover, a straightforward observation shows that $(u,v)$ is a solution of (9) if and only if $(u,v)$ is a fixed point of $\mathcal{A}_t$ when $t = 1$, and this is independent of the choice of $\tilde{P}$.

Simple analysis gives that for $t \in [0,1]$, when $r_0 > d$ and $0 < m < \lambda_1^D(d_v, \Omega_0)$, $\mathcal{A}_t$ has three nonnegative fixed points $(0,0)$, $((r_0-d)/a, 0)$ and $(0, v_m^{d_v})$ which are not positive. Next, we calculate the indexes of $\mathcal{A}_0$ and $\mathcal{A}_1$ at these fixed points, respectively.

**Lemma 3.2:** *Let $r_0 > d$ and $0 < m < \lambda_1^D(d_v, \Omega_0)$. Then the following statements hold.*

(i) $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0, (0,0)) = \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0, ((r_0-d)/a, 0)) = \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0, (0, v_m^{d_v})) = 0$;
(ii) $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, (0,0)) = \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, ((r_0-d)/a, 0)) = 0$;
(iii) $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) = \begin{cases} 0, & \lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right) > 0, \\ 1, & \lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right) \le 0. \end{cases}$

***Proof:*** We only present the proof of (iii) since the other cases can be proved similarly.

For the fixed point $(0, v_m^{d_v})$, on the basis of the definitions of $\mathfrak{M}_{(u^\diamond, v^\diamond)}$, $S_{(u^\diamond, v^\diamond)}$, $\mathfrak{S}_{(u^\diamond, v^\diamond)}$ and $\mathcal{P}$, we obtain $\overline{\mathfrak{M}}_{(0,v_m^{d_v})} = \{(u,v) \in \mathfrak{E} | u \ge 0\}$, $S_{(0,v_m^{d_v})} = \{(u,v) \in \mathfrak{E} \mid u = 0\}$, $\mathfrak{S}_{(0,v_m^{d_v})} = \{(u,v) \in \mathfrak{E} \mid v = 0\}$ and $\mathcal{P}(u,v) = (u,0)$. On account of

$$\mathcal{A}_1'(0, v_m^{d_v}) = \begin{pmatrix} (-d_u\Delta + \tilde{P})^{-1}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right) & 0 \\ (-d_v\Delta + \tilde{P})^{-1}\frac{cp\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} & (-d_v\Delta + \tilde{P})^{-1}(m - 2s(x)v_m^{d_v} + \tilde{P}) \end{pmatrix},$$

and the formula $\mathcal{P}(u,v) = (u,0)$ holds, where $\tilde{P}$ is large and satisfies $r_0/(1+kv_m^{d_v}) - d - p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}) + \tilde{P} > 0$, $m - 2s(x)v_m^{d_v} + \tilde{P} > 0$, we can verify

$$\mathcal{P} \circ \mathcal{A}_1'(0, v_m^{d_v})(u,v)^T = \left((-d_u\Delta + \tilde{P})^{-1}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right)u, 0\right)^T.$$

Suppose $(\varphi, \psi) \in \overline{\mathfrak{M}}_{(0,v_m^{d_v})}$ and $\mathcal{P} \circ \mathcal{A}_1'(0, v_m^{d_v})(\varphi, \psi)^T = \tau(\varphi, \psi)^T$, where $\tau$ is the eigenvalue of $\mathcal{P} \circ \mathcal{A}_1'(0, v_m^{d_v})$. Then

$$(-d_u\Delta + \tilde{P})^{-1}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right)\varphi = \tau\varphi, \quad \varphi \geqslant, \not\equiv 0. \tag{27}$$

Obviously, since $r_0/(1+kv_m^{d_v}) - d - p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}) + \tilde{P} > 0$, one has $\tau \neq 0$. Multiplying both sides of (27) by $-d_u\Delta + \tilde{P}$ and applying some transformations, we get

$$-d_u\Delta\varphi + \left(\frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} - \frac{r_0}{1+kv_m^{d_v}}\right)\varphi + \frac{\tau-1}{\tau}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right)\varphi$$
$$= -d\varphi.$$

This indicates

$$d = \lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} - \frac{\tau-1}{\tau}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right), \Omega\right).$$

If $\lambda_1^N(d_u, r_0/(1+kv_m^{d_v}) - d - p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}), \Omega) > 0$, then by $r_0/(1+kv_m^{d_v}) - d - p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}) + \tilde{P} > 0$, we can choose $\tau > 1$ such that

$$\lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} - \frac{\tau-1}{\tau}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right), \Omega\right)$$
$$< \lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right).$$

This derives that $\mathcal{P} \circ \mathcal{A}_1'(0, v_m^{d_v})$ has eigenvalue larger than 1. Therefore, Lemma 3.1(i) shows $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) = 0$.

If $\lambda_1^N(d_u, r_0/(1+kv_m^{d_v}) - d - p\rho(k)v_m^{d_v}/(1+q_2v_m^{d_v}), \Omega) \leq 0$, then there holds

$$\lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} - \frac{\tau-1}{\tau}\left(\frac{r_0}{1+kv_m^{d_v}} - d - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}} + \tilde{P}\right), \Omega\right)$$
$$\geq \lambda_1^N\left(d_u, \frac{r_0}{1+kv_m^{d_v}} - \frac{p\rho(k)v_m^{d_v}}{1+q_2v_m^{d_v}}, \Omega\right).$$

This means $\tau \leq 1$. Then we get from Lemma 3.1(ii) that $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) = \mathrm{index}_{\mathfrak{E}_{(0,v_m^{d_v})}}(\mathcal{A}_1'(0, v_m^{d_v}), (0,0)) = (-1)^\sigma$, where $\sigma$ is the sum of algebraic multiplicities of all eigenvalues of $\mathcal{A}_1'(0, v_m^{d_v})$ in $\mathfrak{E}_{(0,v_m^{d_v})}$ which are larger than 1. Therefore, in order to obtain the index $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v}))$, we now need to determine the value of $\sigma$.

Denote by $\tau'$ the eigenvalue of $\mathcal{A}_1'(0, v_m^{d_v})$ corresponding to the eigenfunction $(\varphi', \psi') \in \mathfrak{E}_{(0,v_m^{d_v})}$ with $\varphi' \equiv 0$, $\psi' \not\equiv 0$. Then we have

$$(-d_v\Delta + \tilde{P})^{-1}(m - 2s(x)v_m^{d_v} + \tilde{P})\psi' = \tau'\psi'. \tag{28}$$

Since $m - 2s(x)v_m^{d_v} + \tilde{P} > 0$, $\tau' \neq 0$. Multiplying both sides of (28) by $-d_u\Delta + \tilde{P}$ and applying some transformations, we get

$$-d_v\Delta\psi' + 2s(x)v_m^{d_v}\psi' + \frac{\tau'-1}{\tau'}(m - 2s(x)v_m^{d_v} + \tilde{P})\psi' = m\psi'.$$

Then

$$m = \lambda_1^N\left(d_v, 2s(x)v_m^{d_v} + \frac{\tau' - 1}{\tau'}(m - 2s(x)v_m^{d_v} + \tilde{P}), \Omega\right).$$

Meanwhile, we derive from (10) that $m = \lambda_1^N(d_v, s(x)v_m^{d_v}, \Omega) < \lambda_1^N(d_v, 2s(x)v_m^{d_v}, \Omega)$, this implies $(\tau' - 1)/\tau'(m - 2s(x)v_m^{d_v} + \tilde{P}) < 0$. Thus, $\tau' < 1$, it means $\sigma = 0$. Consequently, Lemma 3.1(ii) implies

$$\text{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) = \text{index}_{\mathfrak{E}_{(0,v_m^{d_v})}}(\mathcal{A}_1'(0, v_m^{d_v}), (0,0)) = 1.$$

The proof is completed. ■

Based on Lemma 3.2, we next discuss the uniqueness and stability of positive solution for model (8) when $k \to 0$ and $q_2 \to \infty$. The result is shown below.

**Theorem 3.6:** *Let* $0 < m < \lambda_1^D(d_v, \Omega_0)$ *and* $r_0 > r^\diamond$. *Then there exist a small and a large positive numbers* $\varepsilon$ *and* $q^*$, *respectively, such that for* $0 < k < \varepsilon$ *and* $q_2 > q^*$, *model* (9) *has a unique positive solution* $(u, v)$, *which is non-degenerate and linearly stable.*

***Proof:*** By Theorem 3.5, it is easy to check that model (9) has at least one positive solution when $0 < m < \lambda_1^D(d_v, \Omega_0)$ and $r_0 > r^\diamond$. Next, we prove that any positive solution $(u, v)$ of (9) is non-degenerate and linearly stable for sufficiently small $k$ and sufficiently large $q_2$. This purpose will be achieved by reductio. Assume that there exist a sequence $\{(k_n, q_2^n)\}_{n=1}^\infty$ satisfying $(k_n, q_2^n) \to (0, \infty)$ as $n \to \infty$, and a corresponding positive solution $(u_n, v_n)$ of (9) at $(k, q_2) = (k_n, q_2^n)$, such that for any $n \geq 1$, the linearized problem

$$\begin{cases} -d_u\Delta\phi_n = \left(\dfrac{r_0}{1 + k_nv_n} - d - 2au_n - \dfrac{p\rho(k_n)v_n(1 + q_2^nv_n)}{(1 + q_1u_n + q_2^nv_n)^2}\right)\phi_n \\ -\left(\dfrac{r_0k_nu_n}{(1 + k_nv_n)^2} + \dfrac{p\rho(k_n)u_n(1 + q_1u_n)}{(1 + q_1u_n + q_2^nv_n)^2}\right)\psi_n + \gamma_n\phi_n, & x \in \Omega, \\ -d_v\Delta\psi_n = \left(m - 2s(x)v_n + \dfrac{cp\rho(k_n)u_n(1 + q_1u_n)}{(1 + q_1u_n + q_2^nv_n)^2}\right)\psi_n \\ +\dfrac{cp\rho(k_n)v_n(1 + q_2^nv_n)}{(1 + q_1u_n + q_2^nv_n)^2}\phi_n + \gamma_n\psi_n, & x \in \Omega, \\ \partial_{\mathbf{n}}\phi_n = \partial_{\mathbf{n}}\psi_n = 0, & x \in \partial\Omega \end{cases} \tag{29}$$

of (9) has an eigenvalue solution pair $(\phi_n, \psi_n, \gamma_n)$, where $\gamma_n$ is the eigenvalue and $(\phi_n, \psi_n)$ is the corresponding eigenfunction with $\|\phi_n\|_2 + \|\psi_n\|_2 = 1$ and $\text{Re}\gamma_n \leq 0$. Notice that $\phi_n$ and $\psi_n$ may be complex-valued.

In what follows, we prove that $\{\gamma_n\}_{n=1}^\infty$ is uniformly bounded. At first, we claim that $\{\text{Re}\gamma_n\}_{n=1}^\infty$ is uniformly bounded. Owing to $\text{Re}\gamma_n \leq 0$, we only need to prove that $\{\text{Re}\gamma_n\}_{n=1}^\infty$ is bounded below. Let $\text{Re}\gamma_n \to -\infty$ as $n \to \infty$. Denote $\bar{\phi}_n$ and $\bar{\psi}_n$ the conjugate functions of $\phi_n$ and $\psi_n$, respectively. By the Kato's inequality, one has

$$\begin{aligned}-d_u\Delta|\phi_n| &\le -d_u\mathrm{Re}\left(\frac{\bar{\phi}_n}{|\phi_n|}\Delta\phi_n\right)\\ &\le \left(r_0 - d - 2au_n - \frac{p\rho(k_n)v_n(1+q_2^n v_n)}{(1+q_1u_n+q_2^n v_n)^2}\right)|\phi_n|\\ &\quad + \left(\frac{r_0k_nu_n}{(1+k_nv_n)^2} + \frac{p\rho(k_n)u_n(1+q_1u_n)}{(1+q_1u_n+q_2^n v_n)^2}\right)|\psi_n| + \mathrm{Re}\gamma_n|\phi_n| \qquad (30)\end{aligned}$$

and

$$\begin{aligned}-d_v\Delta|\psi_n| &\le -d_v\mathrm{Re}\left(\frac{\bar{\psi}_n}{|\psi_n|}\Delta\psi_n\right)\\ &\le \left(m - 2s(x)v_n + \frac{cp\rho(k_n)u_n(1+q_1u_n)}{(1+q_1u_n+q_2^n v_n)^2}\right)|\psi_n|\\ &\quad + \frac{cp\rho(k_n)v_n(1+q_2^n v_n)}{(1+q_1u_n+q_2^n v_n)^2}|\phi_n| + \mathrm{Re}\gamma_n|\psi_n|. \qquad (31)\end{aligned}$$

Multiplying both sides of (30) by $|\phi_n|$, then integrating over $\Omega$, and then applying the Hölder inequality and the fact $u_n < \frac{r_0-d}{a}$, we have

$$\begin{aligned}-(r_0+\mathrm{Re}\gamma_n)\int_\Omega|\phi_n|^2\,\mathrm{d}x \le \left(\frac{r_0k_n(r_0-d)}{a} + \frac{p\rho(k_n)}{q_1}\right)\left(\int_\Omega|\phi_n|^2\,\mathrm{d}x\right)^{1/2}\\ \times\left(\int_\Omega|\psi_n|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Since $\mathrm{Re}\gamma_n \to -\infty$ and $k_n \to 0$ as $n \to \infty$, we get that for sufficiently large $n$, there has $\|\phi_n\|_2 \le p/(q_1(-r_0 - \mathrm{Re}\gamma_n))\|\psi_n\|_2$. Let $n \to \infty$ in this inequality. One has $\|\phi_n\|_2 \to 0$ as $n \to \infty$, this implies $\|\psi_n\|_2 \to 1$ as $n \to \infty$.

Similarly, multiplying both sides of (31) by $|\psi_n|$, then integrating by parts over $\Omega$, and then applying the Hölder inequality, we get

$$-\left(m + \frac{cp\rho(k_n)}{q_1} + \mathrm{Re}\gamma_n\right)\int_\Omega|\psi_n|^2\,\mathrm{d}x \le \frac{cp\rho(k_n)}{q_2^n}\left(\int_\Omega|\phi_n|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega|\psi_n|^2\mathrm{d}x\right)^{1/2}.$$

Combining $\mathrm{Re}\gamma_n \to -\infty$, $k_n \to 0$ and $q_2^n \to \infty$ as $n \to \infty$, we obtain $\|\psi_n\|_2 \to 0$ as $n \to \infty$, which contradicts to $\|\psi_n\|_2 \to 1$ as $n \to \infty$ obtained above. So, $\{\mathrm{Re}\gamma_n\}_{n=1}^\infty$ is bounded below.

Now, we derive that $\{\mathrm{Im}\gamma_n\}_{n=1}^\infty$ is uniformly bounded. Assume $|\mathrm{Im}\gamma_n| \to \infty$ as $n \to \infty$. Multiply the first two equations of (29) by $\bar{\phi}_n$ and $\bar{\psi}_n$ respectively and integrate by parts over $\Omega$ to yield

$$|\mathrm{Im}\gamma_n|\int_\Omega|\phi_n|^2\,\mathrm{d}x = \left|\mathrm{Im}\int_\Omega\left(\frac{r_0k_nu_n}{(1+k_nv_n)^2} + \frac{p\rho(k_n)u_n(1+q_1u_n)}{(1+q_1u_n+q_2^n v_n)^2}\right)\bar{\phi}_n\psi_n\,\mathrm{d}x\right|$$

and

$$|\mathrm{Im}\gamma_n|\int_\Omega|\psi_n|^2\,\mathrm{d}x = \left|\mathrm{Im}\int_\Omega\frac{cp\rho(k_n)v_n(1+q_2^n v_n)}{(1+q_1u_n+q_2^n v_n)^2}\phi_n\bar{\psi}_n\,\mathrm{d}x\right|.$$

Applying the Hölder inequality to the above two equations, we have

$$|\mathrm{Im}\gamma_n|\|\phi_n\|_2 \le \left(\frac{r_0k_n(r_0-d)}{a}+\frac{p\rho(k_n)}{q_1}\right)\|\psi_n\|_2 \quad \text{and}$$

$$|\mathrm{Im}\gamma_n|\|\psi_n\|_{2,\Omega} \le \frac{cp\rho(k_n)}{q_2^n}\|\phi_n\|_2.$$

Owing to $|\mathrm{Im}\gamma_n| \to \infty$, $k_n \to 0$ and $q_2^n \to \infty$ as $n \to \infty$, we conclude $\|\phi_n\|_2 \to 0$ and $\|\psi_n\|_2 \to 0$ as $n \to \infty$, contradicting the fact that $\|\phi_n\|_2 + \|\psi_n\|_2 = 1$. Thus, $\{\mathrm{Im}\gamma_n\}_{n=1}^{\infty}$ is uniformly bounded. Summarizing the previous analysis, we conclude that $\{\gamma_n\}_{n=1}^{\infty}$ is uniformly bounded. So, there exists $\gamma$ satisfying $\mathrm{Re}\gamma \le 0$ such that $\gamma_n \to \gamma$ as $n \to \infty$. Next, we derive $\mathrm{Re}\gamma > 0$, this contradiction will show that any positive solution $(u,v)$ of model (9) is stable.

Due to $(\phi_n,\psi_n)$ satisfies (29) and $\|\phi_n\|_2 + \|\psi_n\|_2 = 1$, one has that as $n \to \infty$, there exists $(\phi,\psi)$ such that $(\phi_n,\psi_n) \to (\phi,\psi)$ weakly in $H^1(\Omega)\times H^1(\Omega)$ and strongly in $L^2(\Omega)\times L^2(\Omega)$. Furthermore, since $r_0^{**} > d$, $k_n \to 0$ and $q_2^n \to \infty$ as $n \to \infty$, we have that when $r_0 > r_0^{**}$ and $0 < m < \lambda_1^D(d_v,\Omega_0)$, $(u_n,v_n) \to ((r_0-d)/a, v_m^{d_v})$ as $n \to \infty$. Hence, $(\phi,\psi)$ satisfies

$$\begin{cases} -d_u\Delta\phi = (d-r_0)\phi + \gamma\phi, & x\in\Omega,\\ -d_v\Delta\psi = (m-2s(x)v_m^{d_v})\psi + \gamma\psi, & x\in\Omega,\\ \partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}\psi = 0, & x\in\partial\Omega \end{cases} \tag{32}$$

by letting $n \to \infty$ in (29). According to the definitions of $\varphi$ and $\psi$, we know that at least one of them is not equivalent to zero. Thus, (32) implies $\mathrm{Re}\gamma > 0$, which contradicts to $\mathrm{Re}\gamma \le 0$. Therefore, any positive solution $(u,v)$ of model (9) is non-degenerate and linearly stable.

The remaining proves the uniqueness of positive solution $(u,v)$. Suppose that $\mathcal{A}_t$ and $\mathfrak{D}$ are the same as that of defined in the previous analysis. Then $(u,v)$ is a solution of (9) if and only if $(u,v)$ is a fixed point of $\mathcal{A}_t$ as $t=1$. Clearly, for any $t\in[0,1]$, $\mathcal{A}_t(u,v)\ne(u,v)$, $(u,v)\in\partial\mathfrak{D}$. Hence, $\deg_{\mathfrak{M}}(I-\mathcal{A}_t,\mathfrak{D},(0,0))$ is well-defined, and the homotopy invariance of degree implies $\deg_{\mathfrak{M}}(I-\mathcal{A}_0,\mathfrak{D},(0,0)) = \deg_{\mathfrak{M}}(I-\mathcal{A}_1,\mathfrak{D},(0,0))$.

Due to $r_0 > r_0^{**} > d$ and $0 < m < \lambda_1^D(d_v,\Omega_0)$, we get that when $t=0$, $\mathcal{A}_0$ has four fixed points $(0,0)$, $((r_0-d)/a,0)$, $(0,v_m^{d_v})$ and $((r_0-d)/a,v_m^{d_v})$, and Lemma 3.2(i) shows $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,(0,0)) = \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,((r_0-d)/a,0)) = \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,(0,v_m^{d_v})) = 0$. Moreover, simple analysis gives that the positive fixed point $((r_0-d)/a,v_m^{d_v})$ of $\mathcal{A}_0$ is equivalent to the positive solution of (9) as $k\to 0$ and $q_2\to\infty$. Then the stability result obtained above means that $((r_0-d)/a,v_m^{d_v})$ is stable, it follows that $\mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,((r_0-d)/a,v_m^{d_v})) = 1$. Further, from the additivity property of fixed point index, we obtain

$$\begin{aligned}\deg_{\mathfrak{M}}(I-\mathcal{A}_0,\mathfrak{D},(0,0)) &= \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,(0,0)) + \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,((r_0-d)/a,0))\\ &\quad + \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,(0,v_m^{d_v})) + \mathrm{index}_{\mathfrak{M}}(\mathcal{A}_0,((r_0-d)/a,v_m^{d_v}))\\ &= 0+0+0+1 = 1.\end{aligned}$$

When $t=1$, $\mathcal{A}_1$ has three nonnegative fixed points $(0,0)$, $((r_0-d)/a,0)$ and $(0,v_m^{d_v})$ which are not positive. Since $r_0 > r^{\diamond}$ is equivalent to $\lambda_1^N(d_u,r_0/(1+kv_m^{d_v}) -$

$d - p\rho(k)v_m^{d_v}/(1+q_2 v_m^{d_v}), \Omega) > 0$, Lemma 3.2(ii)–(iii) derives $\text{index}_{\mathfrak{M}}(\mathcal{A}_1, (0,0)) = \text{index}_{\mathfrak{M}}(\mathcal{A}_1, ((r_0 - d)/a, 0)) = \text{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) = 0$. Moreover, by the compactness of $\mathcal{A}_1$, there exist only finite isolated positive fixed points of $\mathcal{A}_1$, which are denoted by $(u_i, v_i)$, $i = 1, \cdots, n$. Additionally, the stability of any positive solution obtained above yields that $\mathcal{A}_1$ has index 1 at each positive fixed point, that is, $\text{index}_{\mathfrak{M}}(\mathcal{A}_1, (u_i, v_i)) = 1$, $i = 1, \cdots, n$. Hence,

$$\begin{aligned} 1 &= \deg_{\mathfrak{M}}(I - \mathcal{A}_0, \mathfrak{D}, (0,0)) = \deg_{\mathfrak{M}}(I - \mathcal{A}_1, \mathfrak{D}, (0,0)) \\ &= \text{index}_{\mathfrak{M}}(\mathcal{A}_1, (0,0)) + \text{index}_{\mathfrak{M}}\left(\mathcal{A}_1, \left(\frac{r_0 - d}{a}, 0\right)\right) \\ &\quad + \text{index}_{\mathfrak{M}}(\mathcal{A}_1, (0, v_m^{d_v})) + \sum_{i=1}^{n} \text{index}_{\mathfrak{M}}(\mathcal{A}_1, (u_i, v_i)) \\ &= 0 + 0 + 0 + n = n. \end{aligned}$$

It holds $n = 1$. Therefore, model (9) has a unique positive solution. The proof is completed. ■

Similar to the discussion of Theorem 3.6, the uniqueness and stability of positive solution for model (9) when $k \to 0$ and $q_1 \to \infty$ can also be obtained, see Theorem 3.7.

**Theorem 3.7:** *Let $0 < m < \lambda_1^D(d_v, \Omega_0)$ and $r_0 > r^\diamond$. Then there exist a small and a large positive numbers $\varepsilon$ and $q^{**}$, respectively, such that for $0 < k < \varepsilon$ and $q_1 > q^{**}$, model* (9) *has a unique positive solution $(u, v)$, which is non-degenerate and linearly stable.*

**Remark 3.5:** Theorems 3.6–3.7 show that when the fear level $k$ is very low and the interference effect $q_2$ ($q_1$) of predator (prey) is quite strong, model (9) has a unique positive solution under certain conditions. However, due to the complexity of model (7), the mathematical techniques in proving Theorem 3.6 are not applicable to it. Therefore, whether the conclusions similar to that of Theorems 3.6–3.7 hold true for (7) remains unknown in the present paper.

### 3.3. Asymptotic behaviors of positive solutions as $r_0 \to \infty$ or $m \to \lambda_1^D(d_v, \Omega_0)$

This subsection concentrates on the asymptotic behaviors of positive solutions for model (9) as $r_0 \to \infty$ or $m \to \lambda_1^D(d_v, \Omega_0)$.

Firstly, the asymptotic behavior of positive solutions as $r_0 \to \infty$ is discussed. The result is as follows.

**Theorem 3.8:** *Assume that $\{r_0^n\}_{n=1}^{\infty}$ is a sequence satisfying $\lim_{n\to\infty} r_0^n = \infty$, and $(u_n, v_n)$ is an arbitrary positive solution of model* (9) *corresponding to $r_0 = r_0^n$. Then the following statements hold.*

(i) $\lim_{n\to\infty} \frac{u_n}{r_0^n} = l_1' \in [0, 1/a]$.

(ii) *If $l_1'$ in* (i) *is positive and* $-cp\rho(k)/q_1 < m < \lambda_1^D(d_v,\Omega_0) - cp\rho(k)/q_1$, *then subject to a subsequence,* $\lim_{n\to\infty} v_n = v_{\bar m}^{d_v}$ *in* $\overline{\Omega}$, *where* $\bar m$ *and* $v_{\bar m}^{d_v}$ *are the same as that of defined in Theorem* 2.6(ii).

(iii) *If $l_1'$ in* (i) *is positive and* $\lambda_1^D(d_v,\Omega_0) - cp\rho(k)/q_1 \le m < \lambda_1^D(d_v,\Omega_0)$, *then*

**(B1)** $\|v_n\|_\infty \to \infty$ *as* $n \to \infty$;

**(B2)** *subject to a subsequence,* $\|v_n\|_\infty/r_0^n \to l_2' \in [0,\infty)$, $\tilde v_n = v_n/\|v_n\|_\infty \to \tilde v$ *weakly in* $H^1(\Omega)$ *and strongly in* $L^P(\Omega)$ *for any* $p > 1$ *as* $n \to \infty$, *where* $\tilde v$ *is a nonnegative function satisfying* $\tilde v = 0$ *for* $x \in \Omega\backslash\overline{\Omega}_0$, *and* $\tilde v|_{\Omega_0} \in H_0^1(\Omega_0)$ *is a positive weak solution of*

$$-d_v\Delta\tilde v = \tilde v\left(m + \frac{cpl_1'\rho(k)}{q_1l_1' + q_2l_2'\tilde v}\right), \quad x \in \Omega_0, \qquad \tilde v = 0, \;\; x \in \partial\Omega_0.$$

***Proof:*** For Part (i), set $w_n = u_n/r_0^n$. Similar to the proof of Theorem 2.6(i), we have $0 \le w_n \le (r_0^n - d)/(ar_0^n)$. It follows that $\lim_{n\to\infty} u_n/r_0^n = \lim_{n\to\infty} w_n = l_1' \in [0, 1/a]$. The proof of Part (i) follows.

The remaining parts can be proved by using a simple variant of the arguments in the proof of Theorem 2.6(ii)–(iii), we omit them here. The proof is completed. ■

Secondly, we consider the asymptotic behavior of positive solutions for model (9) as $m \to \lambda_1^D(d_v,\Omega_0)$. To this end, let's start with some notations and basic facts.

As we all know, $m \to v_m^{d_v}$ is continuous and strictly increasing from $(0, \lambda_1^D(d_v, \Omega_0))$ to $C^1(\overline{\Omega})$. According to Theorem 2.2 in [22], we obtain that $m \to \lambda_1^N(d_u, p\rho(k)v_m^{d_v}/(1 + q_2v_m^{d_v}), \Omega\backslash\overline{\Omega}_0)$ is also continuous and strictly increasing, and

$$\lim_{m\to\lambda_1^D(d_v,\Omega_0)} \lambda_1^N\left(d_u, \frac{p\rho(k)v_m^{d_v}}{1 + q_2v_m^{d_v}}, \Omega\backslash\overline{\Omega}_0\right) = \lambda_1^N\left(d_u, \frac{p\rho(k)V^*}{1 + q_2V^*}, \Omega\backslash\overline{\Omega}_0\right),$$

where $V^*$ is the minimal positive solution of the blow-up problem

$$-d_v\Delta V = mV - s(x)V^2, \;\; x \in \Omega\backslash\overline{\Omega}_0, \quad \partial_{\mathbf n}V = 0, \;\; x \in \partial\Omega, \quad V = \infty, \;\; x \in \partial\Omega_0. \quad (33)$$

Obviously,

$$\lambda_1^N\left(d_u, \frac{p\rho(k)V^*}{1 + q_2V^*}, \Omega\backslash\overline{\Omega}_0\right) \to \lambda_1^N(d_u, p\rho(k)V^*, \Omega\backslash\overline{\Omega}_0) \triangleq \mathfrak{R}^*$$

as $q_2 \to 0$. Make the following assumptions:

(**H1**) There exist positive constants $\vartheta$ and $\epsilon$ such that $s(x)$ satisfies $\lim_{x\to\partial\Omega_0} \frac{s(x)}{(d(x,\partial\Omega))^\vartheta} = \epsilon$;

(**H2**) $\{m_n\}_{n=1}^\infty$ is a sequence satisfying $m_n \to \lambda_1^D(d_v,\Omega_0)$ as $n \to \infty$.

By Theorem 2.8 in [33], (**H1**) guarantees that problem (33) has a unique positive solution $V^*$. The following theorems describe the asymptotic behaviors of positive solutions for model (9) as $m \to \lambda_1^D(d_v,\Omega_0)$.

**Theorem 3.9:** *Let* (***H2***) *hold. Suppose that* $v_n$ *is an arbitrary positive solution of the second equation of model* (9) *with* $m = m_n$. *Then* $\lim_{n\to\infty} v_n = \infty$ *uniformly on* $\overline{\Omega}_0$.

***Proof:*** From the equation of $v_n$, one sees that $v_n$ satisfies

$$-d_v\Delta v_n \geq m_n v_n - s(x)v_n^2, \ \ x \in \Omega, \quad \partial_{\mathbf{n}} v_n = 0, \ \ x \in \partial\Omega.$$

Let $v_{m_n}^{d_v}$ be positive solution of the problem

$$-d_v\Delta v_n = m_n v_n - s(x)v_n^2, \ \ x \in \Omega, \quad \partial_{\mathbf{n}} v_n = 0, \ \ x \in \partial\Omega$$

when $0 < m_n < \lambda_1^D(d_v, \Omega_0)$. Then it follows from Lemma 2.1 in [33] that $v_{m_n}^{d_v} \leq v_n$. Since $m_n \to \lambda_1^D(d_v, \Omega_0)$ as $n \to \infty$, one sees from Theorem 2.2 in [22] that $v_{m_n}^{d_v} \to \infty$ uniformly on $\overline{\Omega}_0$ as $n \to \infty$. This means $v_n \to \infty$ uniformly on $\overline{\Omega}_0$ as $n \to \infty$. The proof follows. ■

**Theorem 3.10:** *Let (**H1**) and (**H2**) hold. Assume that* $\{q_2^n\}_{n=1}^{\infty}$ *is a sequence satisfying* $\lim_{n\to\infty} q_2^n = 0$, *and* $(u_n, v_n)$ *is an arbitrary positive solution of model* (9) *with* $(m, q_2) = (m_n, q_2^n)$. *Then the following statements hold.*

(i) $\lim_{n\to\infty} u_n = 0$ *uniformly on any compact subset of* $\Omega_0$;
(ii) *If* $r_0 = d + \Re^*$ *and* $q_1$ *is sufficiently small, then subject to a subsequence,* $\lim_{n\to\infty}(u_n,\ v_n)|_{\Omega\setminus\overline{\Omega}_0} = (0, V^*)$ *in the space* $L^\infty(\overline{\Omega}\setminus\Omega_0) \times C_{loc}^1(\overline{\Omega}\setminus\overline{\Omega}_0)$, *where* $C_{loc}^1(\overline{\Omega}\setminus\overline{\Omega}_0) = \bigcap_\Lambda C^1(\Lambda)$ *with* $\Lambda$ *running through all the closed subsets of* $\overline{\Omega}\setminus\overline{\Omega}_0$;
(iii) *If* $r_0 > d + \Re^*$, *then subject to a subsequence,* $\lim_{n\to\infty}(u_n, v_n)|_{\Omega\setminus\overline{\Omega}_0} = (u^\circ, v^\circ)$ *in the space* $C_{loc}^1(\overline{\Omega}\setminus\overline{\Omega}_0) \times C_{loc}^1(\overline{\Omega}\setminus\overline{\Omega}_0)$ *with* $(u^\circ, v^\circ)$ *being a positive solution of the boundary blow-up problem*

$$\begin{cases} -d_u\Delta u = \dfrac{r_0}{1+kv}u - du - au^2 - \dfrac{p\rho(k)uv}{1+q_1u+q_2v}, & x \in \Omega\setminus\overline{\Omega}_0, \\ -d_v\Delta v = \lambda_1^D(d_v, \Omega_0)v - s(x)v^2 + \dfrac{cp\rho(k)uv}{1+q_1u+q_2v}, & x \in \Omega\setminus\overline{\Omega}_0, \\ \partial_{\mathbf{n}} u|_{\partial\Omega} = u|_{\partial\Omega_0} = 0, \quad \partial_{\mathbf{n}} v|_{\partial\Omega} = 0, \quad v|_{\partial\Omega_0} = \infty. \end{cases} \tag{34}$$

***Proof:*** (i) Set $\alpha_n = \min_{x\in\overline{\Omega}_0} v_n(x)$. The conclusion in Theorem 3.9 implies $\alpha_n \to \infty$ as $n \to \infty$. From the equation of $u_n$, we find that $u_n$ satisfies

$$\begin{aligned} -d_u\Delta u_n &\leq r_0u_n - du_n - au_n^2 - \frac{p\rho(k)\alpha_n u_n}{1+q_1u_n+q_2^n\alpha_n} \\ &\leq \left(r_0 - d - \frac{p\rho(k)\alpha_n}{1+q_1(r_0-d)/a+q_2^n\alpha_n}\right)u_n - au_n^2. \end{aligned} \tag{35}$$

Let $\Pi_n = |r_0 - d - p\rho(k)\alpha_n/(1+q_1(r_0-d)/a+q_2^n\alpha_n)|$. Denote $\mathfrak{w}_n = \beta\varpi(x)^{-4}/\Pi_n$ with $\varpi(x)$ being a smooth function on $\overline{\Omega}_0$ satisfying $\varpi(x) = 0$ on $\partial\Omega_0$ and $\varpi(x) > 0$ in $\Omega_0$, and $\beta > 0$ being a constant to be determined later. Some straightforward calculations yield that if $\beta$ is chosen large enough, then for all $x \in \Omega_0$, we have

$$\begin{aligned} & d_u \Delta \mathfrak{w}_n + \left( r_0 - d - \frac{p\rho(k)\alpha_n}{1 + q_1(r_0 - d)/a + q_2^n \alpha_n} \right) \mathfrak{w}_n - a\mathfrak{w}_n^2 \\ & = \frac{\beta d_u}{\Pi_n} \left( 20\varpi(x)^{-6} |\nabla \varpi(x)|^2 - 4\varpi(x)^{-5} \Delta \varpi(x) \right) - \beta \varpi(x)^{-4} - \frac{a\beta^2}{\Pi_n^2} \varpi(x)^{-8} \\ & \leq \frac{\beta \varpi(x)^{-6}}{\Pi_n} (20 d_u |\nabla \varpi(x)|^2 - 4 d_u \varpi(x) \Delta \varpi(x) - 2\sqrt{a\beta}) \\ & < 0. \end{aligned}$$

Thus, for all $n \geq 1$, one has

$$-d_u \Delta \mathfrak{w}_n \geq \left( r_0 - d - \frac{p\rho(k)\alpha_n}{1 + q_1(r_0 - d)/a + q_2^n \alpha_n} \right) \mathfrak{w}_n - a\mathfrak{w}_n^2, \quad x \in \Omega_0. \tag{36}$$

Since $\mathfrak{w}_n = \beta \varpi(x)^{-4}/\Pi_n \to \infty$ as $x \to \partial\Omega_0$, $\mathfrak{w}_n > u_n$ on $\partial\Omega_0$. Furthermore, combining (35)–(36) and Lemma 2.1 in [34], we conclude $\mathfrak{w}_n \geq u_n$ on $\Omega_0$. Additionally, since $q_2^n \to 0$ and $\alpha_n \to \infty$ as $n \to \infty$, we have $\Pi_n \to \infty$ as $n \to \infty$, and then $\mathfrak{w}_n \to 0$ uniformly on any compact subset of $\Omega_0$ as $n \to \infty$, the same is true for $u_n$. The proof of (i) follows.

(ii) Since

$$-d_u \Delta u_n \leq r_0 u_n, \ \ x \in \Omega, \quad \partial_{\mathbf{n}} u_n = 0, \ \ x \in \partial\Omega$$

and

$$d_u \int_\Omega |\nabla u_n|^2 \, \mathrm{d}x + \int_\Omega u_n^2 \, \mathrm{d}x \leq (r_0 + 1) \int_\Omega u_n^2 \, \mathrm{d}x \leq (r_0 + 1)|\Omega|,$$

there exists a subsequence of $\{u_n\}_{n=1}^\infty$, also denoted by itself, such that $u_n \to u^\circ$ weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for any $p \geq 1$. Owing to $m_n \to \lambda_1^D(d_v, \Omega_0)$ and $q_2^n \to 0$ as $n \to \infty$, we see from conclusion (i) that $u^\circ = 0$ almost everywhere in $\Omega_0$. Furthermore, from the smoothness assumption on $\partial\Omega_0$, one has $u^\circ|_{\Omega\setminus\overline{\Omega}_0} \in H_0^1(\Omega\setminus\overline{\Omega}_0)$ and $u^\circ|_{\partial\Omega_0} = 0$.

We now show $u^\circ = 0$ almost everywhere in $\Omega$. Otherwise, together with the discussion above, $u^\circ > 0$ in $\Omega\setminus\overline{\Omega}_0$. By the proof of Theorem 3.9, we have $v_n \geq v_{m_n}^{d_v}$ in $\Omega$. Due to $u_n < (r_0 - d)/a$, one can obtain from the equation of $u_n$ that

$$\begin{aligned} r_0 - d &> \lambda_1^N \left( d_u, au_n + \frac{p\rho(k)v_n}{1 + q_1 u_n + q_2^n v_n}, \Omega \right) \\ &\geq \lambda_1^N \left( d_u, au_n + \frac{p\rho(k)v_{m_n}^{d_v}}{1 + q_1(r_0 - d)/a + q_2^n v_{m_n}^{d_v}}, \Omega \right). \end{aligned}$$

Since (**H2**) holds, it follows from Theorem 2.2 in [22] that $v_{m_n}^{d_v} \to V^*$ in $\Omega\backslash\overline{\Omega}_0$. Combining $q_2^n \to 0$ as $n \to \infty$, one concludes that if $q_1$ is chosen suitably small, then

$$\begin{aligned}\lambda_1^N\left(d_u, au_n + \frac{p\rho(k)v_{m_n}^{d_v}}{1+q_1(r_0-d)/a+q_2^n v_{m_n}^{d_v}}, \Omega\backslash\overline{\Omega}_0\right)\\ \to \lambda_1^N\left(d_u, au^\circ + \frac{p\rho(k)V^*}{1+q_1(r_0-d)/a}, \Omega\backslash\overline{\Omega}_0\right)\\ > \lambda_1^N(d_u, p\rho(k)V^*, \Omega\backslash\overline{\Omega}_0) = \mathfrak{R}^*\end{aligned}$$

as $n \to \infty$. This means $r_0 - d > \mathfrak{R}^*$, contradicting the assumption $r_0 - d = \mathfrak{R}^*$. Thus, $u^\circ = 0$ almost everywhere in $\Omega$, and therefore $u_n \to 0$ in $L^p(\Omega)$ for any $p \geq 1$. Furthermore, from $-d_u\Delta u_n \leq r_0 u_n$, we get $0 \leq u_n \leq (-d_u\Delta + I)^{-1}(r_0+1)u_n$. Then the regularity theory of elliptic equation shows $u_n \to 0$ in $L^\infty(\Omega)$.

Next, we prove $v_n \to V^*$ in $C_{loc}^1(\overline{\Omega}\backslash\overline{\Omega}_0)$. For this purpose, we define a sequence of enlarging smooth domains $\Omega_n$ given by $\Omega_n = \{x \in \Omega_0 : d(x, \partial\Omega_0) > \sigma_n\}$, where $\sigma_n$ is a decreasing sequence of positive numbers with $\sigma_n \to 0$ as $n \to \infty$. Suppose that $\sigma_1$ is a sufficiently small positive number such that for any $n \geq 1$, $\Omega_n$ is not empty and $\partial\Omega_n$ is as smooth as $\partial\Omega_0$. Set

$$s_n(x) = s(x) + d(x, \Omega_n), \quad x \in \Omega.$$

Then $s_n(x)$ satisfies (i) $s_n(x) \to s(x)$ as $n \to \infty$ in $L^\infty(\Omega)$; (ii) $s_n(x) > 0$ in $\overline{\Omega}\backslash\overline{\Omega}_n$; (iii) $s_n(x) = 0$ in $\overline{\Omega}_n$; (iv) $s_n(x) \geq s_{n+1}(x)$ for all $x \in \Omega$.

For fixed $\varepsilon > 0$, by Theorem 2.8 in [33], we know that the problem

$$\begin{aligned}&-d_v\Delta\mathcal{H} = (m_n+\varepsilon)\mathcal{H} - s_n(x)\mathcal{H}^2,\ \ x \in \Omega\backslash\overline{\Omega}_n, \quad \partial_{\mathbf{n}}\mathcal{H} = 0,\ \ x \in \partial\Omega,\\ &\mathcal{H} = \infty,\ \ x \in \partial\Omega_n\end{aligned}$$

has a unique positive solution $\mathcal{H}_n$ for each $n$. Further, with the help of Lemma 2.1 in [33], we conclude that in $\Omega\backslash\overline{\Omega}_0$, $\mathcal{H}_n$ is increasing with respect to $n$. It holds that $\mathcal{H}^*(x) = \lim_{n\to\infty}\mathcal{H}_n(x)$ is well-defined over $\Omega\backslash\overline{\Omega}_0$. Then a regularity consideration shows that $\mathcal{H}^*$ is a solution of

$$\begin{aligned}-d_v\Delta\mathcal{H} = (\lambda_1^D(d_v, \Omega_0)+\varepsilon)\mathcal{H} - s(x)\mathcal{H}^2,\ \ x \in \Omega\backslash\overline{\Omega}_0, \quad \partial_{\mathbf{n}}\mathcal{H} = 0,\ \ x \in \partial\Omega,\\ \mathcal{H} = \infty,\ \ x \in \partial\Omega_0.\end{aligned} \tag{37}$$

It is obvious that problem (37) has a unique positive solution $V_\varepsilon^*$ under the assumption (**H1**). Therefore, $\mathcal{H}^* = V_\varepsilon^*$, that is, $\lim_{n\to\infty}\mathcal{H}_n(x) = V_\varepsilon^*$, $x \in \Omega\backslash\overline{\Omega}_0$.

On account of $\|u_n\|_\infty \to 0$ as $n \to \infty$, for all large $n$, $v_n$ satisfies

$$-d_v\Delta v_n = \left(m_n + \frac{cp\rho(k)u_n}{1+q_1u_n+q_2^n v_n}\right)v_n - s(x)v_n^2 \leq (m_n+\varepsilon)v_n - s_n(x)v_n^2.$$

Using Lemma 2.1 in [33], we get that for all large $n$, $v_n \leq \mathcal{H}_n$ in $\Omega\backslash\overline{\Omega}_0 \subset \Omega\backslash\overline{\Omega}_n$. This derives

$$\overline{\lim}_{n\to\infty} v_n(x) \leq \overline{\lim}_{n\to\infty}\mathcal{H}_n = V_\varepsilon^*(x) \tag{38}$$

for $x \in \Omega\backslash\overline{\Omega}_0$. Furthermore, by a simple regularity and compactness consideration, one sees that the uniqueness of $V_\varepsilon^*$ implies that $V_\varepsilon^*$ varies continuously with respect to $\varepsilon$ in the norm $C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$. Letting $\varepsilon \to 0$ in (38), we have

$$\overline{\lim}_{n\to\infty} v_n(x) \leq V^*(x), \quad x \in \Omega\backslash\overline{\Omega}_0.$$

On the other side, owing to $v_n(x) \geq v^{d_v}_{m_n}(x)$ and $v^{d_v}_{m_n}(x) \to V^*(x)$ in $\Omega\backslash\overline{\Omega}_0$ as $n \to \infty$,

$$\underline{\lim}_{n\to\infty} v_n(x) \geq V^*(x), \quad x \in \Omega\backslash\overline{\Omega}_0.$$

Therefore,

$$\lim_{n\to\infty} v_n(x) = V^*(x), \quad x \in \Omega\backslash\overline{\Omega}_0.$$

By making use of the regularity theory of elliptic equation, we conclude $v_n \to V^*$ in $C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$. The proof of (ii) follows.

(iii) On account of $r_0 > d + \mathfrak{R}^*$, it holds from the proof of (ii) that $u^\circ > 0$ in $\Omega\backslash\overline{\Omega}_0$. Due to $u_n < (r_0 - d)/a$, we know that if there exists a constant $\eta$ satisfying $\eta \geq \lambda^D_1(d_v, \Omega_0) + cp\rho(k)(r_0 - d)/(a + q_1(r_0 - d))$, then

$$\begin{aligned} -d_v\Delta v_n &= \left(m_n + \frac{cp\rho(k)u_n}{1 + q_1 u_n + q_2^n v_n}\right) v_n - s(x)v_n^2 \\ &< \left(m_n + \frac{cp\rho(k)(r_0 - d)}{a + q_1(r_0 - d)}\right) v_n - s(x)v_n^2 \\ &\leq \eta v_n - s(x)v_n^2, \end{aligned}$$

and then Lemma 2.1 in [33] derives $v_n \leq V^{d_v}_\eta$ in $\Omega\backslash\overline{\Omega}_0$ with $V^{d_v}_\eta$ being the unique positive solution of (33) with $m = \eta$. Thus, $\{v_n(x)\}_{n=1}^\infty$ is uniformly bounded on any compact subset of $\Omega\backslash\overline{\Omega}_0$.

Denote $\mathcal{B}_n = \{x \in \overline{\Omega} : d(x, \Omega_0) > \varrho_n\}$, where $\varrho_n$ is a decreasing sequence of positive numbers with $\varrho_n \to 0$ as $n \to \infty$. By using the interior and boundary estimates on $\mathcal{B}_{i+1}$, one sees that $\{v_n|_{\mathcal{B}_i}\}$ is compact in $C^1(\mathcal{B}_i)$. Then a standard diagonal process gives that $\{v_n|_{\Omega\backslash\overline{\Omega}_0}\}$ has a subsequence converging to some $v^\circ$ in $C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$, and clearly, $\partial_{\mathbf{n}} v^\circ = 0$ for $x \in \partial\Omega$. Moreover, since $v^{d_v}_{m_n}(x) \leq v_n(x)$ and $v^{d_v}_{m_n}(x) \to V^*(x)$ in $C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$ as $n \to \infty$, we deduce $v^\circ \geq V^*$ in $C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$. So $v^\circ > 0$ in $\Omega\backslash\overline{\Omega}_0$ and $v^\circ|_{\partial\Omega_0} = \infty$. Then by letting $n \to \infty$ in the equations for $u_n$ and $v_n$, we obtain $(u^\circ, v^\circ)|_{\Omega\backslash\overline{\Omega}_0}$ satisfies (34). And the standard interior and boundary regularity indicates $u^\circ|_{\Omega\backslash\overline{\Omega}_0} \in C^1_{loc}(\overline{\Omega}\backslash\overline{\Omega}_0)$. The proof of (iii) is completed. ■

**Remark 3.6:** Theorems 3.9–3.10 imply that when the growth rate $m$ of predator approaches the critical value $\lambda^D_1(d_v, \Omega_0)$, the following phenomena will occur: (i) Predator population blows up in $\Omega_0$ (see Theorem 3.9); (ii) Prey population vanishes in $\Omega_0$ if the interference effect $q_2$ of predator is weak (see Theorem 3.10(i)); (iii) Under the conditions in case (ii), if the interference effect $q_1$ of prey is also weak, then when the growth rate $r_0$ of prey is equal to $d + \mathfrak{R}^*$, the number of predators will reach a certain level, but the prey species will die out in $\Omega\backslash\overline{\Omega}_0$ (see Theorem 3.10(ii)), however, when $r_0$ is larger than $d + \mathfrak{R}^*$, both the number of prey and predator species will reach a certain level in $\Omega\backslash\overline{\Omega}_0$ (see Theorem 3.10(iii)).

We end up this section by pointing out that for (7), we are not able to explore the asymptotic behavior as $m \to \infty$.

## 4. Numerical simulations

In this section, based on Theorems 2.5 and 3.5, we perform some numerical simulations of models (7) and (9) in one-dimensional spaces $\Omega_0 = (0, l\pi/2)$ and $\overline{\Omega}\backslash\overline{\Omega}_0 = (l\pi/2, l\pi]$ $(l > 0)$ to illustrate the existence of positive solutions. Then the effects of fear level, predator-taxis, intra-specific pressure of predator and degeneracy on the positive solutions are analyzed according to these numerical results.

### 4.1. Numerical simulations of model (7)

Let $s(x)$ satisfy

$$s(x) \equiv 0, \quad x \in \overline{\Omega}_0, \quad s(x) = \ln(1 + x), \quad x \in \overline{\Omega}\backslash\overline{\Omega}_0. \tag{39}$$

Given the parameter set

$$\begin{aligned} &l = 0.5, \quad a = 10, \quad \alpha = 10, \quad M = 10, \quad p = 2, \quad c = 0.6, \quad m = -0.1, \\ &c_1 = 0.2, \quad d = 6, \quad q_1 = 1, \quad q_2 = 10, \quad r_0 = 10, \quad d_u = 2, \quad d_v = 0.01. \end{aligned} \tag{40}$$

Then $\overline{\Omega}_0 = [0, 0.785]$ and $\overline{\Omega}\backslash\overline{\Omega}_0 = (0.785, 1.57]$.

Choose the parameter values in (40), we find that when $k = 0.01, 0.1, 1$ or $10$, all conditions in Theorem 2.5(i) are satisfied, thus, model (7) has at least one positive solution. It is worthwhile to point out that due to the degeneracy of function $s(x)$, the forms of positive solutions of (7) in spaces $\overline{\Omega}_0$ and $\overline{\Omega}\backslash\overline{\Omega}_0$ are different, so we perform the numerical simulations of (7) in spaces $\overline{\Omega}_0$ and $\overline{\Omega}\backslash\overline{\Omega}_0$, respectively, see Figures 2–4. In what follows, we give the detailed explanations of Figures 2–4.

Firstly, by the proof of Theorem 2.5(i), the positive solutions of (7) are on the continuum $\Theta_*$ of (7) emitting from the semi-trivial solution $((r_0 - d)/a, 0)$. Since $s(x) \equiv 0, x \in \overline{\Omega}_0$, one sees from the proof of Theorem 2.2(i) that in space $\overline{\Omega}_0$, the positive solutions of (7) bifurcating from $((r_0 - d)/a, 0)$ are its positive constant solutions at $s(x) = 0$.

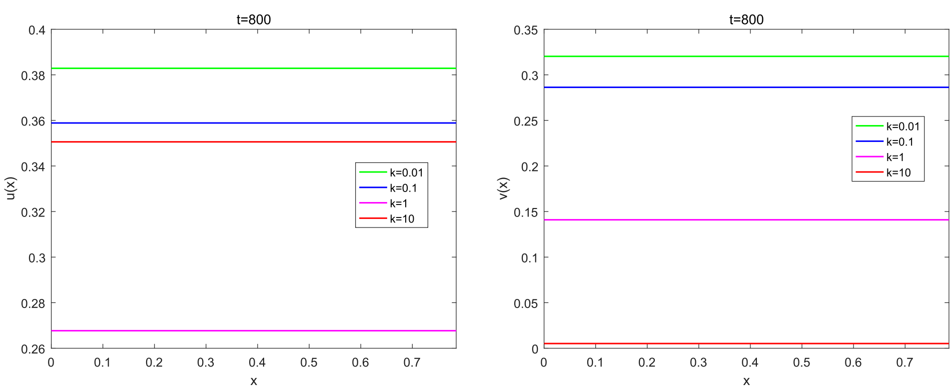



**Figure 2.** Positive solutions of model (7) in the domain $\overline{\Omega}_0$ under different $k$.

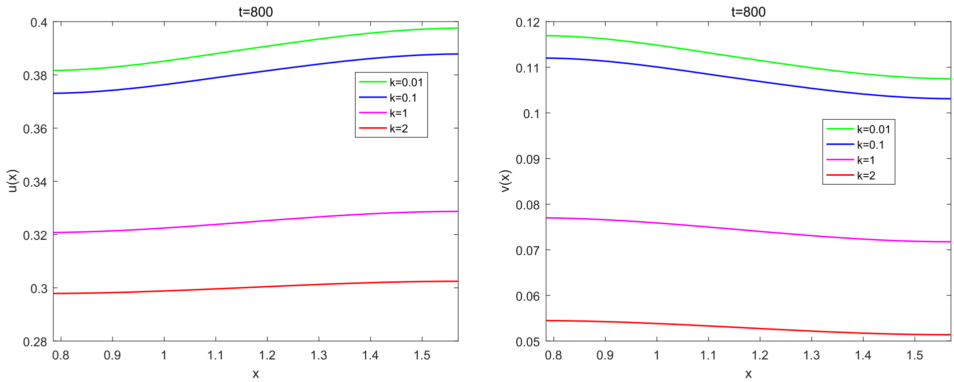



**Figure 3.** Positive solutions of model (7) in the domain $\overline{\Omega}\backslash\overline{\Omega}_0$ under different $k$.

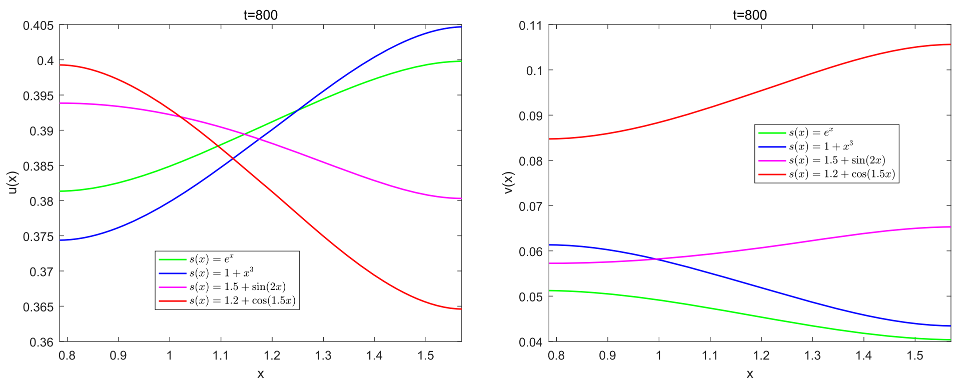



**Figure 4.** Positive solutions of model (7) in the domain $\overline{\Omega}\backslash\overline{\Omega}_0$ under different $s(x)$ with $k = 0.1$.

Some straightforward calculations show that when $k = 0.01, 0.1, 1$ and 10, the corresponding positive constant solutions are $(0.3829, 0.3202)$, $(0.3589, 0.2863)$, $(0.2677, 0.1409)$ and $(0.3506, 0.0052)$, respectively. Figure 2 presents the positive solutions in space $\overline{\Omega}_0$ at $t = 800$ when $k = 0.01, 0.1, 1$ and 10, respectively. Secondly, by the proof of Theorem 2.2(i) again, we can derive that the positive solutions in space $\overline{\Omega}\backslash\overline{\Omega}_0$ are spatially dependent. Figure 3 performs the positive solutions in space $\overline{\Omega}\backslash\overline{\Omega}_0$. Particularly, we observe from Figures 2–3 that regardless of which space the two species are in, their density decreases with the increase of $k$. In addition, to observe the effect of $s(x)$ on the positive solution, we also show the numerical simulations of positive solutions in space $\overline{\Omega}\backslash\overline{\Omega}_0$ when $s(x) = e^x, 1 + x^3, 1.5 + \sin(2x)$ and $1.2 + \cos(1.5x)$, respectively, see Figure 4. We find that the type of function $s(x)$ has an important effect on the trend of species density.

Notice that Figures 2–4 mean that the combination of predator-taxis and degeneracy can cause model (7) to produce positive solutions. Meanwhile, comparing these simulation results, we conclude that under certain conditions, the perturbation of $k$, different function $s(x)$ and spaces $\overline{\Omega}_0$, $\overline{\Omega}\backslash\overline{\Omega}_0$ can lead model (7) to generate *different patterns of the solution.*

### 4.2. Numerical simulations of model (9)

Assume that $s(x)$ satisfies Equation (39). Given the following parameter set

$$\begin{aligned} &l = 0.5, \quad a = 0.5, \quad p = 0.2, \quad c = 1, \quad c_1 = 0.2, \quad m = -0.1, \\ &d = 0.2, \quad q_1 = 0.1, \quad q_2 = 0.1, \quad r_0 = 10, \quad d_u = 0.013, \quad d_v = 0.01. \end{aligned} \tag{41}$$

Then $\overline{\Omega}_0 = [0, 0.785]$ and $\overline{\Omega} \backslash \overline{\Omega}_0 = (0.785, 1.57]$.

Take the parameter value in (41), then one can easy to check that when $k = 0.01, 0.1, 1$ or 10, all conditions in Theorem 3.5(i) are satisfied. This deduces that model (9) has at least one positive solution. Combining Theorem 3.2(i) and the analysis in Section 4.1, one can verify that (9) only has a unique positive constant solution in space $\overline{\Omega}_0$. Numerically, when $k = 0.01, 0.1, 1$ and 10, the corresponding positive constant solutions are $(0.3829, 0.3202)$, $(0.3589, 0.2863)$, $(0.2677, 0.1409)$ and $(0.3506, 0.0052)$, respectively, see Figure 5. Moreover, in $\overline{\Omega} \backslash \overline{\Omega}_0$, the positive solutions of (9) are spatially heterogeneous, as demonstrated in Figure 6. In particular, we observe from Figures 5–6 that with the increase of $k$, the density of two species decreases both in $\overline{\Omega}_0$ and $\overline{\Omega} \backslash \overline{\Omega}_0$.

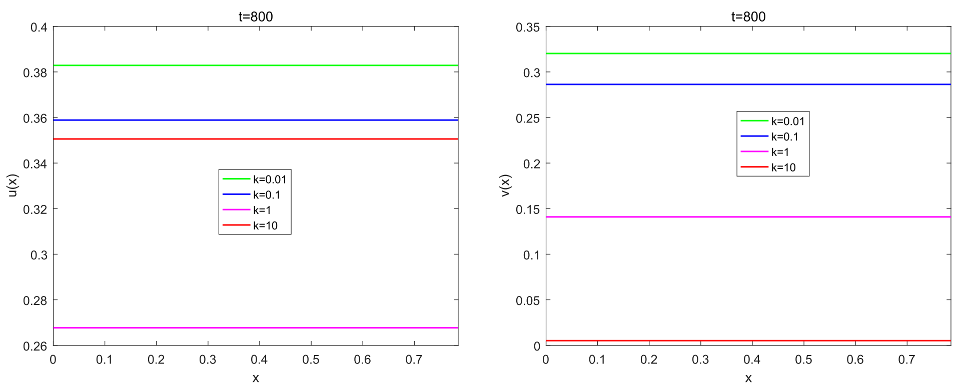



**Figure 5.** Positive solutions of model (9) in the domain $\overline{\Omega}_0$ under different $k$.

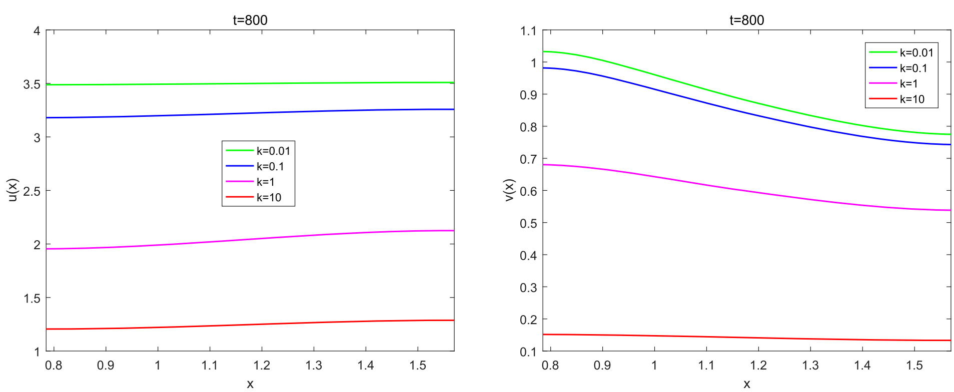



**Figure 6.** Positive solutions of model (9) in the domain $\overline{\Omega} \backslash \overline{\Omega}_0$ under different $k$.

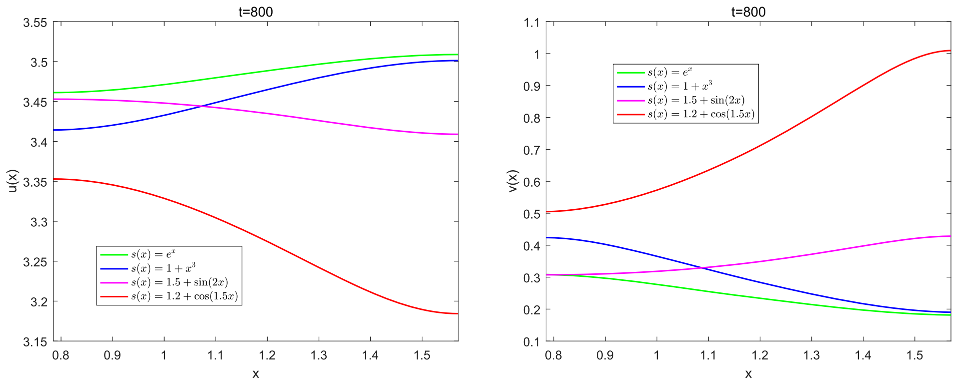



**Figure 7.** Positive solutions of model (9) in the domain $\overline{\Omega}\backslash\overline{\Omega}_0$ under different $s(x)$ with $k = 0.1$.

On the impact of spatially heterogeneous parameter $s(x)$, we also present some numerical simulations of positive solutions for model (9) in space $\overline{\Omega}\backslash\overline{\Omega}_0$ for some particular forms of $s(x)$: $s(x) = e^x, 1 + x^3, 1.5 + \sin(2x)$ and $1.2 + \cos(1.5x)$, respectively in Figure 7. We find that the impact of $s(x)$ on the spatial distribution of species' population is significant. We have to admit that analytic results on such an impact is very challenging and are not explored in this paper.

We point out that the numeric results in Figures 5–7 not only indicate that the individual degeneracy can cause model (9) to produce positive solutions, but also show that the perturbation of $k$, different function $s(x)$ and spaces $\overline{\Omega}_0$, $\overline{\Omega}\backslash\overline{\Omega}_0$ can induce model (9) to generate *different solution patterns*. Moreover, by comparing Figures 2–4 and 5–7, we find that the appearance of *predator-taxis*, in conjunction with the spatial heterogeneity of $s(x)$, can lead to *different spatial patterns of persistence* of the interacting species, particularly of the prey species, as clearly illustrated in the left panels of Figures 4 and 7.

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## Disclosure statement

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