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Chaotic invariant sets of a delayed discrete neural network of two non-identical neurons

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Abstract In this paper, we show that a delayed discrete Hopfield neural network of two nonidentical neurons with no self-connections can demonstrate chaotic behavior in a region away from the origin. To this end, we first transform the model, by a novel way, into an equivalent system which enjoys some nice properties. Then, we identify a chaotic invariant set for this system and show that the system within this set is topologically conjugate to the full shift map on two symbols. This confirms chaos in the sense of Devaney. Our main result is complementary to the results in Kaslik and Balint (2008) and Huang and Zou (2005), where it was shown that chaos may occur in neighborhoods of the origin for the same system. We also present some numeric simulations to demonstrate our theoretical results.

Keywords neural network, Devaney chaos, discrete-time, topological conjugacy

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1 Introduction

Chaotic behaviors in neural networks have attracted more and more attentions due to its potential applications to various practical problems. For example, when using neural networks as computational methods to solve combinatorial optimization problems, chaotic behaviors for the network systems can provide global searching ability, which may prevent the objective function from getting trapped at local extrema [2]. Chaotic dynamics existing in real neurons and neural networks play an important role in neural activities [4].

Among the most frequently used and studied neural networks is the continuous Hopfield neural network, which was first considered in [5]. Its various discrete versions have also been intensively and extensively studied in literature. In particular, for the following simple discrete version:

$$\begin{cases} x(n+1) = \beta x(n) + \alpha f(y(n-k)), \\ y(n+1) = \beta y(n) + \alpha f(x(n-k)), \end{cases} \tag{1.1}$$

where $\alpha > 0$, $\beta \in (0,1)$ and the delay $k \geqslant 1$, Wu and Zhang [10] showed that under some conditions on the activation function $f(x)$, for every positive integer p with $p|2k$, System (1.1) has several distinct

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asymptotically stable p-periodic solutions in a region of the x-y plane away from the origin $(0,0)$. In a recent work, Huang and Zou [6] further showed that under certain technical conditions on the nonlinear function $f(x)$, System (1.1) actually demonstrates Li-Yorke chaotic behavior[1)] in a *neighborhood of the origin*.

Model (1.1) is for a network consisting of two identical neurons with a uniform connection between them. In a more recent work, Kaslik and Balint [7] considered the following generalization of (1.1):

$$\begin{cases} x(n+1) = \beta_1 x(n) + \alpha_{12} f_2(y(n-k_2)), \\ y(n+1) = \beta_2 y(n) + \alpha_{21} f_1(x(n-k_1)), \end{cases} \tag{1.2}$$

where $n \in \mathbb{N}, \beta_i \in (0,1)$ for $i = 1, 2$, and α_{12} and α_{21} are non-zero constants representing connection strengths, and the delays $k_i \geqslant 0$, $i = 1, 2$, are fixed integers. The activation functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are continuously differentiable. In addition to the stability and bifurcation analysis by central manifold theory, Kaslik and Balint [7] also showed that under some conditions, (1.2) may exhibit Li-Yorke chaos in the *vicinity of the origin* as well, and generalize the results reported in [6].

Notice that the chaotic behaviors obtained in [6] and [7] all occur in neighborhoods of the origin $(0,0)$ in the x-y plane. One naturally wonders if chaos is possible in other regions in the x-y plane. This constitutes the purpose of this paper. Note that (1.2) allows distinct neurons and non-uniform connections, and hence, gives us more parameters to play in order to generate chaos. Thus, in this paper, we consider (1.2) and investigate the possibility of chaos for System (1.2) in regions away from the origin. Our method is motivated by the idea of establishing the horseshoe structure in families of generalized Henon-type maps in [8,9], which is also called anti-integrable limit approach for systems with generating functions inspired by Aubry and Abramovici in [1].

The rest of this paper is organized as follows. In Section 2, we construct a map $\Phi(\lambda, \cdot)$ from l_∞ to l_∞; and by applying the implicit function theorem in Banach spaces to this parameterized map, we obtain a uniform result for a family of implicit functions. This result will be used in Section 3 to construct a conjugacy map from the full shift at certain values of the parameter to solutions to (1.2). To achieve this, we rewrite the model (1.2) as a system of difference equations by a novel way which enjoys some nice properties that the rewritings in [6] and [7] do not have. In particular, we are able to obtain an invariant set *away from the origin* for the transformed system, and show that on this invariant set the map representing the transformed system is topologically conjugate to the full shift on the symbolic dynamical system with two symbols. This conjugacy implies chaos[2)] for (1.2) restricted on an invariant set in the sense of Devaney [3]. See Theorem 3.2. Our main results are complementary to the results on chaos in discrete Hopfield neural networks obtained in [6,7]. At the end, we present a particular example and its numeric simulations, which confirm the chaotic phenomena predicted by our theoretical results.

2 Preliminaries

To proceed, we need some preparations. The following implicit function theorem will be needed in the sequel.

Lemma 2.1 (Implicit function theorem [11]). *Let (Λ, d) be a metric space, Y and X be Banach spaces, and $U \subset \Lambda \times Y$ be open and nonempty. Suppose that $F : U \to X$ is a continuous map and that there exists a point $(\lambda_0, y_0) \in U$ with the following conditions*:

(i) $F(\lambda_0, y_0) = 0$;

(ii) $DF_y(\lambda, y)$ *is continuous at* (λ_0, y_0), *where* $DF_y(\lambda, y)$ *is the Fréchet partial derivative of* $F(\lambda, y)$ *with respect to* y;

(iii) $DF_y(\lambda_0, y_0) : Y \to X$ *is an invertible linear map.*

Then there exist open balls $B_{\delta_0}(y_0) = \{y \in Y : \|y - y_0\| < \delta_0\}$ *and* $B_{r_0}(\lambda_0) = \{\lambda \in \Lambda : d(\lambda, \lambda_0) < r_0\}$,

1) Let (X, d) be a metric space and $g : X \to X$ be continuous. g is called Li-Yorke chaos if there exists a uncountable set S in X such that $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \forall\, x,\ y \in S, x \neq y$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \forall\, x, y \in S$.

2) g is said to be Devaney chaos if g is topologically transitive and has density of periodic points.

where $\delta_0 > 0, r_0 > 0$, *such that for any* $\lambda \in B_{r_0}(\lambda_0)$, $F(\lambda, y) = 0$ *has a unique continuous solution* $y = h(\lambda) \in B_{\delta_0}(y_0)$ *with* $h(\lambda_0) = y_0$.

Without loss of generality, let us assume $k_2 \geqslant k_1$. Letting $\alpha = \alpha_{21}$ and $\alpha_{12} = C\alpha$, (1.2) is rewritten as

$$\begin{cases} x(n+1) = \beta_1 x(n) + \alpha C f_2(y(n-k_2)), \\ y(n+1) = \beta_2 y(n) + \alpha f_1(x(n-k_1)), \end{cases} \quad n \in \mathbb{Z}. \tag{2.1}$$

Note that C is non-zero since the connecting coefficients α_{12} and α_{21} are non-zero.

In what follows, we will need some assumptions on the activation functions f_1 and f_2:

(H1) $f_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable for $i = 1, 2$; f_2 has a simple zero point $\bar{x}$, and f_1 has two distinct simple zero points $x', x'' \in \mathbb{R}$, i.e., $f_1(x') = f_1(x'') = f_2(\bar{x}) = 0$ with $f_1'(x') \neq 0$, $f_1'(x'') \neq 0$ and $f_2'(\bar{x}) \neq 0$.

Now, we introduce some notations. Let l_∞ be the usual space of bounded real sequences endowed with the supreme norm. And let $\sigma : l_\infty \to l_\infty$ be the forward shift map defined by, for $y = \{y_n\} \in l_\infty$,

$$(\sigma y)_n = y_{n+1}, \quad n \in \mathbb{Z}. \tag{2.2}$$

Obviously, σ is invertible with its inverse σ^{-1} given by

$$(\sigma^{-1} y)_n = y_{n-1}, \quad n \in \mathbb{Z}.$$

Moreover, for any $k \in \mathbb{Z}$,

$$(\sigma^k y)_n = y_{n+k}, \quad n \in \mathbb{Z}.$$

Motivated by (2.1), for any $\lambda \in \mathbb{R}$, we define $\Phi(\lambda, \cdot) : l_\infty \to l_\infty$ by

$$\begin{cases} \Phi(\lambda, W)_{2n+1} = \lambda(-w_{2n+1} + \beta_1 w_{2n-1}) + C f_2(w_{2(n-k_2)}), \\ \Phi(\lambda, W)_{2n+2} = \lambda(-w_{2n+2} + \beta_2 w_{2n}) + f_1(w_{2(n-k_1)-1}), \end{cases} \quad \forall\, W = (w_n) \in l_\infty.$$

Then, $W = \{w_n\} \in l_\infty$ such that $(x_n, y_n) = (w_{2n+1}, w_{2n})_{n \in \mathbb{Z}}$ is a solution to (2.1) if and only if W solves $\Phi(1/\alpha, W) = 0$.

Let

$$\Gamma = \{W = (w_n) \in l_\infty \mid w_{2n} = \bar{x}, w_{2n+1} = x' \text{ or } x'', n \in \mathbb{Z}\}, \tag{2.3}$$

where $\bar{x}$, x' and x'' are given by the condition (H1). Making use of Lemma 2.1, we can prove the following lemma.

Lemma 2.2. *Assume that* (H1) *holds. Then*

(i) *there are* $r_0 > 0$ *and* $\delta_0 > 0$ *such that for every* $\bar{W} \in \Gamma$ *and every* $\lambda \in B_{r_0}(0)$, *there is a unique* $W = W(\lambda) \in B_{\delta_0}(\bar{W})$ *satisfying* $\Phi(\lambda, W(\lambda)) = 0$;

(ii) *for every* $\delta \in (0, \delta_0)$, *there is* $r \in (0, r_0)$ *such that for every* $\lambda \in \bar{B}_r(0) = \{\lambda : d(\lambda, 0) \leqslant r\}$ *and every* $\bar{W} \in \Gamma$, *there is a unique* $W(\lambda)$ *with* $W(\lambda) \in B_\delta(\bar{W})$ *and* $\Phi(\lambda, W(\lambda)) = 0$.

Proof. Firstly, for any $\bar{W} \in \Gamma$, we have $\Phi(0, \bar{W}) = 0$, verifying Lemma 2.1(i). Condition (ii) in Lemma 2.1 is ensured by Assumption (H1). Finally, denote by $D\Phi_W(0, \bar{W})$ the Fréchet derivative of $\Phi(0, W)$ with respect to W at $\bar{W}$. Then, calculations show that

$$\begin{cases} (D\Phi_W(0, \bar{W})W)_{2n+1} = C f_2'(\bar{w}_{2(n-k_2)}) w_{2(n-k_2)}, \\ (D\Phi_W(0, \bar{W})W)_{2n+2} = f_1'(\bar{w}_{2(n-k_1)-1}) w_{2(n-k_1)-1}, \end{cases} \quad \forall\, n \in \mathbb{Z}.$$

By (H1), $C f_2'(\bar{w}_{2(n-k_2)}) = C f_2'(\bar{x}) \neq 0$ and $f_1'(\bar{w}_{2(n-k_1)-1}) = f_1'(x')$ or $f_1'(x'')$ which also does not vanish. This implies that $D\Phi_W(0, \bar{W})$ is an invertible linear operator, which verifies Lemma 2.1(iii). Therefore, for every $\bar{W} \in \Gamma$, by Lemma 2.1, there are $r = r(\bar{W}) > 0$ and $\delta = \delta(\bar{W}) > 0$ such that for every $\lambda \in B_r(0)$, there exists a unique $W = W(\lambda) \in B_\delta(\bar{W})$ satisfying $\Phi(\lambda, W(\lambda)) = 0$.

To prove (i), it suffices to show that the above $r = r(\bar{W}) > 0$ and $\delta = \delta(\bar{W}) > 0$ can be chosen to be independent of $\bar{W}$. This can be achieved by showing that there exist $r_0 > 0$ and $\delta_0 > 0$ which are independent of $\bar{W} \in \Gamma$ but can play the same role as $r = r(\bar{W}) > 0$ and $\delta = \delta(\bar{W}) > 0$ do for every $\bar{W} \in \Gamma$. To this end, let

$$M = \frac{1}{\min\{|f_1'(x')|, |f_1'(x'')|, |Cf_2'(\bar{x})|\}}, \quad b = \max\{\beta_1, \beta_2\}.$$

Note that (H1) implies that $0 < M < \infty$ and $b > 0$. Then, for any $\bar{W} \in \Gamma$, one has $\|(D\Phi_W(0, \bar{W}))^{-1}\| \leqslant M$. Since $f_1'(x)$ is continuous at $x = x'$ and x'' and $Cf_2'(x)$ is continuous at $x = \bar{x}$, there exists δ_0 such that

$$\begin{cases} |f_1'(x) - f_1'(x')| \leqslant \dfrac{1}{4M}, & \text{for } x \in B_{\delta_0}(x'), \\ |f_1'(x) - f_1'(x'')| \leqslant \dfrac{1}{4M}, & \text{for } x \in B_{\delta_0}(x''), \\ |Cf_2'(x) - Cf_2'(\bar{x})| \leqslant \dfrac{1}{4M}, & \text{for } x \in B_{\delta_0}(\bar{x}). \end{cases} \tag{2.4}$$

Note that, for $\bar{W} = (\bar{w}_n) \in \Gamma$ and any $W = (w_n) \in l_\infty$, we have

$$\begin{cases} (D\Phi_W(\lambda, W) - D\Phi_W(0, \bar{W})Y)_{2n+1} \\ \quad = \lambda(-y_{2n+1} + \beta_1 y_{2n-1}) + (Cf_2'(w_{2(n-k_2)}) - Cf_2'(\bar{w}_{2(n-k_2)}))y_{2(n-k_2)}, \\ (D\Phi_W(\lambda, W) - D\Phi_W(0, \bar{W})Y)_{2n+2} \\ \quad = \lambda(-y_{2n+2} + \beta_2 y_{2n}) + (f_1'(w_{2(n-k_1)-1}) - f_1'(\bar{w}_{2(n-k_1)-1}))y_{2(n-k_1)-1}, \end{cases} \quad \forall n \in \mathbb{Z}. \tag{2.5}$$

Choose $r_1 = \frac{1}{4M(1+b)}$. It then follows from (2.5) that for any $\bar{W} \in \Gamma$, $W \in l_\infty$ with $\|W - \bar{W}\| \leqslant \delta_0$ and $|\lambda| \leqslant r_1$, we have

$$\|D\Phi_W(\lambda, W) - D\Phi_W(0, \bar{W})\| \leqslant |\lambda|(1 + b) + \frac{1}{4M} \leqslant \frac{1}{2M}.$$

We further choose $r_2 = \frac{\delta_0}{2M(1+b)}$. Then, for $|\lambda| \leqslant r_2$, by the definition of $\Phi(\lambda, \cdot)$, we have

$$\|\Phi(\lambda, \bar{W})\| \leqslant |\lambda|(1 + b) \leqslant \frac{\delta_0}{2M}.$$

Recall that in the proof of Lemma 2.1 (see [11]), the constants r_0 and δ_0 are decided so that $\|D\Phi_W(\lambda, W) - D\Phi_W(0, \bar{W})\|$ and $\|\Phi(\lambda, \bar{W})\|$ are bounded, when W is in the δ_0-neighborhood of $\bar{W}$ and λ is in the r_0-neighborhood of 0. Let $r_0 = \min\{r_1, r_2\}$. Then the constants r_0 and δ_0 obviously serve the purpose stated in (i), and the proof of (i) is completed.

(ii) follows from (i) by letting

$$r = \min\left\{\frac{1}{4M(1+b)}, \frac{\delta}{2M(1+b)}\right\}$$

for every $\delta < \delta_0$. This completes the proof. □

By Lemma 2.2, for sufficiently large $\alpha > 0$, we can define a map $T_\alpha : \Gamma \to l_\infty$ by

$$T_\alpha(\bar{W}) = W\left(\frac{1}{\alpha}\right),$$

where $W(\frac{1}{\alpha})$ is the unique solution to $\Phi(\frac{1}{\alpha}, W) = 0$ satisfying $\|W(\frac{1}{\alpha}) - \bar{W})\| \leqslant \delta$. The following lemma reveals a nice property of the map T_α.

Lemma 2.3. *For sufficiently large $\alpha > 0$, T_α commutes with the shift map σ^2, i.e.,*

$$\sigma^2 \circ T_\alpha = T_\alpha \circ \sigma^2.$$

Moreover, $\sigma^2(\Gamma_\alpha) = \Gamma_\alpha$, where $\Gamma_\alpha = T_\alpha(\Gamma)$.

Proof. Note that if W is a solution to $\Phi(1/\alpha, W) = 0$, so is $\sigma^2(W)$. Then for any $\bar{W} \in \Gamma$, it follows that $\sigma^2 \circ T_\alpha(\bar{W}) = \sigma^2(W(\frac{1}{\alpha}))$ is a solution to $\Phi(1/\alpha, W) = 0$. On the other hand, $\|W(\frac{1}{\alpha}) - \bar{W}\| \leqslant \delta$ by Lemma 2.2, which leads to $\|\sigma^2(T_\alpha(\bar{W}) - \sigma^2(\bar{W})\| = \|\sigma^2(W(\frac{1}{\alpha})) - \sigma^2(\bar{W})\| = \|W(\frac{1}{\alpha}) - \bar{W}\| \leqslant \delta$. Hence, by the uniqueness of $W(\lambda)$ in Lemma 2.2, we have $\sigma^2(T_\alpha(\bar{W})) = T_\alpha(\sigma^2(\bar{W}))$. Note that $\sigma^2(\Gamma) = \Gamma$, it follows that $\sigma^2(\Gamma_\alpha) = \Gamma_\alpha$. □

Let $\eta(n) = (\eta_1(n), \ldots, \eta_{k_1+k_2+2}(n))$ where

$$\begin{cases} \eta_{2j+1}(n) = x(n - k_1 + j), \quad \forall\, j = 0, 1, 2, \ldots, k_1, \\ \eta_{2j+2}(n) = y(n - k_2 + j), \quad \forall\, j = 0, 1, 2, \ldots, k_1, \\ \eta_{2k_1+2+i}(n) = y(n + k_1 - k_2 + i), \quad \forall\, i = 1, 2, \ldots, k_2 - k_1, \end{cases} \qquad \forall\, n \in \mathbb{Z}. \tag{2.6}$$

Then, (2.1) is further rewritten as the following discrete dynamical system on $\mathbb{R}^{k_1+k_2+2}$:

$$\eta(n+1) = F_\alpha(\eta(n)), \quad n \in \mathbb{Z}, \tag{2.7}$$

where $F_\alpha : \mathbb{R}^{k_2+k_1+2} \to \mathbb{R}^{k_2+k_1+2}$ is given by

$$F_\alpha \begin{pmatrix} \eta_1(n) \\ \eta_2(n) \\ \vdots \\ \eta_{2k_1+1}(n) \\ \eta_{2k_1+2}(n) \\ \vdots \\ \eta_{k_2+k_1+1}(n) \\ \eta_{k_2+k_1+2}(n) \end{pmatrix} = \begin{pmatrix} \eta_3(n) \\ \eta_4(n) \\ \vdots \\ \beta_1 \eta_{2k_1+1}(n) + C\alpha f_2(\eta_2(n)) \\ \eta_{2k_1+3}(n) \\ \vdots \\ \eta_{k_2+k_1+2}(n) \\ \beta \eta_{k_2+k_1+2}(n) + \alpha f_1(\eta_1(n)) \end{pmatrix}.$$

We remark that this transformation is different from the ones used in [6,7]. One novelty of this array is that it enables us to identify an invariant set for the transformed system (2.7), as will be seen in Lemma 2.4 below. To describe this invariant set, we introduce a family of projections from l_∞ onto $\mathbb{R}^{k_1+k_2+2}$, which plays the key in the proof of the main theorem.

For every $k \in \mathbb{Z}$, define $\Pi_k : l_\infty \to \mathbb{R}^{k_1+k_2+2}$ as

$$\Pi_k(W) = \eta(k), \quad \forall\, W \in l_\infty, \tag{2.8}$$

where, for $W = (w_n) \in l_\infty$, $\eta(k) = (\eta_1(k), \ldots, \eta_{k_1+k_2+2}(k)) \in \mathbb{R}^{k_1+k_2+2}$ is defined by

$$\begin{aligned} &\eta_{2j+1}(k) = w_{2(k-k_1)+2j-1}, \quad j = 0, 1, 2, \ldots, k_1, \\ &\eta_{2j+2}(k) = w_{2(k-k_2)+2j}, \quad j = 0, 1, 2, \ldots, k_1, \\ &\eta_{2k_1+2+i}(k) = w_{2(k-k_2+k_1+i)}, \quad i = 1, 2, \ldots, k_2 - k_1. \end{aligned}$$

The following facts follow easily from the definitions and Lemma 2.3.

Lemma 2.4. *Let $\Lambda_\alpha = \Pi(\Gamma_\alpha)$, then Λ_α is invariant for F_α. Here $\Pi = \Pi_0$.*

Proof. For each $\eta(0) \in \Lambda_\alpha$, there exists $W \in \Gamma_\alpha$ such that $\Pi(W) = \eta(0)$. Therefore,

$$F_\alpha(\eta(0)) = \eta(1) = \Pi(\sigma^2(W)) \in \Pi(\sigma^2(\Gamma_\alpha)) = \Pi(\Gamma_\alpha) = \Lambda_\alpha.$$

This proves $F_\alpha(\Lambda_\alpha) \subset \Lambda_\alpha$.

On the other hand, By Lemma 2.3, we have $\sigma^2(\Gamma_\alpha) = \Gamma_\alpha$. Thus there exists $W' \in \Gamma_\alpha$ such that $W = \sigma^2(W')$. Since

$$\eta(0) = \Pi(W) = \Pi(\sigma^2(W')) = \eta'(1) = F_\alpha(\eta'(0)) = F_\alpha(\Pi(W')) \in F_\alpha(\Lambda_\alpha),$$

$\Lambda_\alpha \subset F_\alpha(\Lambda_\alpha)$ is proved. Hence $F_\alpha(\Lambda_\alpha) = \Lambda_\alpha$. The conclusion of the lemma holds on. □

3 Main results

In this section, we shall show that the dynamical system $(\Lambda_\alpha, F_\alpha)$ is topological conjugate to a symbolic dynamical system. To proceed, we define

$$\Sigma_k = \{(\cdots i_{-1} i_0 i_1 \cdots) \mid i_n \in \{1, 2, \ldots, k\},\ n \in \mathbb{Z}\},$$

which is a symbolic space with k symbols. Equipping it with the usual metric

$$d(s,t) = \max\{2^{-|n|} | t_n \neq s_n, n \in \mathbb{Z}\}, \quad t = (\cdots t_{-1} t_0 t_1 \cdots), \quad s = (\cdots s_{-1} s_0 s_1 \cdots) \in \Sigma_k,$$

Σ_k becomes a compact metric space. The shift map σ defined in (2.2) can be viewed as a shift map on Σ_k. Thus, (Σ_k, σ) defines a symbolic dynamical system with k symbols and has a particular name: a Markov shift.

In Section 2, the set Γ defined in (2.3) is a subset of l_∞. Now we treat it as a subset of Σ_3. It is easy to see that Γ is closed under the topology on Σ_3 and $\sigma^2(\Gamma) \subset \Gamma$. Moreover, we have

Lemma 3.1. *σ^2 restricted on Γ is topological conjugate to (Σ_2, σ).*

Proof. Define $g : \Gamma \to \Sigma_2$ by

$$g(W) = (\cdots w_{-2n-1} w_{-2n+1} \cdots w_{-1} w_1 w_3 \cdots w_{2n-1} w_{2n+1} \cdots),$$

for

$$W = (\cdots w_{-2n} w_{-2n+1} w_{-2n+2} \cdots w_{-2} w_{-1} w_0 w_1 w_2 \cdots w_{2n-2} w_{2n-1} w_{2n} \cdots) \in \Gamma.$$

That is, $g(W)$ is obtained by deleting the even indexed elements in W, and hence it is easy to see that g is homeomorphism and

$$g \circ \sigma^2 = \sigma \circ g,$$

completing the proof. □

We are now in the position to state and prove our main result.

Theorem 3.2. *Assume that* (H1) *holds. Then there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$, $(\Lambda_\alpha, F_\alpha)$ is topologically conjugate to the full shift map (Σ_2, σ); and therefore, F_α restricted on Λ_α is chaotic in the sense of Devaney.*

Proof. By Lemma 3.1, it suffices to prove that there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$, $(\Lambda_\alpha, F_\alpha)$ is topological conjugate to (Γ, σ^2). Note here that Γ is a closed invariant set under σ^2.

Let $\|\cdot\|_*$ denote the supreme norm on the Euclidean space $\mathbb{R}^{k_1+k_2+2}$. That is,

$$\|\eta\|_* = \sup_{1 \leqslant i \leqslant k_1+k_2+2} |\eta_i|, \quad \text{for } \eta = (\eta_1, \ldots, \eta_{k_1+k_2+2}) \in \mathbb{R}^{k_1+k_2+2}.$$

Let $\Omega = \Pi(\Gamma)$, where $\Pi = \Pi_0$ is defined in (2.8). Then Ω is a finite set in $\mathbb{R}^{k_1+k_2+2}$ with cardinality of Ω being 2^{k_1+1}. Denote it by

$$\Omega = \{\xi_1, \xi_2, \ldots, \xi_{2^{k_1+1}}\}.$$

Let δ_0 and r_0 be given as in Lemma 2.2, and let $\delta \in (0, \delta_0)$ be small enough such that the family of closed balls $\{A_i = \bar{B}(\xi_i, \delta)\}_{i=1}^{2^{k_1+1}}$ in $\mathbb{R}^{k_1+k_2+2}$ are piecewise disjoint.

For the given δ and any $\bar{W} = (\bar{w}_n) \in \Gamma$, by Lemma 2.2(ii), there exists a $\alpha_0 = \frac{1}{r} > 0$ such that for every $\alpha > \alpha_0$ there exists a unique $T_\alpha(\bar{W}) = W(\frac{1}{\alpha})$ satisfying $\|W(\frac{1}{\alpha}) - \bar{W}\| \leqslant \delta$ and $\Phi(\frac{1}{\alpha}, W(\frac{1}{\alpha})) = 0$.

For $\alpha > \alpha_0$, define $h : \Gamma \to \Lambda_\alpha$ by $h = \Pi \circ T_\alpha$. We claim that h is a conjugacy from σ^2 to F_α. To prove our claim, it suffices to show that both h and h^{-1} are continuous and

$$h \circ \sigma^2 = F_\alpha \circ h, \quad \text{on } \Gamma. \tag{3.1}$$

Let

$$S = \{s = (\ldots, s_{-1}, s_0, s_1, \ldots) \mid s_i \in \{1, 2, \ldots, 2^{k_1+1}\},\ \xi_{s_i} = \Pi_i(\bar{W}) \text{ for some } \bar{W} \in \Gamma\}$$

be the subset of the symbolic space with symbols $\{1, 2, \dots, 2^{k_1+1}\}$. For any $s = (\dots, s_{-1}, s_0, s_1, \dots) \in S$, we define

$$V_{s_{-i}\cdots s_0\cdots s_j} = F_\alpha^{-j}(A_{s_j}) \cap \cdots \cap A_{s_0} \cap \cdots \cap F_\alpha^{i}(A_{s_i}), \quad i > 0, \quad j > 0,$$
$$V_s = \bigcap_{i>0, j>0} V_{s_{-i}\cdots s_0\cdots s_j}.$$

We claim the following two facts:

(a) for each $s \in S$, V_s contains only one point;

(b) $\bigcup_{s\in S} V_s = \Lambda_\alpha$.

In fact, for each $s \in S$, note that

$$V_{s_{-i}\cdots s_0\cdots s_j} = \{\eta \in \mathbb{R}^{k_1+k_2+2} \mid F_\alpha^{-i}(\eta) \in A_{s_i}, \dots, \eta \in A_{s_0}, \dots, F_\alpha^{j}(\eta) \in A_{s_j}\}.$$

From the definition of s, there is a unique $\bar{W} \in \Gamma$ such that $\Pi_i(\bar{W}) = \xi_{s_i} \in \Omega, \forall\, i \in \mathbb{Z}$. By Lemma 2.2, there is a unique $T_\alpha(\bar{W}) = W(\frac{1}{\alpha})$ with $\|W(\frac{1}{\alpha}) - \bar{W}\| \leqslant \delta$ and $\Phi(\frac{1}{\alpha}, W(\frac{1}{\alpha})) = 0$. So $\{\Pi_n(W(\frac{1}{\alpha})) = \eta(n)\}_{n\in\mathbb{Z}}$ is a bounded global orbit of F_α, which implies that $\eta(n) = F_\alpha^n(\eta(0)) \in A_{s_n}, \forall\, n \in \mathbb{Z}$. Thus $\eta(0) \in V_s$ and so V_s is nonempty.

On the other hand, for any $\eta' \in V_s$, we know that $F_\alpha^n(\eta') \in A_{s_n}$ for $n \in \mathbb{Z}$ and $\{F_\alpha^n(\eta')\}_{n\in\mathbb{Z}}$ is a bounded global orbit of F_α. Thus there exists $W \in l_\infty$ such that $\Pi_n(W) = F_\alpha^n(\eta')$ with $\|W - \bar{W}\| \leqslant \delta$ and $\Phi(\frac{1}{\alpha}, W) = 0$. Again by Lemma 2.2, we have $W = T_\alpha(\bar{W})$ and hence, $\eta' = \eta(0)$, confirming (a).

For (b), let $\eta \in \Lambda_\alpha$. Then there exists a $\bar{W} \in \Gamma$ such that $\eta = \Pi(T_\alpha(\bar{W}))$. Let $s = (\cdots s_{-1}s_0s_1\cdots) \in S$ be the corresponding sequence of $\bar{W}$. Similar to the above argument, we have $\eta \in V_s$. Therefore,

$$\Lambda_\alpha \subset \bigcup_{s\in S} V_s.$$

The converse inclusion follows from the fact that each V_s contains only one point which belongs to Λ_α, proving (b).

From the definition of h, it is obvious that h is surjective. Therefore, it follows from Lemma 2.2 and Claim (a) that h is bijective. We now prove the continuity of h. Let $\bar{W} \in \Gamma$, and $\bar{s} = (\dots, \bar{s}_{-1}, \bar{s}_0, \bar{s}_1, \dots) \in S$ be its corresponding subindex sequence. It follows from Claim (a) that

$$\lim_{i,j\to+\infty} \mathrm{diam}(V_{s_{-i}\cdots s_0\cdots s_j}) = 0,$$

where the notation $\mathrm{diam}(\cdot)$ denotes the diameter of a set. For any $\varepsilon > 0$, there exists an integer n such that $\mathrm{diam}(V_{\bar{s}_{-n}\cdots\bar{s}_0\cdots\bar{s}_n}) < \varepsilon$. Let $\delta_1 = 1/2^{n+k_1+1}$. Then for any $\tilde{W} \in \Gamma$ with $d(\tilde{W}, \bar{W}) < \delta_1$, it follows that $\tilde{W}$ agrees with $\bar{W}$ in the terms with indices from $i = -n - k_1$ to $i = n + k_1$, which implies that the sequence $\tilde{s} \in S$ corresponding to $\tilde{W}$ agrees with $\bar{s} \in S$ corresponding to $\bar{W}$ in the terms with indices from $i = -n$ to $i = n$. Thus $h(\tilde{W}), h(\bar{W}) \in V_{\bar{s}_{-n}\cdots\bar{s}_0\cdots\bar{s}_n}$ and $\|h(\tilde{W}) - h(\bar{W})\| < \varepsilon$. This shows the continuity of h.

Since Γ is compact, Λ_α is Hausdorff, and $h: \Gamma \to \Lambda_\alpha$ is continuous and bijective, we conclude that h is a homeomorphism.

Finally, for any $\bar{W} \in \Gamma$, we have

$$h(\bar{W}) = \Pi \circ T_\alpha(\bar{W}) = \eta(0) = (\eta_1(0), \eta_2(0), \dots, \eta_{k_1+k_2+2}(0))^{\mathrm{T}}.$$

Thus

$$\begin{aligned} F_\alpha(h(\bar{W})) &= (\eta_1(1), \eta_2(1), \dots, \eta_{k_1+k_2+2}(1))^{\mathrm{T}} \\ &= \Pi \circ \sigma^2 \circ T_\alpha(\bar{W}) \\ &= \Pi \circ T_\alpha \circ \sigma^2(\bar{W}) \quad \text{(by Lemma 2.3)} \\ &= h \circ \sigma^2(\bar{W}). \end{aligned}$$

This shows that (3.1) holds. Therefore, h is a conjugacy from σ^2 to F_α and thus, (Γ, σ^2) is topological conjugate to $(\Lambda_\alpha, F_\alpha)$. This completes the proof. □

We have several remarks about Theorem 3.2.

Remark 3.3. Theorem 3.2 assures the F_α is chaotic on Λ_α for α sufficiently large. But Λ_α is contained in a small neighborhood of $\{\xi_1, \dots, \xi_{2^{k_1+1}}\}$, which is far away from the origin as long as the zero points of f_1 and f_2 are nonzero. In this case, the delayed discrete Hopfield neural network can demonstrate chaotic behavior outside the neighborhood of the origin. This is complementary to the results in Kaslik and Balint [7], where it was shown that the same system can have chaotic behavior near the origin.

Remark 3.4. We can easily extend Theorem 3.2 to a more general case. Indeed, we can show that if f_1 and f_2 have $m_1 > 1$ and $m_2 > 1$ simple zeros respectively, then there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$, F_α on Λ_α is topologically conjugate to the full shift map σ on $\Sigma_{m_1 m_2}$ in the system (1.2).

Remark 3.5. We can also apply this method to more general cases of discrete-time, delayed neural networks with more than two neurons with or without self-connections.

To illustrate the result of Theorem 3.2, let us consider the following discrete network of two identical neurons with exciteatory interactions:

$$\begin{cases} x(n) = \beta x(n-1) + \alpha f(y(n-k)), \\ y(n) = \beta y(n-1) + \alpha f(x(n-k)), \end{cases} \qquad k \geqslant 1, \tag{3.2}$$

which was considered by Huang and Zou [6]. In that paper, it was shown that for α large enough, the system has chaotic behavior *near the origin* under the assumptions that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and has two distinct simple zero points. By Theorem 3.2, under the same assumptions, we see that F_α on Λ_α is topologically conjugate to the shift map σ on $\Sigma_4 = \Sigma_{2\cdot 2}$, implying that this system also demonstrates chaotic behavior in a region *away from the neighborhood of the origin*.

4 Some simulations

In this section, we provide some numeric simulation results which support our theoretical result obtained in Section 3. To this end, we choose $f_1(t) = \sin(t)$ and $f_2(t) = \tanh(t)$, and set $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{3}{4}$, $\alpha_{12} = a$, $\alpha_{21} = b$, $k_1 = 1$, $k_2 = 2$. With these specifications, (2.1) becomes

$$\begin{cases} x(n+1) = \dfrac{1}{4}x(n) + a\tanh(y(n-2)), \\ y(n+1) = \dfrac{3}{4}y(n) + b\sin(x(n-1)), \end{cases} \qquad \forall\, n \geqslant 2. \tag{4.1}$$

Figures 1 and 2 give the simulation results.

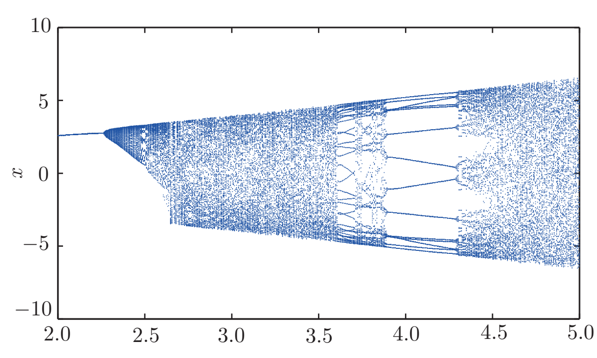

Figure 1 Bifurcation diagram for system (4.1) for $a \in (2, 5)$ and $b = 1$ in the (a, x)-plane

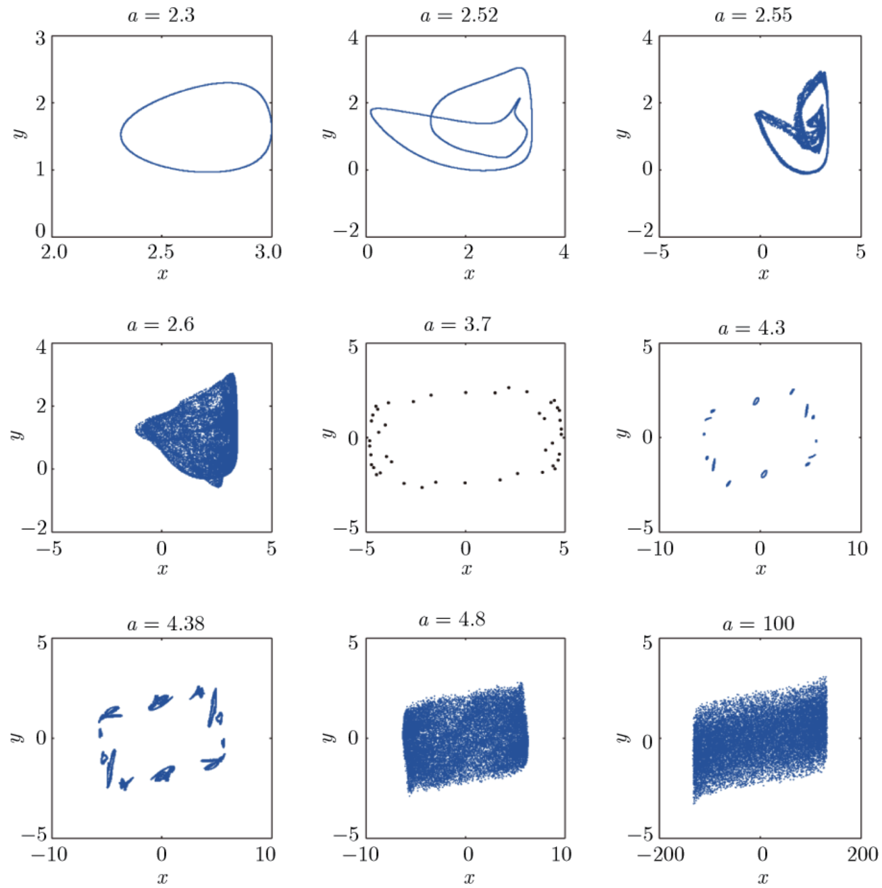

Figure 2 Simulations for system (4.1) in x-y plane for $a \in (2, 100)$, $b = 1$

In Figure 1, for each a value, the initial conditions were reset to $y_0 = 0.01$, $y_1 = 0.02$, $y_2 = 0.1$, $x_1 = 6.2$, $x_2 = 0.02$ and 10^4 time steps were iterated before plotting the data which consists of 10^4 points per a value. The plotting is for x vs the parameter a. Plotting for y vs a is similar and hence is omitted.

In Figure 2, we show the results on the x-y plane. For each a value, the initial conditions were reset to $y_0 = 0.01$, $y_1 = 0.02$, $y_2 = 0.1$, $x_1 = 6.2$, $x_2 = 0.02$ and 2×10^4 time steps were iterated before plotting the data which consists of 2×10^4 points per a value. Kaslik and Balint [7] showed that under some conditions, the system (4.1) may exhibit chaos in the *vicinity of the origin* in [7]. But the result in [7] cannot well explain some simulation results in Figures 1–2. For example, for $a = 3.7, 4.3$ and 4.38, respectively, the simulation results in Figure 2 show that there are chaotic invariant sets for the system (4.1) which are in some disjoint closed sets outside the vicinity of the origin and they actually correspond to the disjoint closed sets constructed in the proof of Theorem 3.2 in this paper. Also, for $a = 2.6, 4.8$ and particularly for $a = 100$, the chaotic invariant sets is seen to be quite large (although they all include the the origin) and hence, contain points far away from origin.

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