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Map dynamics versus dynamics of associated delay reaction–diffusion equations with a Neumann condition

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In this paper, we consider a class of delay reaction–diffusion equations (DRDEs) with a parameter $\varepsilon > 0$. A homogeneous Neumann boundary condition and non-negative initial functions are posed to the equation. By letting $\varepsilon \to 0$, such an equation is formally reduced to a scalar difference equation (or map dynamical system). The main concern is the relation of the absolute (or delay-independent) global stability of a steady state of the equation and the dynamics of the nonlinear map in the equation. By employing the idea of attracting intervals for solution semiflows of the DRDEs, we prove that the globally stable dynamics of the map indeed ensures the delay-independent global stability of a constant steady state of the DRDEs. We also give a counterexample to show that the delay-independent global stability of DRDEs cannot guarantee the globally stable dynamics of the map. Finally, we apply the abstract results to the diffusive delay Nicholson blowfly equation and the diffusive Mackey–Glass haematopoiesis equation. The resulting criteria for both model equations are amazingly simple and are optimal in some sense (although there is no existing result to compare with for the latter).

Keywords: delay-independent global stability; delay reaction–diffusion equation; map dynamical system

1. Introduction

In the absence of spatial heterogeneity, many model equations from biology and other areas are of the form of the following delay differential equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\delta u(t) + f(u(t-\tau)). \tag{1.1}$$

Here, the time delay $\tau > 0$ may account for various contexts depending on the practical problem under consideration. A prototype of such equations is the well-known delayed Nicholson blowfly equation (Nicholson 1954; Gurney *et al.* 1980)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\delta u(t) + \beta u(t-\tau)\mathrm{e}^{-au(t-\tau)}, \tag{1.2}$$

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where $u(t)$ is the adult population of the fly and $\tau > 0$ explains the maturation time of the fly (Gurney *et al.* 1980; Cooke *et al.* 1999). Equation (1.2) has been very well studied (e.g. Kulenovic & Ladas 1987; Karakostas *et al.* 1992; Kuang 1993; So & Yu 1994; Cooke *et al.* 1999; Györi & Trofimchuk 1999; Faria 2006). The more general form (1.1) has also been extensively and intensively studied (e.g. Kuang 1993; Walther 1995; Cooke *et al.* 1999; Györi & Trofimchuk 1999; Krisztin *et al.* 1999; Faria 2006; Röst & Wu 2007; Krisztin 2008; Liz & Röst 2009, in press; and references therein). Depending on the nonlinear function f, equation (1.1) can demonstrate very rich and complicated dynamics (e.g. Walther 1995; Krisztin *et al.* 1999; Krisztin 2008). For equation (1.2), the global dynamics can be summarized as given below (e.g. Kuang 1993; Cooke *et al.* 1999; Györi & Trofimchuk 1999; Wei & Li 2005; Faria 2006; Berezansky *et al.* 2010)

- (i) when $0 < \beta/\delta \le 1$, $u = 0$ is the only equilibrium of equation (1.2) that is globally asymptotically stable for any $\tau \ge 0$;
- (ii) when $1 < \beta/\delta$, $u = 0$ becomes unstable and there is a unique positive equilibrium $u = (1/a)\ln(\beta/\delta) =: u_1$;
- (iii) when $1 < \beta/\delta \le e^2$, u_1 is globally asymptotically stable for all positive solutions, *regardless* of the value of $\tau > 0$ and
- (iv) when $\beta/\delta > e^2$, u_1 remains stable for small $\tau > 0$, but larger values of $\tau > 0$ will destroy the stability of u_1 giving rise to periodic solutions around u_1 via Hopf bifurcation.

When the spatial heterogeneity becomes an issue (e.g. random diffusion in population dynamics), the model equation (1.1) is modified to the following delay partial differential equation:

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + f(u(t-\tau,x)), \tag{1.3}$$

and accordingly, equation (1.2) is replaced by

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \beta u(t-\tau,x)\mathrm{e}^{-au(t-\tau,x)}. \tag{1.4}$$

In recent years, there have been many works dealing with delay reaction–diffusion equations, particularly the diffusive Nicholson blowflies equation (1.4) and various versions of diffusive delay logistic equations that serve as the models for population dynamics and ecological problems (Busenberg & Mahaffy 1985; Friesecke 1993; Yang & So 1996; Huang 1998; So & Yang 1998; So & Zou 2001; Mei *et al.* 2004; Yi & Zou 2008; Yi *et al.* 2009). These equations fall into the category of partial functional differential equations (PFDEs) and the monograph by Wu (1996) serves as a good source for fundamental theory of PFDEs.

For equations (1.3) and (1.4), depending on the practical situation, one may face a bounded or an unbounded spatial domain. When an unbounded domain is considered, travelling wavefront solutions are an important topic and have been discussed by many researchers (e.g. So & Zou 2001; So *et al.* 2001; Wu & Zou 2001; Mei *et al.* 2004).

When a bounded domain Ω is considered, boundary conditions need to be posed on the boundary $\partial\Omega$ of the domain, depending on the practical scenarios. Taking population dynamics as an example, the homogeneous Dirichlet boundary condition represents the situation where the boundary is hostile, while the homogeneous Neumann boundary condition accounts for the case when the domain is isolated from outside. In this paper, we are only interested in the Neumann boundary condition, that is, we consider the following boundary initial value problem (BIVP):

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}(t,x)=d\Delta u(t,x)-\delta u(t,x)+f(u(t-\tau,x)),\\
&\qquad\qquad \text{for } (t,x)\in(0,\infty)\times\Omega,\\
&\frac{\partial u}{\partial \nu}=0,\quad \text{for } (t,x)\in(0,\infty)\times\partial\Omega\\
\text{and}\quad &u(\theta,x)=\phi(\theta,x),\quad \text{for } (\theta,x)\in[-\tau,0]\times\bar{\Omega}.
\end{aligned}\right\}\tag{1.5}$$

Here, $\Omega\subseteq R^m$ is a bounded domain with smooth boundary $\partial\Omega$, Δ is the Laplacian operator, $(\partial/\partial\nu)$ represents the differentiation in the direction of the outward normal to $\partial\Omega$, parameters $d,\delta>0$ and $f:R\longrightarrow R$ is a continuous function.

When $f(u)=\beta u\mathrm{e}^{-au}$ in equation (1.5), Yang & So (1996) showed that

- (i) when $0<\beta/\delta\le 1$, $u=0$ is the only steady state of equation (1.5) that is globally asymptotically stable for any $\tau>0$;
- (ii) when $1<\beta/\delta$, $u=0$ becomes unstable and there is a positive constant steady state $u=(1/a)\ln(\beta/\delta)=: u_1$;
- (iii) when $1<\beta/\delta\le e$, u_1 is globally asymptotically stable for all positive solutions, *regardless* of the value of $\tau>0$ and
- (iv) when $\beta/\delta>e^2$, u_1 remains stable for small $\tau>0$, but larger values of $\tau>0$ will destroy the stability of u_1 giving rise to periodic solutions around u_1 via Hopf bifurcation.

Note that comparing with the results (i)–(iv) for equation (1.2), there is a gap $(e,e^2]$ for the ratio β/δ for equation (1.5). In a recent work (Yi & Zou 2008), we have filled this gap by showing that

- (iii)′ when $1<\beta/\delta\le e^2$, u_1 is globally attractive for all positive solutions, *regardless* of the value of $\tau>0$.

From (i)–(iii)′–(iv) for equation (1.5) (as well as (i)–(iv) for equation (1.2)), we see that $\beta/\delta=e^2$ is a threshold value in terms of the *delay-independent global stability* of u_1. This makes one wonder what happens if the Ricker function $\beta u\mathrm{e}^{-au}$ is replaced by another reproduction function $f(u)$ satisfying the general requirements for a birth function. This motivates us to consider the impact of the map dynamics of $f(\cdot)$ on the dynamics of the BIVP (1.5).

For convenience, by rescaling

$$\left.\begin{aligned}
&\frac{t}{\tau}\longrightarrow t,\quad \frac{x}{\sqrt{d\tau}}\longrightarrow x,\quad \frac{1}{\sqrt{d\tau}}\Omega\longrightarrow\Omega,\quad \tau\delta\longrightarrow\mu,\\
&u(\tau t,\sqrt{d\tau}x)\longrightarrow u(t,x)\quad\text{and}\quad \frac{1}{\delta}f(\cdot)\longrightarrow f(\cdot),
\end{aligned}\right\}\tag{1.6}$$

we transform equation (1.5) to the following:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu f(u(t-1,x)), \\ &\qquad\qquad \text{for } (t,x) \in (0,\infty) \times \Omega, \\ &\frac{\partial u}{\partial \nu} = 0, \quad \text{for } (t,x) \in (0,\infty) \times \partial\Omega \\ \text{and}\quad &u(\theta,x) = \phi(\theta,x), \quad \text{for } (\theta,x) \in [-\tau,0] \times \bar{\Omega}. \end{aligned}\right\} \tag{1.7}$$

Letting $\varepsilon = (1/\mu)$, equation (1.7) is further transformed to the following problem:

$$\left.\begin{aligned} &\varepsilon\frac{\partial u}{\partial t}(t,x) = \varepsilon\Delta u(t,x) - u(t,x) + f(u(t-1,x)), \\ &\qquad\qquad \text{for } (t,x) \in (0,\infty) \times \Omega, \\ &\frac{\partial u}{\partial \nu} = 0, \quad \text{for } (t,x) \in (0,\infty) \times \partial\Omega \\ \text{and}\quad &u(\theta,x) = \phi(\theta,x), \quad \text{for } (\theta,x) \in [-\tau,0] \times \bar{\Omega}. \end{aligned}\right\} \tag{1.8}$$

Since we are interested in the delay-independent dynamics of equation (1.7), we may formally let $\varepsilon \to 0^+$ in equation (1.8), leading to the following equation:

$$u(t,x) = f(u(t-1,x)), \quad \text{for } (t,x) \in (0,\infty) \times \Omega. \tag{1.9}$$

Obviously, the asymptotic behaviour of the solutions of equation (1.9) is determined by the dynamics of

$$u_{n+1} = f(u_n), \tag{1.10}$$

which is governed by the one-dimensional map $f(\cdot)$ and is sometimes referred to as the dynamics of the map $f(\cdot)$.

A question arises naturally: can one determine the dynamics of BIVP (1.8) by the properties of equation (1.10) (or dynamics of map f) and in what sense? More concretely, relating to our main motivation stated above, what dynamics of equation (1.10) would imply the delay-independent global stability of a steady state for equation (1.8). We will address this question by using a dynamical system approach. In §3, by employing the idea of attracting intervals for solution semiflows of equation (1.8), we prove that the globally stable dynamics for equation (1.10) indeed ensures the delay-independent global stability of a constant steady state for the BIVP (1.8). A counterexample is also given to show that the delay-independent global stability of equation (1.8) cannot guarantee the globally stable dynamics of equation (1.10). To achieve this, we need to make use of some existing results on globally stable dynamics for equation (1.10), which are collected in §2 as a preliminary section. Then, by applying these results, we obtain some very simple conditions that can assure the globally stable dynamics for equation (1.10). In §4, we apply the main results in §3 to the delayed diffusive Nicholson blowflies equation and another model equation arising from biology. For the former, we re-confirm the existing optimal results summarized in (i)–(ii)–(iii)′, while for the latter, we also obtain respective conditions for the delay-independent global stability of the trivial steady state and that of a positive steady state.

We point out that the idea of relating the dynamics of a map to the dynamics of a delay *ordinary* differential equation has been used by some other researchers. For example, by treating the delay ordinary differential equation

$$\varepsilon u'(t) = -u(t) + f(u(t-1)) \tag{1.11}$$

as a singular perturbation of equation (1.10), Mallet-Paret & Nussbaum (1986), Ivanov & Sharkovsky (1992), Hale & Verduyn Lunel (1993, §12.7) and Liz (2004) obtained some information on the dynamics of equation (1.11) based on that of equation (1.10). Our work can be considered a first attempt to further extend this idea to the delay *partial* differential equations.

2. Some existing results on equation (1.10)

Let $I \subseteq R$ be a closed (possibly infinite) interval and $f : I \longrightarrow I$ be a continuous function. Let $I_f = \bigcap_{n\geq 1} \overline{f^n(I)}$. Then, either $I_f = \emptyset$, or $I_f \subseteq I$ is a closed (possibly infinite) interval (e.g. Hale 1988). In §3, we will see that I_f actually attracts the solutions of equation (1.8). Therefore, the set I_f plays a crucial role in determining the asymptotic behaviours of solutions of equation (1.8). For convenience of applications in later sections, we collect some known results on I_f below, with some remarks on these results.

Firstly, by employing the main theorem in Coppel (1955) coupled with proposition 1.2 in Mallet-Paret & Nussbaum (1986), one can obtain the following result.

Proposition 2.1. *Let I be a compact interval. Assume that*

(H1) *there is a $u^* \in I$ such that $\{u \in I : f^2(u) = u\} = \{u^*\}$.*

Then, $I_f = \{u^*\}$.

Remark 2.2. Generally, in proposition 2.1, we cannot omit the assumption that I is compact. For example, let $I = R$ and $f : R \longrightarrow R$ be given by $f(u) = (1/2)u$ for all $u \in R$. By taking $u^* = 0$ and $I = R$, it is obvious that f satisfies the assumption (H1). However, by the definition of I_f, we easily obtain $I_f = R \neq \{0\}$.

Remark 2.3. If $I_f = \{u^*\}$ for some $u^* \in I$, then by the definition of I_f, $\{u \in I : f^n(u) = u\} = \{u^*\}$ for every positive integer n and hence $\{u \in I : f^2(u) = u\} = \{u^*\}$. Thus, by proposition 2.1, we know that in the case of a compact interval I, $I_f = \{u^*\}$ if and only the assumption (H1) holds; that is, u^* is a globally stable fixed point of f in I if and only if the assumption (H1) holds.

The following result can be obtained from Coppel (1955).

Proposition 2.4. *Let I be compact. Assume that*

(H2) *there exists an $u^* \in I$ such that $|f(u) - u^*| < |u - u^*|$ for all $u \in I \setminus \{u^*\}$.*

Then, (H1) *holds.*

Remark 2.5. Assumption (H1) does not imply (H2). This can be seen by the function $f:[-1,2]\longrightarrow[-1,2]$ defined by

$$f(u)=\begin{cases}-2u, & u\le 0,\\ 0, & u>0.\end{cases}$$

Obviously, this function does not satisfy (H2) on $I=[-1,2]$. However, it can be easily shown that $I_f=\{0\}$. Thus, by remark 2.3, f satisfies (H1).

3. On the global stability of equation (1.7)

For convenience, we begin by introducing some notations. Let Ω be a bounded domain in R^m with smooth boundary $\partial\Omega$, Δ be the Laplacian operator and $(\partial/\partial\nu)$ be the derivative in the outward normal direction of $\partial\Omega$. Let R (R_+, respectively) be the set of all real (non-negative, respectively) numbers. Let $C=C(\bar{\Omega},R)$ and $X=C([-1,0]\times\bar{\Omega},R)$ be equipped with the usual supremum norm $\|\cdot\|$. Also, let $C_+=C(\bar{\Omega},R_+)$ and $X_+=C([-1,0]\times\bar{\Omega},R_+)$.

For any $\phi,\ \psi\in X$, we write $\phi\ge_X\psi$ if $\phi-\psi\in X_+$, $\phi>_X\psi$ if $\phi\ge\psi$ and $\phi\ne\psi$, $\phi\gg_X\psi$ if $\phi-\psi\in\mathrm{Int}(X_+)$. Similarly, for any $\xi,\ \eta\in C$, we write $\xi\ge_C\eta$ if $\xi-\eta\in C_+$, $\xi>_C\eta$ if $\xi\ge_C\eta$ and $\xi\ne\eta$, $\xi\gg_C\eta$ if $\xi-\eta\in\mathrm{Int}(C_+)$. For simplicity of notations, when there is no confusion about the spaces, we write $\ge$, $>$ and $\gg$ for $\ge_*$, $>_*$ and $\gg_*$, respectively, where the asterisk stands for X or C.

For a real interval I, let $I+[-1,0]=\{t+\theta: t\in I \text{ and } \theta\in[-1,0]\}$. For $u:(I+[-1,0])\times\bar{\Omega}\to R$ and $t\in I$, we write $u_t(\cdot)(\cdot)$ for the element of X defined by $u_t(x)(\theta)=u(t+\theta,x)$, for $-1\le\theta\le 0$ and $x\in\bar{\Omega}$. For any $k\in R$, we still denote by k the constant functions in C and X taking value k when no confusion arises.

Let $\mu>0$, I be a real interval and $f:I\longrightarrow I$ be a continuous function. Consider the following scalar delayed reaction–diffusion equation:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}(t,x)=\Delta u(t,x)-\mu u(t,x)+\mu f(u(t-1,x)),\\ &\qquad\qquad \text{for } (t,x)\in(0,\infty)\times\Omega,\\ &\frac{\partial u}{\partial\nu}=0,\quad \text{for } (t,x)\in(0,\infty)\times\partial\Omega\\ \text{and}\quad &u(\theta,x)=\phi(\theta,x),\quad \text{for } (\theta,x)\in[-1,0]\times\bar{\Omega},\end{aligned}\right\}\tag{3.1}$$

where $\phi\in C([-1,0]\times\bar{\Omega},I)$.

Let $T(t)(t\ge 0)$ be the semigroup on C generated by the closure operator of $\Delta-\mu\mathrm{Id}$ under the homogeneous Neumann boundary condition, where Id is the identity operator on C. Then, by the general results on this semigroup (e.g. Wu 1996), we have the following.

Lemma 3.1. *Let $T(\cdot)$ be defined as above. Then, the following statements are true:*

(i) $T(t)(t\ge 0)$ *is an analytical strongly continuous semigroup on* C*;*
(ii) $T(t)a=a\mathrm{e}^{-\mu t}$ *for all* $t\in R_+$ *and* $a\in R$ *and*
(iii) $T(t)(C_+\setminus\{0\})\subseteq\mathrm{Int}(C_+)$ *for all* $t>0$.

We first consider a *bounded and closed* $I=[a,b]\subseteq R$. Denote by $\tilde{f}$ the expansion of f from I to R, that is, $\tilde{f}:R\longrightarrow R$ is defined by

$$\tilde{f}(u)=\begin{cases} f(a), & u<a,\\ f(u), & a\le u\le b,\\ f(b), & u>b.\end{cases}$$

Then, $\tilde{f}$ is a continuous function on R, $\tilde{f}(R)=\tilde{f}(I)$, $\tilde{f}|_I\equiv f$ and $\cap_{n\geq 1}\overline{\tilde{f}^n(R)}=I_f$. Moreover, define $F:X\to C$ by $F(\phi)(x)=\mu\tilde{f}(\phi(-1,x))$ for all $x\in\bar{\Omega}$ and $\phi\in X$. Associated with equation (3.1) is the following integral equation with the given initial function:

$$\left.\begin{aligned} &u(t)=T(t)\phi(0,\cdot)+\int_0^t T(t-s)F(u_s)\mathrm{d}s, \quad t\geq 0,\\ &u_0=\phi\in X. \end{aligned}\right\} \tag{3.2}$$

For a given $\phi\in X$, by the step argument and the definition of F, one can solve equation (3.2) inductively on $[0,1],[1,2],\ldots$, giving a unique solution of equation (3.2) defined for all $t\geq 0$. Denote this solution of equation (3.2) by $u^{\phi}(t,x)$, which is the mild solution of equation (3.1) in the sense of Martin & Smith (1990, 1991).

Since the semigroup $T(t)$ is analytical, by corollary 2.2.5 in Wu (1996), we know that the mild solution $u^{\phi}(t,x)$ of equation (3.1) is also the classical solution of equation (3.1) (e.g. Travis & Webb 1974; Martin & Smith 1990, 1991; Wu 1996) for all $t>1$. From now on, we will not distinguish 'mild' and 'classical', and simply use the word 'solution'. This allows us to study the dynamics of equation (3.1) via that of equation (3.2).

Define the map $U:R_+\times X\longrightarrow X$ by $U(t,\phi)=(u^{\phi})_t$ for $(t,\phi)\in R_+\times X$, then $U(t,\cdot)$ is a semiflow on X called the solution semiflow of equation (3.2). We first introduce some terminologies from dynamical systems theory.

Definition 3.2. An element $\phi\in X$ is called an equilibrium of U if $U(t,\phi)=\phi$ for all $t\geq 0$. A subset M of X is said to be positively invariant under U if $U(t,\phi)\in M$ for every $\phi\in M$ and $t\geq 0$.

Definition 3.3. Let u^* be an equilibrium and M be a positively invariant set of the semiflow U.

- (i) we say that u^* is a stable equilibrium in M if, for every neighbourhood V of u^* in M, there exists a neighbourhood W of u^* in M such that $U(t,\phi)\in V$ for all $(t,\phi)\in R_+\times W$;
- (ii) we say that u^* is globally attractive in M if $\lim_{t\to+\infty}U(t,\phi)=u^*$ for all $\phi\in M$ and
- (iii) we say u^* is globally asymptotically stable in M if u^* is a stable equilibrium as well globally attractive in M.

As usual, we will omit M in definition 3.2 if $M=X$.

Remark 3.4. If u^* is a globally asymptotically stable equilibrium in M and $u^*\in M'\subseteq M$ is another positively invariant set of the semiflow U, then u^* is also globally asymptotically stable in M'.

Lemma 3.5. *If* $\phi \in C([-1,0]\times\bar{\Omega}, I)$, *then* $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega}, I)$ *for all* $t\in R_+$, *namely,* $C([-1,0]\times\bar{\Omega}, I)$ *is a positively invariant set of* U.

Proof. We firstly notice that $I=[a,b]$ where $a,b\in R$. We now show that

$$u^{\phi}(t,x)\geq a, \quad \text{for all } t\in R_+ \text{ and } x\in\bar{\Omega}.$$

Indeed, from the definition of $u^{\phi}(t,x)$, we can obtain

$$u^{\phi}(t,x)=\big[T(t)\phi(0,\cdot)\big](x)+\mu\int_0^t\big[T(t-s)\tilde{f}(u^{\phi}(s-1,\cdot))\big](x)\mathrm{d}s.$$

Let $t_1=\sup\{t\geq 0: u^{\phi}(s,x)\geq a$ for all $s\in[0,t]$ and $x\in\bar{\Omega}\}$. For each $(t,x)\in[t_1,t_1+1]\times\bar{\Omega}$, we have

$$\begin{aligned}
u^{\phi}(t,x)&\geq\big[T(t)a\big](x)+\mu\int_0^t\big[T(t-s)a\big](x)\mathrm{d}s\\
&\geq \mathrm{e}^{-\mu t}a+\mu\int_0^t \mathrm{e}^{-\mu(t-s)}a\,\mathrm{d}s\\
&=\mathrm{e}^{-\mu t}a+\mu a\int_0^t \mathrm{e}^{-\mu(t-s)}\mathrm{d}s\\
&=\mathrm{e}^{-\mu t}a+a(1-\mathrm{e}^{-\mu t})\\
&=a,
\end{aligned}$$

where the first inequality follows from lemma 3.1(iii) and the second inequality follows from lemma 3.1(ii). This contradicts the choice of t_1. Similarly, we can prove

$$u^{\phi}(t,x)\leq b \text{ for all } t\in R_+ \text{ and } x\in\bar{\Omega}. \qquad \blacksquare$$

Remark 3.6. By the same argument as for the case $I=[a,b]$, we can prove a similar result of lemma 3.5 for the case $I=[a,\infty)$ or $(-\infty,a]$ or $(-\infty,-\infty)$.

Let $A\subseteq X$. For $\varepsilon>0$, the ε-neighbourhood of A is defined by $O(A,\varepsilon)=\{\psi\in X: ||\psi-\phi||<\varepsilon$ for some $\phi\in A\}$; for $\phi\in X$, the distance between ϕ and A is defined by $\mathrm{dist}(\phi,A)=\inf\{||\phi-\psi||:\psi\in A\}$.

Let $P(t)\in X$ for large t. As is customary, by $\lim_{t\to+\infty}\mathrm{dist}(P(t),A)=0$, we mean that for any $\varepsilon>0$, there exists $T=T_\varepsilon>0$ such that $P(t)\in O(A,\varepsilon)$ for all $t>T$.

Lemma 3.7. *Let* $\phi\in X$ *and* $J\equiv[c,d], K\equiv[c^*,d^*]$ *be real closed intervals. Assume that* $(u^{\phi})_t\in C([-1,0]\times\bar{\Omega},J)$ *and* $\tilde{f}(u^{\phi}(t-1,x))\in K$ *for all* $t\in R_+$ *and* $x\in\bar{\Omega}$. *Then,* $\lim_{t\to+\infty}\mathrm{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},K))=0$.

Proof. It suffices to show that for any $\varepsilon > 0$, there exists $T > 0$ such that $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(c^*-\varepsilon,d^*+\varepsilon))$ for all $t > T$. Firstly, by equation (3.2) and (ii,iii) in lemma 3.1, we know that for all $t \geq 0$ and $x \in \bar{\Omega}$,

$$\begin{aligned} u^{\phi}(t,x) &= \big[T(t)\phi(0,\cdot)\big](x) + \mu\int_0^t \big[T(t-s)\tilde{f}(u^{\phi}(s-1,\cdot))\big](x)\mathrm{d}s \\ &\geq \big[T(t)c\big](x) + \mu\int_0^t \big[T(t-s)c^*\big](x)\mathrm{d}s \\ &\geq \mathrm{e}^{-\mu t}c + \mu\int_0^t \mathrm{e}^{-\mu(t-s)}c^*\mathrm{d}s \\ &= \mathrm{e}^{-\mu t}c + \mu c^*\int_0^t \mathrm{e}^{-\mu(t-s)}\mathrm{d}s \\ &= \mathrm{e}^{-\mu t}c + (1-\mathrm{e}^{-\mu t})c^*. \end{aligned}$$

Hence, for any $\varepsilon > 0$, taking $T_1 = 1 + \max\{0,(1/\mu)\ln((1+|c^*-c|)/\varepsilon)\}$, we have $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(c^*-\varepsilon,+\infty))$ for all $t > T_1$. A similar argument also shows that for any $\varepsilon > 0$, there exists $T_2 > 0$ such that $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(-\infty,d^*+\varepsilon))$ for all $t > T_2$. So, $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(c^*-\varepsilon,d^*+\varepsilon))$ for all $t > T \equiv \max\{T_1,T_2\}$. ■

Lemma 3.8. *Let $\phi \in X$ and $J \equiv [c,d]$ be real closed intervals. If* $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},J)) = 0$, *then* $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},\tilde{f}(J))) = 0$. *Consequently,* $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},\tilde{f}^n(J))) = 0$ *for every non-negative integer n.*

Proof. Obviously, there exist $c^*, d^* \in R$ such that $\tilde{f}(J) = [c^*,d^*]$. For any $\varepsilon > 0$, the continuity of $\tilde{f}$ implies that there exists $\delta \in (0,(\varepsilon/3))$ such that $\tilde{f}([c-\delta,d+\delta]) \subseteq [c^*-(\varepsilon/3), d^*+(\varepsilon/3)]$. Now, assume that $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},J)) = 0$. Then, there exists $T_1 > 0$ such that $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(c-\delta,d+\delta))$ for all $t \geq T_1$. Thus, by the choices of δ and T_1, we obtain that $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},[c-\delta,d+\delta])$ and $\tilde{f}(u^{\phi}(t-1,x)) \in [c^*-(\varepsilon/3), d^*+(\varepsilon/3)]$ for all $t > T_1$ and $x \in \bar{\Omega}$.

On the other hand, by the semigroup property of the solution semiflow, we know $(u^{(u^{\phi})_{T_1+1}})_t = (u^{\phi})_{(t+T_1+1)}$ for all $t \geq 0$. Applying lemma 3.7 to $(u^{\phi})_{T_1+1}$, we infer that there exists $T_2 > 0$ such that $(u^{(u^{\phi})_{T_1+1}})_t \in C([-1,0]\times\bar{\Omega},(c^*-\varepsilon,d^*+\varepsilon))$ for all $t > T_2$. Taking $T = 1 + T_1 + T_2$, we have shown that $(u^{\phi})_t \in C([-1,0]\times\bar{\Omega},(c^*-\varepsilon,d^*+\varepsilon))$ for all $t > T$. This proves $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},\tilde{f}(J))) = 0$. The second conclusion is a consequence of inductive use of the first conclusion. ■

Proposition 3.9. *For each $\phi \in X$,* $\lim_{t\to+\infty}\operatorname{dist}((u^{\phi})_t, C([-1,0]\times\bar{\Omega},I_f)) = 0$.

Proof. By the continuity of ϕ, there are $a', b' \in R$ such that $[a, b] \subseteq I' = [a', b']$ and $\phi \in C([-1, 0] \times \bar{\Omega}, I')$. Let $a^* = \inf I_f$, $b^* = \sup I_f$, $a_n = \inf \tilde{f}^n(I')$ and $b_n = \sup \tilde{f}^n(I')$ for all $n \geq 0$ with $a_0 = a'$ and $b_0 = b'$. It is obvious that $a_n = \inf f^n(I)$ and $b_n = \sup f^n(I)$ for all $n \geq 1$.

For arbitrarily given $\varepsilon > 0$, we need to show that there exists $T = T_\varepsilon > 0$, such that $(u^\phi)_t \in C([-1, 0] \times \bar{\Omega}, (a^* - \varepsilon, b^* + \varepsilon))$ for all $t > T$. By the definition of I_f, there is an integer k such that $[a_k - (\varepsilon/3), b_k + (\varepsilon/3)] \subseteq (a^* - \varepsilon, b^* + \varepsilon)$. Thus, it suffices to prove that there exists $T^* = T^*_\varepsilon > 0$ such that $(u^\phi)_t \in C([-1, 0] \times \bar{\Omega}, [a_k - (\varepsilon/3), b_k + (\varepsilon/3)])$ for all $t > T^*$. This is implied by $\lim_{t\to+\infty} \operatorname{dist}((u^\phi)_t, C([-1, 0] \times \bar{\Omega}, [a_k, b_k])) = 0$, which is a consequence of applying lemmas 3.5 and 3.8 to $(u^\phi)_t$. ∎

We are now in a position to state and prove our first main result.

Theorem 3.10. *Assume that* (H1) *holds. Then,* u^* *is a globally asymptotically stable equilibrium in* $C([-1, 0] \times \bar{\Omega}, I)$.

Proof. Lemma 3.5 implies that $C([-1, 0] \times \bar{\Omega}, I)$ is a positively invariant set under U. It suffices to prove that u^* is a globally asymptotically stable equilibrium.

Note that proposition 2.1 implies $I_f = \{u^*\}$. By proposition 3.9, we have $\lim_{t\to+\infty} (u^\phi)_t = u^*$ for all $\phi \in X$, that is, u^* is a globally attractive equilibrium.

Next we prove the stability of u^*. Indeed, for every neighbourhood V of u^*, there exists $\varepsilon > 0$, such that $V \supseteq C([-1, 0] \times \bar{\Omega}, (u^* - \varepsilon, u^* + \varepsilon))$. By proposition 2.1, there exists an open interval $J \subseteq R$ such that the length of J is less than $(\varepsilon/2)$, $u^* \in J$ and $\tilde{f}(\bar{J}) \subseteq \bar{J}$.

By applying lemma 3.5 to $\tilde{f}|_{\bar{J}}$, we have $(u^\phi)_t \in C([-1, 0] \times \bar{\Omega}, \bar{J})$ for all $t \in R_+$ and $\phi \in C([-1, 0] \times \bar{\Omega}, J)$. Thus, $||(u^\phi)_t - u^*|| < \varepsilon$ for all $\phi \in C([-1, 0] \times \bar{\Omega}, J)$ and $t \in R_+$. Therefore, $(u^\phi)_t \in V$ for all $\phi \in C([-1, 0] \times \bar{\Omega}, J)$ and $t \in R_+$, implying the stability of u^*. ∎

As a result of proposition 2.4 and theorem 3.10, we have the following corollary.

Corollary 3.11. *Assume that* (H2) *holds. Then,* u^* *is a globally asymptotically stable equilibrium in* $C([-1, 0] \times \bar{\Omega}, I)$.

The above results are all for $I = [a, b]$, a compact interval. In the case that I is only a closed interval (that is $I = [a, \infty)$ or $I = (-\infty, b]$ or even $I = (-\infty, \infty)$), we have the following result.

Theorem 3.12. *Let* $I \subseteq R$ *be a closed interval (may be unbounded). Assume that* (H1) *or* (H2) *holds. Moreover, suppose that* $|f(u)| \leq |u|$ *for all sufficient large* $u \in I$ *(in particular, it is the case when* $f(I)$ *is bounded). Then,* u^* *is a globally asymptotically stable equilibrium in* $C([-1, 0] \times \bar{\Omega}, I)$.

Proof. Without loss of generality, we may assume that (H1) holds. Lemma 3.5 (see remark 3.6) implies that $C([-1,0]\times\bar{\Omega}, I)$ is a positively invariant set under U. It suffices to prove that u^* is a globally asymptotically stable equilibrium. $\tilde{f}: R \longrightarrow R$ is defined by

$$\tilde{f}(u)=\begin{cases} f(\inf I), & u<\inf I \text{ and } u\in R,\\ f(u), & u\in I,\\ f(\sup I), & u>\sup I \text{ and } u\in R.\end{cases}$$

Suppose $\phi\in X$. Then there exists $A>0$ such that $\tilde{f}([-A,A])\subseteq[-A,A]$, $u^*\in(-A,A)$ and $\phi\in C([-1,0]\times\bar{\Omega},[-A,A])$. By applying theorem 3.10 to $[-A,A]$ and $\tilde{f}|_{[-A,A]}$, we know that $\lim_{t\to+\infty}(u^\phi)_t=u^*$ and u^* is a stable equilibrium. From the arbitrariness of ϕ, we conclude that u^* is a globally asymptotically stable equilibrium. ■

When the interval I is *even not closed* in R, we have the following results.

Theorem 3.13. *Let $I\subseteq R$ be a real interval and $f(I)\subseteq I$. Assume that* (H1) *or* (H2) *holds. Suppose that f can be continuously extended to $\bar{I}$. Additionally, assume that there exist $a,b\in R$ such that one of the following conditions holds:*

(i) $I=(a,b)$ *and there exist sequences* $\{a_k\}$ *and* $\{b_k\}$ *in* I *such that* $\lim_{k\to+\infty}a_k=a$, $\lim_{k\to+\infty}b_k=b$ *and* $f([a_k,b_k])\subseteq[a_k,b_k]$;
(ii) $I=[a,b)$ *and there exists sequence* $\{b_k\}$ *in* I *such that* $\lim_{k\to+\infty}b_k=b$ *and* $f([a,b_k])\subseteq[a,b_k]$;
(iii) $I=(a,b]$ *and there exists sequence* $\{a_k\}$ *in* I *such that* $\lim_{k\to+\infty}a_k=a$ *and* $f([a_k,b])\subseteq[a_k,b]$;
(iv) $I=(a,+\infty)$ *and there exist sequences* $\{a_k\}$ *and* $\{b_k\}$ *in* I *such that* $\lim_{k\to+\infty}a_k=a$, $\lim_{k\to+\infty}b_k=+\infty$ *and* $f([a_k,b_k])\subseteq[a_k,b_k]$;
(v) $I=(-\infty,b)$ *and there exist sequences* $\{a_k\}$ *and* $\{b_k\}$ *in* I *such that* $\lim_{k\to+\infty}a_k=-\infty$, $\lim_{k\to+\infty}b_k=b$ *and* $f([a_k,b_k])\subseteq[a_k,b_k]$ *and*
(vi) $I=(-\infty,+\infty)$ *and there exist sequences* $\{a_k\}$ *and* $\{b_k\}$ *in* I *such that* $\lim_{k\to+\infty}a_k=-\infty$, $\lim_{k\to+\infty}b_k=+\infty$ *and* $f([a_k,b_k])\subseteq[a_k,b_k]$.

Then, u^* is a globally asymptotically stable equilibrium in $\{u^*\}\bigcup[C([-1,0]\times\bar{\Omega},\bar{I})\setminus\{\inf I,\sup I\}]$.

Proof. We only give the proofs for (i) and (iv), since the rest of the theorem can be proved by a similar argument.

Assume that the assumption (i) holds. Clearly, $a<f(x)<b$ for all $x\in I$. By the continuous extension of f to $\bar{I}=[a,b]$ implies $f([a,b])\subseteq[a,b]$. Thus, lemma 3.5 shows that $C([-1,0]\times\bar{\Omega},\bar{I})$ is positively invariant.

Let $\phi\in C([-1,0]\times\bar{\Omega},\bar{I})\setminus\{a,b\}$. We now claim that there exists $(t_1,x_1)\in[0,1]\times\bar{\Omega}$ such that $U(t_1,\phi)(0,x_1)>a$. Otherwise, $U(1,\phi)=a$. In view of

$\phi > a$ and $f((a,b)) \subseteq (a,b)$, we have $f(\phi(\theta^{**}, x^{**})) > a$ for some $(\theta^{**}, x^{**}) \in [-1,0] \times \bar{\Omega}$. It follows from equation (3.2) and lemma 3.1(ii, iii) that for any $(t,x) \in [1+\theta^{**}, 1] \times \bar{\Omega}$,

$$\begin{aligned}
U(t,\phi)(0,x) = (u^{\phi})_t(0,x) = u^{\phi}(t,x) \\
&= \big[T(t)\phi(0,\cdot)\big](x) + \mu \int_0^t \big[T(t-s) f(u^{\phi}(s-1,\cdot))\big](x)\mathrm{d}s \\
&= \mathrm{e}^{-\mu t} a + \mu \int_0^t \big[T(t-s) f(\phi(s-1,\cdot))\big](x)\mathrm{d}s \\
&> \mathrm{e}^{-\mu t} a + \mu \int_0^t \big[T(t-s) a\big](x)\mathrm{d}s \\
&= \mathrm{e}^{-\mu t} a + \int_0^t \mu \mathrm{e}^{-\mu(t-s)} a \,\mathrm{d}s \\
&= \mathrm{e}^{-\mu t} a + (1-\mathrm{e}^{-\mu t}) a \\
&= a,
\end{aligned}$$

a contradiction to $U(1,\phi) = a$.

By the above claim and lemma 3.1, and making use of the semigroup property, we obtain that for any $(t,x) \in (t_1, +\infty) \times \bar{\Omega}$,

$$\begin{aligned}
u^{\phi}(t,x) &= U(t-t_1, U(t_1,\phi))(0,x) \\
&= \big[T(t-t_1) U(t_1,\phi)(0,\cdot)\big](x) + \mu \int_0^{t-t_1} \big[T(t-s-t_1) \\
&\quad \times f(u^{\phi}(s+t_1-1,\cdot))\big](x)\mathrm{d}s \\
&> \mathrm{e}^{-\mu(t-t_1)} a + \mu \int_0^{t-t_1} \big[T(t-s-t_1) f(u^{\phi}(s+t_1-1,\cdot))\big](x)\mathrm{d}s \\
&\geq \mathrm{e}^{-\mu(t-t_1)} a + \int_0^{t-t_1} \mu \mathrm{e}^{-\mu(t-s-t_1)} a \,\mathrm{d}s \\
&= \mathrm{e}^{-\mu(t-t_1)} a + (1-\mathrm{e}^{-\mu(t-t_1)}) a \\
&= a.
\end{aligned}$$

Thus, $U(2,\phi) \gg a$. A similar argument also shows $U(2,\phi) \ll b$.

By the assumption (i), there exists $k_0 > 1$ such that $u^* \in (a_{k_0}, b_{k_0})$ and $U(2,\phi) \in C([-1,0] \times \bar{\Omega}, (a_{k_0}, b_{k_0}))$. Now applying theorem 3.10 to $f|_{[a_{k_0}, b_{k_0}]}$, we can deduce the conclusion of theorem 3.13.

Assume that the assumption (iv) holds. Suppose that $\phi \in C([-1,0] \times \bar{\Omega}, \bar{I}) \setminus \{a\}$. An argument similar to that of the above implies $U_2(\phi) \gg a$. By the assumption (iv), there exists $k_0 > 1$ such that $u^* \in (a_{k_0}, b_{k_0})$ and $U(2,\phi) \in C([-1,0] \times \bar{\Omega}, (a_{k_0}, b_{k_0}))$, and thus, by applying theorem 3.10 to $f|_{[a_{k_0}, b_{k_0}]}$, we can also obtain the conclusion of theorem 3.13. ■

Remark 3.14. If one of the conditions of theorem 3.13(i–vi) holds, then by proposition 2.1, we easily find that u^* is a globally stable equilibrium in I for the interval dynamical system $\{f^n\}_{n\geq 0}$.

Next we construct an example showing that the global asymptotic stability of a steady state for equation (1.8) does not imply the globally stable dynamics for equation (1.10).

Example 3.15. Define $f:[-1,1]\longrightarrow[-1,1]$ by

$$f(u)=\begin{cases}-2u-1, & u\in\left[-1,-\frac{1}{2}\right],\\ 0, & u\in\left[-\frac{1}{2},\frac{1}{2}\right],\\ -2u+1, & u\in\left[\frac{1}{2},1\right].\end{cases}\tag{3.3}$$

Note that f satisfies neither (H1) nor (H2) in §3. Moreover, it is easily seen that $I_f=[-1,1]$, and in particular, $\{-1,1\}$ a two-period orbit. Thus, this map does not give the globally stable dynamics for equation (1.10). However, we have the following result.

Theorem 3.16. *Let f be defined by equation (3.3) and $I=[-1,1]$. Then, 0 is a globally asymptotically stable equilibrium of equation (3.1) in $C([-1,0]\times\bar{\Omega},[-1,1])$.*

Proof. By simple computations, we obtain the following results:

(i) $f(u)<u$, for all $u\in(0,1]$;
(ii) $f(u)>u$, for all $u\in[-1,0)$ and
(iii) $|f(u)|<|u|$, for all $u\in(-1,1)\setminus\{0\}$.

This means that $f|_{(-1,1)}$ satisfies all the conditions in theorem 3.13(i). Hence, by theorem 3.13(i), we conclude that 0 is a globally asymptotically stable equilibrium in $C([-1,0]\times\bar{\Omega},[-1,1])\setminus\{-1,1\}$. Moreover, from the definition of f and U, we may deduce $U(1,\pm 1)\in C([-1,0]\times\bar{\Omega},[-1,1])\setminus\{-1,1\}$. Hence, 0 is a globally asymptotically stable equilibrium in $C([-1,0]\times\bar{\Omega},[-1,1])$. ■

The above example shows that the delay-independent globally asymptotical stability of a steady state u^* for equation (3.1) in general does not imply the global stability of u^* for equation (1.10); however, it does give some local information of equation (1.10) near u^* under the extra condition of differentiability, as is shown in the following proposition.

Proposition 3.17. *Let u^* be a constant steady state of equation (3.1) and f is differentiable in a neighborhood of u^*. If u^* is globally asymptotically stable (even locally stable) for equation (3.1) in $\{u^*\}\cup(C([-1,0]\times\bar{\Omega},I)\setminus\{\inf I,\sup I\})$ regardless of the value of $\mu>0$, then $|f'(u^*)|\leq 1$.*

Proof. We only need to show that $|f'(u^*)| > 1$ will contradict the delay-independent stability of u^* for equation (3.1). To this end, we consider the subclass of solutions to equation (3.1) governed by the following equation:

$$\left.\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} = -\mu u(t) + \mu f(u(t-1)), \quad t > 0 \\ \text{and} \quad &u(\theta) = \phi(\theta), \quad \theta \in [-1, 0], \end{aligned}\right\} \tag{3.4}$$

i.e. spatially independent solutions of equation (3.1). Obviously, instability of u^* for equation (3.4) implies instability of u^* for equation (3.1).

It is known that the stability/instability of u^* for equation (3.4) is determined by the characteristic equation

$$\lambda + \mu - \mu f'(u^*)\mathrm{e}^{-\lambda} = 0. \tag{3.5}$$

By the well-known result of Hayes (1950) (also see the appendix of Hale & Verduyn Lunel (1993)), one knows that if $f'(u^*) > 1$, then equation (3.5) has a positive real root; and if $f'(u^*) < -1$, then equation (3.5) will have complex roots with positive real parts when μ is large. That is, if $|f'(u^*)| > 1$, then u^* cannot be *delay-independently* asymptotically stable. ■

4. Applications

In this section, we apply the results obtained in §3 to the diffusive delay Nicholson blowflies equation and the diffusive Mackey–Glass haematopoiesis equation arising from population biology.

Example 4.1. Consider the Nicholson blowflies equation with diffusion

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \beta u(t-\tau,x)\mathrm{e}^{-au(t-\tau,x)}, \\ &\quad \text{for } (t,x) \in (0,+\infty) \times \Omega, \\ &\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0 \\ \text{and} \quad &u(\theta,x) = \phi(\theta,x), \quad \text{for } (\theta,x) \in [-1,0] \times \bar{\Omega}, \end{aligned}\right\} \tag{4.1}$$

where $a, d, \beta, \delta, \tau \in (0, +\infty)$ and $\phi \in X$.

Rescaling equation (4.1) by

$$\frac{t}{\tau} \to t, \quad \frac{x}{\sqrt{d\tau}} \to x, \quad \frac{1}{\sqrt{d\tau}}\Omega \to \Omega, \quad \tau\delta \to \mu \quad \text{and} \quad au(\tau t, \sqrt{d\tau}x) \to u(t,x), \tag{4.2}$$

equation (4.1) is transformed to the following system corresponding to equation (3.1):

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu[(\beta/\delta)u(t-1,x)\mathrm{e}^{-u(t-1,x)}],\\
&\quad \text{for } (t,x)\in(0,\infty)\times\Omega,\\
&\frac{\partial u}{\partial \nu}(t,x) = 0, \quad \text{on } \Gamma \equiv (0,\infty)\times\partial\Omega\\
\text{and}\qquad & u(\theta,x) = \phi(\theta,x), \quad \text{for } (\theta,x)\in[-1,0]\times\bar{\Omega}.
\end{aligned}\right\} \tag{4.3}$$

Applying the results in §3 to this model, we can reproduce the conclusions for this model mentioned in the introduction.

Theorem 4.2. *If* $\beta/\delta \in (0, e^2]$, *then the following statements are true:*

(i) *if* $\beta/\delta \le 1$, *then* 0 *is globally asymptotically stable in* X_+ *and*
(ii) *if* $\beta/\delta > 1$, *then* $\ln(\beta/\delta)$ *is globally asymptotically stable in* $X_+ \setminus \{0\}$.

Proof. Let $f: R_+ \longrightarrow R_+$ be defined by $f(u) = (\beta/\delta)u\mathrm{e}^{-u}$ for all $u \in R_+$.

Proof of (i). Suppose $\beta/\delta \le 1$ holds. If $u > 0$, then $|f(u)| < |u|$, that is $|f(u) - 0| < |u - 0|$. Thus, the assumption (H2) holds. By theorem 3.12, we conclude that 0 is globally asymptotically stable in X_+.

Proof of (ii). Suppose $\beta/\delta > 1$ holds. Then, by direct but careful use of calculus (also see remark 4.7 below), one can obtain

$$\{u \in (0, +\infty) : f^2(u) = u\} = \left\{\ln\frac{\beta}{\delta}\right\}. \tag{4.4}$$

Note that for any $\varepsilon \in (0, \ln(\beta/\delta))$, we can verify that $f([\varepsilon, +\infty)) \subseteq [\varepsilon, 1 + \beta/(\delta e)]$. Thus, theorem 3.13(iv) implies that $\ln(\beta/\delta)$ is globally asymptotically stable in $X_+ \setminus \{0\}$. ■

Remark 4.3. We have seen from (i)–(ii)–(iii)′–(iv) in §1 that the trivial solution $u = 0$ is globally asymptotically stable in X_+ regardless of μ if and only if $\beta/\delta \in (0, 1]$; and the positive steady state $u^* = \ln(\beta/\delta)$ exists and is globally asymptotically stable in $X_+ \setminus \{0\}$ regardless of μ if and only if $\beta/\delta \in (1, e^2]$. We can also obtain these conclusions by theorem 4.2 and proposition 3.17. Indeed, let $f(u) = (\beta/\delta)u\mathrm{e}^{-u}$, $u \in R_+$, with $\beta, \delta \in (0, \infty)$, simple computation leads to $f'(0) = \beta/\delta$ and $f'(\ln(\beta/\delta)) = 1 - \ln(\beta/\delta)$. Thus, $|f'(0)| \le 1$ if and only if $\beta/\delta \in (0, 1]$; $\ln(\beta/\delta) > 0$ and $|f'(\ln(\beta/\delta))| \le 1$ if and only if $\beta/\delta \in (1, e^2]$. In other words, our criteria for delay-independent global stability of a steady state of the equation for equation (4.1) are optimal.

Example 4.4. Consider the following scalar delay diffusive equation:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \mu u(t,x) + \mu\frac{pu(t-1,x)}{1 + (u(t-1,x))^n},\\
&\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\\
\text{and}\qquad & u(\theta,x) = \phi(\theta,x), \quad \text{for } (\theta,x)\in[-1,0]\times\bar{\Omega},
\end{aligned}\right\} \tag{4.5}$$

where p, μ and n are all positive constants.

Model (4.5) is a result of rescaling the diffusive version of the original model system proposed by Mackey & Glass (1977) to model blood-cell production. The original model without diffusion has been studied by many researchers. Among other topics is the stability of a positive equilibrium, justifying a long-term stable blood concentration level. See Kuang (1993), Tang & Zou (2003) and references therein. However, the diffusive version (4.5) has not been discussed yet, to our knowledge. Applying the results in §3 to this model, we obtain the following theorem.

Theorem 4.5. *We have the following statements:*

(i) *if* $p \le 1$, *then* 0 *is globally asymptotically stable in* X_+ *for all* $\mu > 0$ *and*
(ii) *if either* $(p > 1$ *and* $0 < n \le 2)$ *or* $(1 < p \le n/(n-2)$ *and* $n > 2)$, *then* $(p-1)^{1/n}$ *is globally asymptotically stable in* $X_+ \setminus \{0\}$ *for all* $\mu > 0$.

Proof. Define $f : R_+ \longrightarrow R_+$ by $f(u) = pu/(1+u^n)$ for all $u \in R_+$.

Proof of (i). Obviously, $|f(u)| = |f(u) - f(0)| < |u - 0|$ when $u > 0$. By theorem 3.12, we have 0 to be globally asymptotically stable in X_+.

Proof of (ii). By subtle use of calculus (also see remark 4.7 below), one can verify

$$\{u \in (0, +\infty) : f^2(u) = u\} = \{(p-1)^{1/n}\} \tag{4.6}$$

under the conditions of (ii) (see also remark 4.7 below). Thus, (H1) holds for equation (4.5). Note that for any $\varepsilon \in (0, (p-1)^{1/n})$, we have $f([\varepsilon, +\infty)) \subseteq [\varepsilon, p+1]$. Thus, theorem 3.13(iv) implies that $(p-1)^{1/n}$ is globally asymptotically stable in $X_+ \setminus \{0\}$. ■

Remark 4.6. Let $f : R_+ \longrightarrow R_+$ be defined by $f(u) = pu/(1+u^n)$ for all $u \in R_+$, where $p, n \in (0, +\infty)$. Then by a simple computation, we have $f'(0) = p$ and $f'((p-1)^{1/n}) = (n+p-np)/p$. Thus, $|f'(0)| \le 1$ if and only if $p \le 1$; and $((p-1)^{1/n} > 0$ and $|f'((p-1)^{1/n}))| \le 1)$ if and only if either $(p > 1$ and $0 < n \le 2)$ or $(1 < p \le n/(n-2)$ and $n > 2)$. So, by proposition 3.17 and theorem 4.5, we easily find that 0 ($(p-1)^{1/n}$, respectively) is a globally asymptotically stable equilibrium for equation (4.5) in X_+ ($X_+ \setminus \{0\}$, respectively), *regardless* of the value of $\mu > 0$ if and only if $p \le 1$ (either $(p > 1$ and $0 < n \le 2)$, or $(1 < p \le n/(n-2)$ and $n > 2)$, respectively). In other words, our criteria for global asymptotic stability of an equilibrium of equation (4.5), *regardless* of the value of $\mu > 0$ are also optimal and cannot be improved.

Remark 4.7. In the proofs of theorems 4.2 and 4.5, we have mentioned that careful and subtle use of calculus can verify equations (4.4) and (4.6). In fact, they can also be obtained by making use of some of the existing results on global stability of fixed points of maps, e.g. those in Gopalsamy *et al.* (1998), Györi & Trofimchuk (1999), Cull (2007) and Liz (2007). In particular, as one referee pointed out, since the nonlinearities in equations (4.3) and (4.5) have negative Schwarzian derivatives, those results obtained using Schwarzian derivatives are especially easy to verify (e.g. Singer 1978; Liz 2007; Liz & Röst in press; and references therein). For example, by theorem 2.3, proposition 2.5 and corollary 2.7 in Liz (2007), together with some straightforward calculations, equations (4.4) and (4.6) can be easily obtained.

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