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STABILITY AND BIFURCATION OF BIDIRECTIONAL ASSOCIATIVE MEMORY NEURAL NETWORKS WITH DELAYED SELF-FEEDBACK

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Some delay independent and delay dependent conditions are derived for the global stability of the bidirectional associative memory neural networks with delayed self-feedback. Regarding the self-connection delay as the parameter to be varied, the linear stability and Hopf bifurcation analysis are carried out. An algorithm to determine the direction and stability of the Hopf bifurcations is also worked out. Some examples and numerical simulations are presented.

Keywords: Bidirectional associative memory; delay; Hopf bifurcation; neural networks; stability.

1. Introduction

Bidirectional associate memorial (BAM) neural networks are a type of network with the neurons arrayed in two layers. Networks with such a bidirectional structure have practical applications in storing paired patterns or memories and possess the ability of searching the desired patterns via both directions: forward and backward. See [Gopalsamy & He, 1994a; Kosko, 1987, 1988, 1990; Mohamad, 2001] for details about the applications on learning and associative memories of neural networks. A BAM neural network can be described by the following system of ordinary differential equations (see e.g. [Kosko, 1987, 1988, 1990])

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + I_i \\ \dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + J_i. \end{cases} \tag{1}$$

where $i \in N(1,n) = \{1, 2, \ldots, n\}$, and a_{ij}, b_{ij}, $i, j \in N(1,n) := \{1, 2, \ldots, n\}$ are the connection weights between the neurons in two layers: the I-layer and the J-layer. On the I-layer, the neurons with states denoted by $x_i(t)$ receive the inputs I_i from outside and the inputs outputted from those neurons in the J-layer via activation functions (input–output functions) f_i; while on the J-layer, the neurons whose associated states denoted by $y_i(t)$ receive the exterior inputs J_i and the inputs outputted from those neurons in the I-layer via activation functions g_i.

Realizing the ubiquitous existence of delay in neural networks (accounting, e.g. for the finite speed of switch and the transmission delay in networks), Gopalsamy and He [1994a] incorporated time delays into the model and considered the following system of delay differential equations

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t - \tau_{ij})) + I_i \\ \dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - r_{ij})) + J_i. \end{cases} \tag{2}$$

A diagonally dominant and delay independent criterion for global stability of (2) was established in [Gopalsamy & He, 1994a]. Recently, Mohamad [2001] addressed the exponential stability of (2). More recently, Wang and Zou [2004] considered a special case of (2) when all delays in each layer are identical, i.e. $\tau_{ij} = \tau$ and $r_{ij} = r$, and performed local stability and Hopf bifurcation analysis for (2) by using $\mu = \tau + r$ as the bifurcation parameter.

On the other hand, recent work [van den Driessche *et al.*, 2001; van den Driessche & Zou, 1998; Wang & Zou, 2001] has shown that inhibitory self-connections play a role in stabilizing a network under some conditions on delays. This motivates us to incorporate inhibitory self-connections terms into the model system (2), and consider the following system

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + c_{ii} g_i(x_i(t - d_{ii})) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} a_{ij} f_j(y_j(t - \tau_{ij})) + I_i \\ \dot{y}_i(t) = -y_i(t) + l_{ii} f_i(y_i(t - m_{ii})) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{n} b_{ij} g_j(x_j(t - r_{ij})) + J_i. \end{cases} \tag{3}$$

We point out that although system (3) can be mathematically regarded as a Hopfield type neural network, which was extensively investigated recently (see, for example, [Bélair, 1993; Cao & Wu, 1996; Feng & Plamondon, 2000; Forti, 1994; Gopalsamy & He, 1994b; Guan *et al.*, 2000; Hopfield, 1984; Lu, 2000; Marcus & Westervelt, 1989; Matsuoka, 1992; Tank & Hopfield, 1986; van den Driessche & Zou, 1998; Wu, 1999]), with dimension $2n$, we will retain the model (3) as it stands since we do not want to alter the bidirectional interplay of the input–output nature of the two layers. We also point out that with the presence of multiple delays, the characteristic equations of the system usually become very complicated and this would make the stability and bifurcation analysis extremely hard and challenging. Due to such a complexity, no work has been accomplished for bifurcation analysis of *general* Hopfield type neural networks *with multiple delays*. However, attempts have been made in some special cases, among which are the following (in addition to [Wang & Zou, 2004] mentioned above): Chen *et al.* [2000], Wu [1999], Wu and Zou [1995] studied networks with a single delay case and some symmetric connection structure; for planar systems, i.e. the networks with two neurons, we refer to [Chen *et al.*, 2000; Chen & Wu, 2001a, 2001b; Faria, 2000; Olien & Bélair, 1997; Ruan & Wei, 1999; Wei & Ruan, 1999] for related bifurcation analysis, in which it was assumed that either two delays are equal or there are two different delays but there is no self-connection (this is the case of $n = 1$ in (2)); for networks with ring structure and one or two distinct delays, Campbell [1999], Campbell *et al.* [1999], Ncube *et al.* [2003], Shayer and Campbell [2001] obtained some detailed results for occurrence of the bifurcations. The mentioned papers have already shown that networks with delays can demonstrate very rich and interesting dynamics.

In this paper, first, we give two sets of delay independent conditions for global stability of system (3) in Sec. 2. For clarity of presentation, the proofs are deferred to Sec. 6. Some delay dependent conditions are also obtained in Sec. 2, which demonstrate the stabilization role of the inhibitory self-connections. Then, in Sec. 3, we analyze the local stability of (3) and thereby, study the Hopf bifurcation. As in [Wang & Zou, 2004] we also assume identical delay in each layer and take advantage of the connection structure of the BAM networks (3) and the Schur complementary theorem. Yet, unlike in [Wang & Zou, 2004] where self-connections are absent, here we assume that all delays in the self-connections are identical and use this identical delay as the bifurcation parameter. In Sec. 4, we give an algorithm for computing the direction and stability of the Hopf bifurcation, which are obtained by applying the theory in [Hassard *et al.*, 1981] or [Kazarinoff *et al.*, 1978]. In Sec. 5, we give some examples to show the feasibility of our results; in particular, we show that our results can be applied to networks of even number of neurons with ring structure. Some numerical simulations are also given in this section.

2. Global Stability of (3)

We assume that (3) has at least one equilibrium. Indeed, by using the Brouwer fixed point theorem [Deimling, 1985], we can establish

Lemma 2.1. *Suppose the activation functions f_i, g_i, $i \in N(1, n)$ are continuous and bounded, then (3) has at least one equilibrium.*

Hence we can always perform a transformation such that the origin is the equilibrium of the new system. Therefore, without loss of generality, in what follows, we assume that $I_i = J_i = 0$ and $f_i(0) = g_i(0) = 0$ for $i \in N(1,n)$. Then (3) reduces to

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + c_{ii}g_i(x_i(t-d_{ii})) \\ \qquad\qquad + \sum_{j=1}^{n} a_{ij}f_j(y_j(t-\tau_{ij})) \\ \dot{y}_i(t) = -y_i(t) + l_{ii}f_i(y_i(t-m_{ii})) \\ \qquad\qquad + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-r_{ij})). \end{cases} \tag{4}$$

In this section, we assume that all the activation functions $f_i,\ g_i,\ i \in N(1,n)$ are Lipschitz continuous with Lipschitz constants $\mathrm{Lip}(f_i),\ \mathrm{Lip}(g_i),\ i \in N(1,n)$.

Theorem 2.1. *Assume that there exist some $p_i > 0$, $q_i > 0,\ i \in N(1,n)$ such that*

$$\begin{cases} |c_{ii}|\mathrm{Lip}(g_i)p_i + \mathrm{Lip}(g_i)\sum_{j=1}^{n}|b_{ji}|q_j < p_i \\ |l_{ii}|\mathrm{Lip}(f_i)q_i + \mathrm{Lip}(f_i)\sum_{j=1}^{n}|a_{ji}|p_j < q_i. \end{cases} \tag{5}$$

Then the zero solution of (4) is globally exponentially stable.

Theorem 2.2. *Assume that there are some real positive numbers $p_i,\ q_i,\ \xi,\ \eta_i,\ i \in N(1,n)$ such that*

$$\begin{cases} p_i\left(|c_{ii}| + |c_{ii}|\mathrm{Lip}^2(g_i) + \sum_{j=1}^{n}|a_{ij}|\xi_j\right) \\ \qquad + \frac{\mathrm{Lip}^2(g_i)}{\eta_i}\sum_{j=1}^{n}|b_{ji}|q_j < 2p_i, \\ q_i\left(|l_{ii}| + |l_{ii}|\mathrm{Lip}^2(f_i) + \sum_{j=1}^{n}|b_{ij}|\eta_j\right) \\ \qquad + \frac{\mathrm{Lip}^2(f_i)}{\xi_i}\sum_{j=1}^{n}|a_{ji}|p_j < 2q_i \end{cases} \tag{6}$$

holds. Then system (4) is globally asymptotically stable.

In the above two theorems, the signs of c_{ii} and $l_{ii},\ i \in N(1,n)$, do not play a role in conditions (5) and (6). In order to identify the stabilization role of inhibitory self-connections (i.e. $c_{ii} < 0$ and $l_{ii} < 0$), we first consider the case when all delays are absent.

Lemma 2.2. *Assume that there are some positive real numbers $p_i,\ q_i$ such that*

$$\begin{cases} p_ic_{ii} + \frac{1}{2}\sum_{j=1}^{n}(p_i|a_{ij}| + q_j|b_{ji}|) < \frac{p_i}{\mathrm{Lip}(g_i)} \\ q_il_{ii} + \frac{1}{2}\sum_{j=1}^{n}(q_i|b_{ij}| + p_j|a_{ji}|) < \frac{q_i}{\mathrm{Lip}(f_i)} \end{cases} \tag{7}$$

hold for $i \in N(1,n)$. Then system (4) admits a unique equilibrium which is globally asymptotically stable if no delay is present.

Proof. This lemma can be proved by using the following Liapunov function

$$V(t) = \sum_{i=1}^{n}\left(p_i\int_0^{x_i} g_i(s)ds + q_i\int_0^{y_i} f_i(s)ds\right). \tag{8}$$

■

Taking

$$V(t) = \sum_{i=1}^{n}(p_i|x_i(t)| + q_i|y_i(t)|), \tag{9}$$

we may establish

Lemma 2.3. *Assume that there are some positive constants p_i, q_i such that*

$$\begin{cases} p_ic_{ii} + \sum_{j=1}^{n}q_j|b_{ji}| < \frac{p_i}{\mathrm{Lip}(g_i)} \\ q_il_{ii} + \sum_{j=1}^{n}p_j|a_{ji}| < \frac{q_i}{\mathrm{Lip}(f_i)} \end{cases} \tag{10}$$

hold for $i \in N(1,n)$ and no delay is present in (4). Then system (4) admits a unique equilibrium which is globally exponentially stable.

As in [van den Driessche *et al.*, 2001; van den Driessche & Zou, 1998], we may expect that if (7) or (10) holds, then the global stability remains true when the self-delays (i.e. d_{ii} and $m_{ii},\ i \in N(1,n)$) are sufficiently small in (4).

Applying the delay dependent stability results established in [Wang, 2004] to (4), we have

Theorem 2.3. *Assume that the activation functions $f_i,\ g_i,\ i \in N(1,n)$ are nondecreasing and*

Lipschitz continuous and the delays d_{ii}, m_{ii} corresponding to $c_{ii} < 0$, $l_{ii} < 0$ for $i \in N(1,n)$ satisfy

$$d_{ii} \le \frac{1}{d^*}, \quad m_{ii} \le \frac{1}{m^*}, \tag{11}$$

where d^* and m^* are the unique positive roots of equations

$$\begin{aligned} 1 + \frac{1}{d} - \ln\frac{d}{|c_{ii}|\mathrm{Lip}(g_i)} = 0, \\ 1 + \frac{1}{m} - \ln\frac{m}{|l_{ii}|\mathrm{Lip}(f_i)} = 0, \end{aligned} \tag{12}$$

respectively. If for some positive constants $p_i, q_i, i \in N(1,n)$, either (7) or (10) holds, then the trivial solution of system (4) is globally attractive.

Remark 2.1. Note that in the above theorem, conditions (7) and (10) are much weaker than conditions (5) and (6) in Theorems 2.1 and 2.2. This allows us to easily design a stable BAM network with negative self-feedback for real applications.

3. Linear Stability and Hopf Bifurcation

Note that in Theorem 2.3, under condition (7) or (10), only restrictions on the self-delays d_{ii} and m_{ii} are imposed, allowing the other delays τ_{ij} and r_{ij} as arbitrary. This suggests that the self-delays may destroy the stability of the network when they are increased to some level. In this section, we will confirm the destabilization of the network by the self-delays via local stability and Hopf bifurcation analysis. Throughout this section, we assume that all the activation functions are sufficiently smooth and we will explore the impact of increasing the self-delays. For simplicity, we assume, from now on, that

$$c_{ii} = l_{ii} = \beta; \quad d_{ii} = m_{ii} = \sigma; \quad \tau_{ij} = \tau_1; \quad r_{ij} = \tau_2, \quad i, j \in N(1,n).$$

The linearization of (3) at the origin is

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \beta x_i(t-\sigma) + \displaystyle\sum_{j=1}^{n} a_{ij} y_j(t-\tau_1) \\ \dot{y}_i(t) = -y_i(t) + \beta y_i(t-\sigma) + \displaystyle\sum_{j=1}^{n} b_{ij} x_j(t-\tau_2) \end{cases} \tag{13}$$

Denote the $n \times n$ identity matrix by E_n, $A = (a_{ij})$, $B = (b_{ij})$ and $\tau = (\tau_1 + \tau_2)/2$ and let

$$W = \begin{pmatrix} (z + 1 - \beta e^{-z\sigma})E_n & -e^{-z\tau_1}A \\ -e^{-z\tau_2}B & (z + 1 - \beta e^{-z\sigma})E_n \end{pmatrix}$$

and

$$W^* = \begin{pmatrix} -e^{-z\tau_2}B & (z + 1 - \beta e^{-z\sigma})E_n \\ (z + 1 - \beta e^{-z\sigma})E_n & -e^{-z\tau_1}A \end{pmatrix}.$$

Then the associated characteristic equation of (13) is given by

$$\det W = 0. \tag{14}$$

Note that

$$\det W = (-1)^n \det W^*.$$

In what follows, we assume that

$$\det B \ne 0, \tag{15}$$

which implies that $e^{-z\tau_2}B$ is nonsingular. Then from Theorem 1.23 of [Fielder, 1986] we have

$$\det W^* = \det(e^{-z\tau_2}B)\det\left[\frac{W^*}{e^{-z\tau_2}B}\right],$$

where $[W^*/e^{-z\tau_2}B]$ is the Schur complement of the block $e^{-z\tau_2}B$ in W^* (see, e.g. [Fielder, 1986]). Therefore, (14) is equivalent to

$$\det[(z + 1 - \beta e^{-z\sigma})^2 E_n - e^{-2z\tau}BA] = 0. \tag{16}$$

It is easily seen that z is a solution of (16) if and only if there is a $\lambda \in \sigma(BA)$ such that

$$(z + 1 - \beta e^{-z\sigma})^2 - \lambda e^{-2z\tau} = 0. \tag{17}$$

Hence, if λ_j, $j \in N(1,n)$ are eigenvalues of BA, then (14) is equivalent to n scalar equations

$$(z + 1 - \beta e^{-z\sigma})^2 - \lambda_j e^{-2z\tau} = 0, \quad j \in N(1,n). \tag{18}$$

For any $\lambda_j \in \sigma(BA)$, $j \in N(1,n)$, we can write it as

$$\lambda_j = |\lambda_j| e^{i\theta_j}, \quad \theta_j \in [0, 2\pi),$$

and then (18) is equivalent to

$$z + 1 - \beta e^{-z\sigma} \pm \sqrt{|\lambda_j|} e^{-z\tau} e^{i\frac{\theta_j}{2}} = 0. \tag{19}$$

Though the left-hand side of (19) is a polynomial with a principal term, it is very difficult to apply Bellman and Cook's general result, namely, Theorem 13.7 of [Bellman & Cooke, 1963], to get sufficient conditions ensuring all zeros of (19) have negative real parts. In the following, we will use a quite straightforward method to derive some conditions obtaining the linear stability of system (3).

Let $z = \mu + i\omega$, then (19) is equivalent to

$$\begin{cases} R(\mu,\omega) := \mu + 1 - \beta e^{-\mu\sigma}\cos(\omega\sigma) \\ \qquad\qquad \pm \sqrt{|\lambda_j|}e^{-\mu\tau}\cos\left(\omega\tau - \dfrac{\theta_j}{2}\right) = 0 \\ I(\mu,\omega) := \omega + \beta e^{-\mu\sigma}\sin(\omega\sigma) \\ \qquad\qquad \mp \sqrt{|\lambda_j|}e^{-\mu\tau}\sin\left(\omega\tau - \dfrac{\theta_j}{2}\right) = 0. \end{cases}$$

Noticing that

$$R(\mu,\omega) \geq 1 - |\beta| - \sqrt{|\lambda_j|},$$
$$\text{for all } \mu \geq 0,\ \sigma \geq 0,\ \tau \geq 0,$$

we immediately have

Theorem 3.1. *Assume that (15) holds. If*

$$\sqrt{|\lambda|} + |\beta| < 1, \quad \sigma \geq 0, \quad \tau \geq 0, \tag{20}$$

where

$$|\lambda| := \max\{|\lambda_j|,\ \lambda_j \in \sigma(BA)\},$$

then all roots of (18) have negative real parts, and hence the trivial solution of (3) is asymptotically stable.

From $R(\mu,\omega) = 0$ and $I(\mu,\omega) = 0$, we obtain

$$\mu = -1 + \beta e^{-\mu\sigma}\cos(\omega\sigma) \mp \sqrt{|\lambda_j|}e^{-\mu\tau}\cos\left(\omega\tau - \frac{\theta_j}{2}\right), \tag{21}$$

$$\omega = -\beta e^{-\mu\sigma}\sin(\omega\sigma) \pm \sqrt{|\lambda_j|}e^{-\mu\tau}\sin\left(\omega\tau - \frac{\theta_j}{2}\right). \tag{22}$$

and hence,

$$(\mu + 1 - \beta e^{-\mu\sigma}\cos(\omega\sigma))^2 + (\omega + \beta e^{-\mu\sigma}\sin(\omega\sigma))^2 = |\lambda_j|e^{-2\mu\tau},$$

or

$$(\mu+1)^2 + \omega^2 - 2\beta e^{-\mu\sigma}[(\mu+1)\cos(\omega\sigma) - \omega\sin(\omega\sigma)] + \beta^2 e^{-2\mu\sigma} - |\lambda_j|e^{-2\mu\tau} = 0. \tag{23}$$

If we assume that

$$\beta < 0, \quad \text{and} \quad \sqrt{|\lambda|} < -\beta, \quad \sigma \in \left[0, \frac{1}{-2\beta}\right], \tag{24}$$

then, it follows from (22) that

$$\omega < -2\beta \quad \text{for } \mu \geq 0,\ \tau \geq 0, \quad \text{and} \quad \omega\sigma \in [0,1].$$

Letting the left-hand side of (23) be $M(\mu)$, we then have

$$\begin{aligned} M(0) &= 1 + \omega^2 - 2\beta(\cos(\omega\sigma) - \omega\sin(\omega\sigma)) \\ &\quad + \beta^2 - |\lambda_j| \\ &= 1 + \beta^2 - |\lambda_j| - 2\beta\cos(\omega\sigma) + \omega^2 \\ &\quad + 2\beta\omega\sin(\omega\sigma) \\ &> \omega^2 + 2\beta\omega(\omega\sigma) \\ &= \omega^2(1 + 2\beta\sigma) \\ &\geq 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left.\frac{dM(\mu)}{d\mu}\right|_{\mu\geq 0} &= 2\{(\mu+1)[1 + \beta\sigma e^{-\mu\sigma}\cos(\omega\sigma)] \\ &\quad - \beta e^{-\mu\sigma}[\cos(\omega\sigma) + \beta\sigma e^{-\mu\sigma}] \\ &\quad + |\lambda_j|\tau e^{-2\mu\tau} - \beta\sigma\omega e^{-\mu\sigma}\sin(\omega\sigma)\} \\ &\geq 0. \end{aligned}$$

This shows that $M(\mu) > 0$ for all $\mu > 0$ and thus we have

Theorem 3.2. *If (15) and (24) hold, then for all $\tau \geq 0$, the trivial solution of (3) is asymptotically stable.*

In the following, we will regard σ as the parameter and try to find its critical value at which the bifurcation occurs.

Letting $\sigma = 0$ in (19), we have

$$R(\mu,\omega) = \mu + 1 - \beta \pm \sqrt{|\lambda_j|}e^{-\mu\tau}\cos(\omega\tau - \theta_j/2)$$

and hence

$$R(\mu,\omega) \geq 1 - \beta - \sqrt{|\lambda_j|} \quad \text{for all } \mu \geq 0,$$

which indicates

Lemma 3.1. *If*

$$\beta < 1 - \sqrt{|\lambda_j|}, \tag{25}$$

then all roots of (19) have negative real parts at $\sigma = 0$ for all $\tau \geq 0$.

Next we investigate if $\sigma > 0$ will destroy the stability. Theorem 3.1 and Lemma 3.1 suggest that in order to explore the possibility that $\sigma > 0$ destroys the stability, we need to assume that (25) and $|\beta| + \sqrt{\lambda_j} \geq 1$ hold, or equivalently,

$$\beta < -\left|1 - \sqrt{|\lambda_j|}\right|. \tag{26}$$

Under this assumption, we know for any fixed $\tau \geq 0$, all roots of (19) have negative real parts when $\sigma = 0$ and it is possible for some roots having non-negative real parts when $\sigma > 0$. It follows from [Campbell *et al.*, 1999] that the only way to achieve this is by way of crossing the imaginary axis.

Note that $z = 0$ cannot be a root of (19) due to (26) and $z = i\omega$ with $\omega > 0$ is a root of (19) if and only if

$$\begin{cases} \beta\cos(\omega\sigma) = 1 \pm \sqrt{|\lambda_j|}\cos\left(\omega\tau - \dfrac{\theta_j}{2}\right) \\ \beta\sin(\omega\sigma) = -\omega \pm \sqrt{|\lambda_j|}\sin\left(\omega\tau - \dfrac{\theta_j}{2}\right). \end{cases} \tag{27}$$

From which we have

$$\beta^2 = 1 + |\lambda_j| + \omega^2 \pm 2\sqrt{|\lambda_j|} \times \left(\cos\left(\omega\tau - \frac{\theta_j}{2}\right) - \omega\sin\left(\omega\tau - \frac{\theta_j}{2}\right)\right). \tag{28}$$

Equation (28) can have either finitely many or no root for $\omega > 0$. In the case of finitely many roots, we denote them by $\omega_l^{\pm}(\lambda_j)$, $l = 1, 2, \ldots, m$. It follows from (27) that

$$\sigma = \frac{1}{\omega_l^{\pm}(\lambda_j)} \times \left(\arccos\frac{1 \pm \sqrt{|\lambda_j|}\cos\left(\omega_l^{\pm}(\lambda_j)\tau - \dfrac{\theta_j}{2}\right)}{\beta} + 2k\pi\right) =: \sigma_{l,j}^{\pm}(k), \tag{29}$$

where $k \in N(0) = \mathbb{N}$. In the case where (28) has no root, we denote the corresponding $\sigma_{l,j}^{\pm}(0) = \infty$. The above analysis and a direct calculation give

Lemma 3.2. *Assume that (26) holds. Then*

(i) *all roots of (19) have negative real parts for any fixed $\tau \geq 0$ and for*

$$\sigma \in [0, \sigma(\lambda_j)); \tag{30}$$

(ii) *Eq. (19) has a pair of simple purely imaginary roots and all other roots have negative real parts at $\sigma = \sigma(\lambda_j)$;*

(iii) *at least one root of (19) has positive real part if*

$$\sigma > \sigma(\lambda_j). \tag{31}$$

Here $\sigma(\lambda_j) := \min\{\sigma_{l,j}^{+}(0), \sigma_{l,j}^{-}(0), l \in N(1,m)\}$. Moreover,

$$\left.\frac{d\,\mathrm{Re}(z)}{d\sigma}\right|_{z=i\omega} \neq 0$$

if and only if

$$\tau \neq \tau^{\pm}(\lambda_j, \omega),$$

where $\tau^{\pm}(\lambda_j, \omega)$ is the solution of

$$\omega\left(1 \mp \tau\sqrt{|\lambda_j|}\cos\left(\omega\tau - \frac{\theta_j}{2}\right)\right) = \pm\sqrt{|\lambda_j|}(1 + \tau)\sin\left(\omega\tau - \frac{\theta_j}{2}\right), \tag{32}$$

and in the case that (32) has no solution, we denote $\tau^{\pm}(\lambda_j, \omega) = \infty$.

Let

$$\sigma^* = \min\{\sigma(\lambda_j), j \in N(1,n)\} = \sigma(\lambda_{j_0}), \quad \text{for some } j_0 \in N(1,n).$$

Then σ^* is the first critical value at which Hopf bifurcation possibly occurs. Corresponding to such value, we denote $i\omega$ by $i\omega_0$, λ_{j_0} by λ_0, and σ by σ_0. Summarizing the above analysis and applying the standard Hopf bifurcation Theorem in [Hassard *et al.*, 1981], we have

Theorem 3.3. *Assume that (15) holds. Let $|\lambda| = \max\{|\lambda_j| : j \in N(1,n)\}$.*

(I) *If*

$$\beta < 1 - \sqrt{|\lambda|}, \tag{33}$$

then the trivial solution of (3) is asymptotically stable at $\sigma = 0$ for all $\tau \geq 0$;

(II) *If*

$$\beta < -|1 - \sqrt{|\lambda|}|, \tag{34}$$

then the trivial solution of (3) is asymptotically stable for $\sigma \in [0, \sigma_0)$ and unstable if $\sigma > \sigma_0$.

(III) *Hopf bifurcation occurs at $\sigma = \sigma_0$ provided*

$$m(\lambda_0) = 1, \quad \tau \neq \tau^{\pm}(\lambda_0, \omega_0),$$

where $m(\lambda_0)$ is the multiplicity of λ_0 being an eigenvalue of the matrix BA.

4. Direction and Stability of the Hopf Bifurcation at $\tau = 0$

The direction and stability of the Hopf bifurcation established in the previous section is not easy to confirm and thus in this section, we will focus on a special case: $\tau = 0$, and give the Hopf bifurcation theorem and an algorithm for direction and stability. Note that (3) is now reduced to

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \beta s_{1,i}(x_i(t-\sigma)) \\ \qquad + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) \\ \dot{y}_i(t) = -y_i(t) + \beta s_{2,i}(y_i(t-\sigma)) \\ \qquad + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) \end{cases} \tag{35}$$

Let

$$\omega_j^{\pm} = \sqrt{\beta^2 - \left(1 \pm \sqrt{|\lambda_j|}\cos\frac{\theta_j}{2}\right)^2} \mp \sqrt{|\lambda_j|}\sin\frac{\theta_j}{2}$$

and

$$\sigma_j^{\pm}(0) = \frac{1}{\omega_j^{\pm}} \arccos \frac{1 \pm \sqrt{|\lambda_j|}\cos\frac{\theta_j}{2}}{\beta}.$$

If $\omega_j^{\pm} \notin \mathbb{R}^+$, we denote the corresponding $\sigma_j^{\pm}(0) = \infty$, where $\lambda_j = |\lambda_j|e^{i\theta_j}$ is the jth eigenvalue of BA and $j = 1, 2, \ldots, n$. Let $\sigma(\lambda_j) = \min\{\sigma_j^+(0), \sigma_j^-(0)\}$ and

$$\begin{aligned} \sigma_0 &= \min\{\sigma(\lambda_j), j \in N(1,n)\} \\ &= \sigma(\lambda_{j_0}) \quad \text{for some } j_0 \in N(1,n). \end{aligned}$$

For this special case, one can easily show that the condition $\tau \neq \tau^{\pm}(\lambda_j, \omega)$ holds. Thus, Theorem 3.3 reads in this case as following

Theorem 4.1. *Assume that* (*15*) *holds.*

(i) *If*

$$\beta < 1 - \sqrt{|\lambda|}, \tag{36}$$

then the trivial solution of (*35*) *is asymptotically stable at* $\sigma = 0$.

(ii) *If*

$$\beta < -|1 - \sqrt{|\lambda|}|, \tag{37}$$

then the trivial solution of (*35*) *is asymptotically stable for* $\sigma \in [0, \sigma_0)$ *and unstable if* $\sigma > \sigma_0$.

(iii) *Hopf bifurcation occurs at* $\sigma = \sigma_0$ *provided* $m(\lambda_0) = 1$.

We assume for simplicity that the activation functions in (35) satisfy

$$f_i''(0) = g_i''(0) = s_{1,i}''(0) = s_{2,i}''(0) = 0, \quad \text{for } i \in N(1,n).$$

Prototype of such functions includes $\tanh(x)$ and $\arctan(x)$ which have been widely used as activation functions in neural networks. Then the Taylor expansion of (35) at zero has the form

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \beta x_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} y_j(t) \\ \qquad + \gamma_i x_i^3(t-\sigma) + \sum_{j=1}^{n} a_{ij}^* y_j^3(t) + \text{h.o.t.} \\ \dot{y}_i(t) = -y_i(t) + \beta y_i(t-\sigma) + \sum_{j=1}^{n} b_{ij} x_j(t) \\ \qquad + \alpha_i y_i^3(t-\sigma) + \sum_{j=1}^{n} b_{ij}^* x_j^3(t) + \text{h.o.t.} \end{cases} \tag{38}$$

where h.o.t. stands for the high order terms, $\gamma_i = \beta s_{1,i}'''(0)/6$, $\alpha_i = \beta s_{2,i}'''(0)/6$, $a_{ij}^* = a_{ij} f_j'''(0)/6$, $b_{ij}^* = b_{ij} g_j'''(0)/6$, $i, j \in N(1,n)$. Let $\sigma = \sigma_0 + \mu$, then Theorem 4.1 implies that Hopf bifurcation occurs at $\mu = 0$. By using the general theory developed in [Hassard *et al.*, 1981], we can derive a specific algorithm to determine the direction and stability of such Hopf bifurcation as below. Note that a direct calculation shows that

$$\left.\frac{d\,\mathrm{Re}(z)}{d\sigma}\right|_{z=i\omega_0} > 0.$$

Our algorithm is given as follows:

Algorithm

1. Put $\alpha_0 := 1 + i\omega_0 - \beta e^{-i\omega_0\sigma_0}$;
2. Find an eigenvector $Q = (q_1, q_2, \ldots, q_n)^T$ for matrix BA corresponding to its eigenvalue λ_0, i.e.

$$(\lambda_0 E_n - BA)Q = 0;$$

3. Let

$$P = \alpha_0 B^{-1} Q, \quad P^* = \overline{\alpha}_0 B^{-1} Q,$$

where $P = (p_1, p_2, \ldots, p_n)^T$, $P^* = (p_1^*, p_2^*, \ldots, p_n^*)^T$, $\overline{\alpha}_0$ is the conjugate of α_0;

4. Compute D, which is defined by

$$D = \frac{1}{(1+\beta\sigma_0 e^{-i\omega_0\sigma_0})\sum_{j=1}^{n}(\overline{q}_j p_j + \overline{p}_j^* q_j)},$$

5. Let

$$C_1(0) = 3D\left\{\sum_{j=1}^{n} q_j\left(\gamma_j|p_j|^2 p_j e^{-i\omega_0\sigma_0} + \sum_{k=1}^{n} a_{jk}^*|q_k|^2 q_k\right) + \sum_{j=1}^{n} p_j^*\left(\alpha_j|q_j|^2 q_j e^{-i\omega_0\sigma_0} + \sum_{k=1}^{n} b_{jk}^*|p_k|^2 p_k\right)\right\},$$

6. Let

$$\mu_2 = -\mathrm{Re}(C_1(0)).$$

Then we have

Theorem 4.2. *If $\mu_2 > 0$ (< 0), then the Hopf bifurcation of (35) at $\sigma = \sigma_0$ is supercritical (subcritical) and the periodic solutions of (35) bifurcating from Hopf bifurcation value are asymptotically orbitally stable (unstable).*

The derivation of the above algorithm is standard but tedious (using the results in [Hassard *et al.*, 1981]), so we omit the details here.

5. Some Examples and Numerical Simulations

Example 5.1. Consider the BAM neural model with three delays

Consider the following BAM neural network with two neurons on each layer

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \beta f(x_1(t-\sigma)) + a_{11}f(y_1(t-\tau_1)) \\ \qquad + a_{12}f(y_2(t-\tau_1)) \\ \dot{x}_2(t) = -x_2(t) + \beta f(x_2(t-\sigma)) + a_{21}f(y_1(t-\tau_1)) \\ \qquad + a_{22}f(y_2(t-\tau_1)) \\ \dot{y}_1(t) = -y_1(t) + \beta f(y_1(t-\sigma)) + b_{11}f(x_1(t-\tau_2)) \\ \qquad + b_{12}f(x_2(t-\tau_2)) \\ \dot{y}_2(t) = -y_2(t) + \beta f(y_2(t-\sigma)) + b_{21}f(x_1(t-\tau_2)) \\ \qquad + b_{22}f(x_2(t-\tau_2)) \end{cases} \tag{39}$$

where $f(x) = \tanh x$.

Corollary 5.1. *If*

$$\begin{aligned} |\beta|q_1 + |a_{11}|p_1 + |a_{21}|p_2 < q_1, \\ |\beta|q_2 + |a_{12}|p_1 + |a_{22}|p_2 < q_2 \end{aligned} \tag{40}$$

and

$$\begin{aligned} |\beta|p_1 + |b_{11}|q_1 + |b_{21}|q_2 < p_1, \\ |\beta|p_2 + |b_{12}|q_1 + |b_{22}|q_2 < p_2 \end{aligned} \tag{41}$$

hold for some positive p_i, $q_i, i = 1, 2$, then the zero solution of (39) is globally asymptotically stable for all $\sigma \geq 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

Take $\beta = -2$, and

$$A = \begin{pmatrix} 1.0 & -1.0 \\ -1.0 & 1.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.8 & 1.0 \\ 1.0 & -2.0 \end{pmatrix}.$$

Then from Theorem 2.3 we know that the zero solution of (39) is globally attractive for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ provided $\sigma \leq 0.1572$. The eigenvalues of matrix BA are: $\lambda_1 = 0.1391$, $\lambda_2 = -3.7391$. If $\tau = (\tau_1+\tau_2)/2 = 0$, then a direct calculation gives $\sigma_0 = 0.5598$ with the associated $\lambda_0 = 0.1391$, $\omega_0 = 1.4543$. This shows that the zero solution of (39) is asymptotically stable when $\sigma \in [0, 0.5598), \tau_1 = \tau_2 = 0$ and local periodic solutions appear via Hopf bifurcation near $\sigma = 0.5598$. The numerical simulations are shown in Figs. 1 and 2. If $\tau_1 + \tau_2 = 0.02$, we can compute that $\sigma_0 = 0.5544$ and the

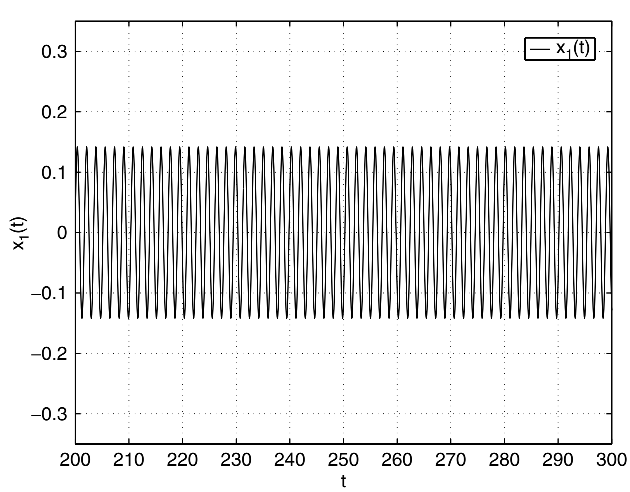

Fig. 1. Hopf bifurcation occurs when σ is near the critical value σ_0, here we use $\tau = 0$, $\sigma = 0.58$ and just give the first component $x_1(t)$ versus t. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

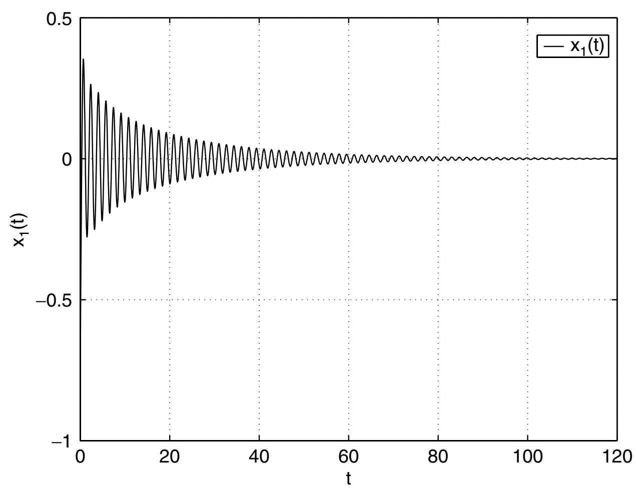

Fig. 2. Locally stable solution of (39) is obtained when $\sigma < \sigma_0$, here $\tau_1 = \tau_2 = 0$, $\sigma = 0.55$ and $x_1(t)$ versus t is shown. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

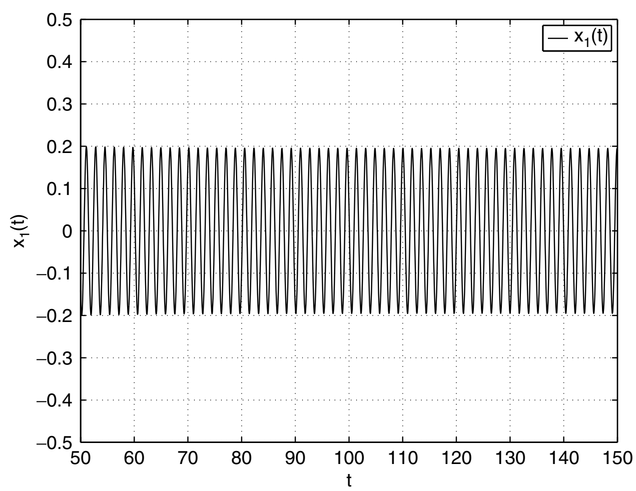

Fig. 3. Long time behavior of solution of (39) which bifurcates from the zero solution when σ is near the critical value σ_0, here we use $\tau_1 = 0.008$, $\tau_2 = 0.012$, $\sigma = 0.57$. The component $x_1(t)$ is shown here and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

associated $\lambda_0 = -3.7391$, $\omega_0 = 3.7038$. This implies that in this case the zero solution of (39) is asymptotically stable when $\sigma \in [0, 0.5544)$ and Hopf bifurcation occurs around $\sigma = 0.5544$. The numerical simulations, are given in Figs. 3 and 4. We acknowledge that all numerical simulations presented here were performed by the DDE23 Solver developed by Shampine and Thompson [2001].

Example 5.2. Ring structured neural network models.

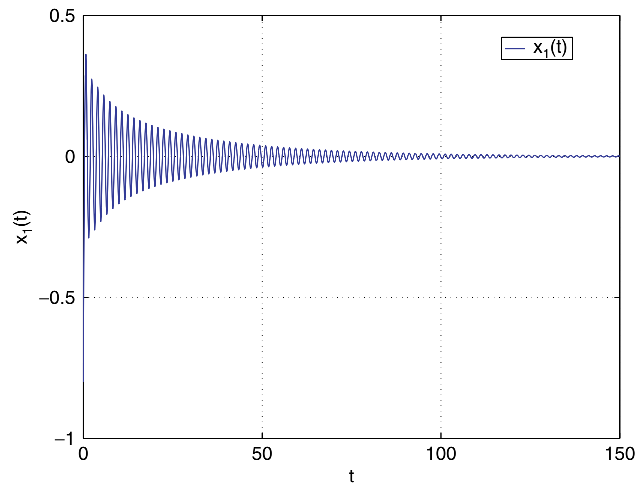

Fig. 4. The zero solution of (39) is locally stable when $\sigma < \sigma_0$, here $\tau_1 = 0.008$, $\tau_2 = 0.012$, $\sigma = 0.54$. The behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

A general neural network model with a special connection architecture, i.e. ring structure, was investigated by Campbell [1999]. A simplified such model takes the form

$$\dot{u}_j(t) = -u_j(t) + s_j(u_j(t-\sigma)) + h_j(u_{j-1}(t-\tau)), \tag{42}$$

where $j = 1, 2, \ldots, k$ and $u_0 = u_k$. In the case where $k = 4$, the local stability and Hopf bifurcation of (42) was discussed in [Campbell *et al.*, 1999]. Note that we can topologically regard (42) as a simple BAM model when the number of neurons k is an even number. For example, a ring of six neurons shown in Fig. 5 can be reorganized as a BAM neural model with $n = 3$ shown in Fig. 6.

For a general even number, $k = 2n$. Let

$$x_j(t) = u_{2j-1}(t), \quad y_j(t) = u_{2j}(t), \quad j = 1, 2, \ldots, n.$$

Then we can rewrite (42) as

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + s_{2j-1}(x_j(t-\sigma)) \\ \qquad\qquad + h_{2j-1}(y_{j-1}(t-\tau)) \\ \dot{y}_j(t) = -y_j(t) + s_{2j}(y_j(t-\sigma)) + h_{2j}(x_j(t-\tau)), \end{cases} \tag{43}$$

where $y_0(t) = y_n(t)$. For convenience, we may further rewrite (42) as

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + s_{1,j}(x_j(t-\sigma)) \\ \qquad\qquad + f_{j-1}(y_{j-1}(t-\tau)) \\ \dot{y}_j(t) = -y_j(t) + s_{2,j}(y_j(t-\sigma)) + g_j(x_j(t-\tau)), \end{cases} \tag{44}$$

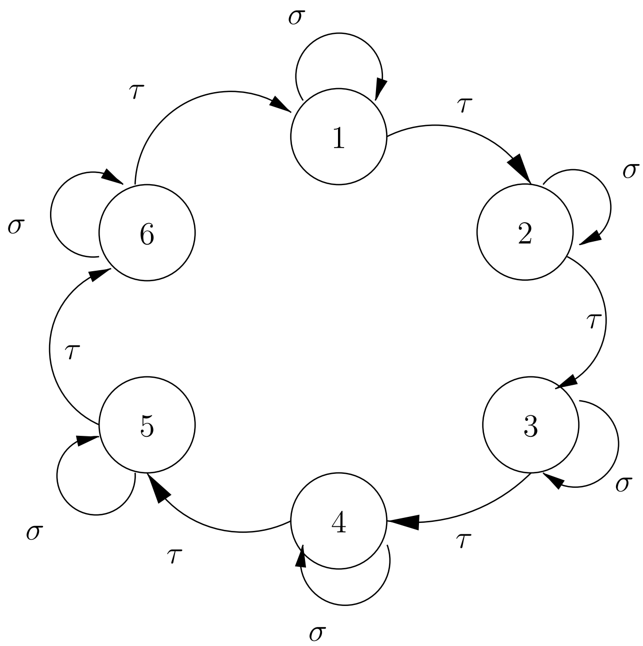

Fig. 5. A ring of six neurons.

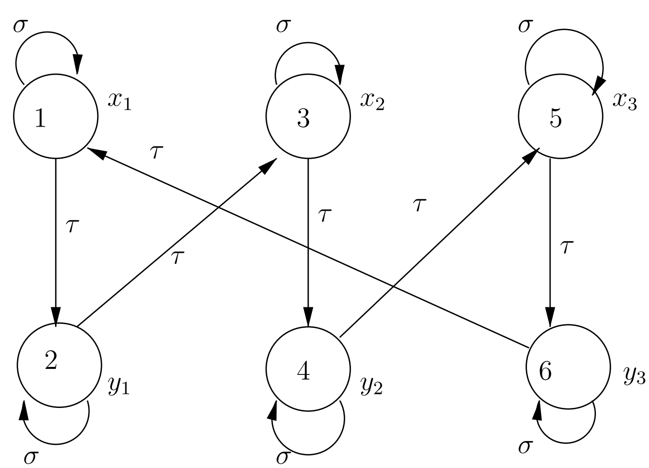

Fig. 6. The BAM neural network obtained from the ring of six neurons.

Without loss of generality, we can assume that zero is an equilibrium of (44), then its linearization at zero is

$$\begin{cases} \dot{x}_j(t) = -x_j(t) + a_j x_j(t-\sigma) + b_{j-1} y_{j-1}(t-\tau) \\ \dot{y}_j(t) = -y_j(t) + a_{j+n} y_j(t-\sigma) + b_{j+n} x_j(t-\tau), \end{cases} \tag{45}$$

where $a_j = s'_{1,j}(0)$, $a_{j+n} = s'_{2,j}(0)$, $b_{j+n} = g'_j(0)$, $j = 1, 2, \ldots, n$ and $b_j = f'_j(0)$, $j = 1, 2, \ldots, n-1$, $b_0 = b_n = f'_0(0)$. If we let $a_j = \beta$ for $j = 1, 2, \ldots, 2n$, and denote

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & b_n \\ b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \ldots & b_{n-1} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} b_{n+1} & 0 & \ldots & 0 \\ 0 & b_{n+2} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & b_{2n} \end{pmatrix}.$$

then we can apply our results to this model to discuss the local stability and Hopf bifurcation, regarding the self-connection delay σ as the parameter. Note that in [Campbell, 1999], β works as the parameter, and in [Campbell *et al.*, 1999], τ does that job. Using our main results, we can obtain the bifurcation analysis by varying σ and this together with [Campbell, 1999] and [Campbell *et al.*, 1999] can enrich the bifurcation analysis for the neural networks with ring structure.

In the following, we restrict our attention to a special case: $\tau = 0$ and $b_j = b$ for $j = 1, 2\ldots, 2n$. We then have

$$BA = \begin{pmatrix} 0 & 0 & 0 & \ldots & b^2 \\ b^2 & 0 & 0 & \ldots & 0 \\ 0 & b^2 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \ldots & b^2 & 0 \end{pmatrix}_{n\times n},$$

which implies that

$$\sigma(BA) = \{\lambda_j, j = 1, 2, \ldots, n\}$$

with $\lambda_j = b^2 e^{i\theta_j}$, $\theta_j = (j-1)2\pi/n$. (In particular, if $n = 2$, this corresponds to the model investigated in [Campbell *et al.*, 1999] and we have $\lambda_1 = b^2$ and $\lambda_2 = -b^2$.) Let

$$\omega_j^1 = \sqrt{\beta^2 - \left(1 + |b|\cos\frac{\theta_j}{2}\right)^2} - |b|\sin\frac{\theta_j}{2},$$

$$\omega_j^2 = \sqrt{\beta^2 - \left(1 - |b|\cos\frac{\theta_j}{2}\right)^2} + |b|\sin\frac{\theta_j}{2},$$

and

$$\sigma_j^1 = \frac{1}{\omega_j^1}\arccos\frac{1+|b|\cos\dfrac{\theta_j}{2}}{\beta},$$
$$\sigma_j^2 = \frac{1}{\omega_j^2}\arccos\frac{1-|b|\cos\dfrac{\theta_j}{2}}{\beta},$$

for $j \in N(1,n)$. If $\omega_j^s \notin \mathbb{R}^+, s = 1,2, j \in N(1,n)$, we denote the corresponding $\sigma_j^s = +\infty$. Set $\sigma_0 = \min\{\sigma_j^s : j \in N(1,n),\ s = 1,2\} = \sigma_{j_0}^{s_0}$ for some $j_0 \in N(1,n)$ and $s_0 \in \{1,2\}$. We denote $\lambda_0 = \lambda_{j_0} = b^2 e^{i\theta_0}$ and $\omega_0 = \omega_{j_0}^{s_0}$. Letting

$$\alpha_0 = 1 + i\omega_0 - \beta e^{-i\omega_0\sigma_0}, \quad q_j = e^{-i(j\theta_0)},$$
$$p_j = \frac{\alpha_0}{b}q_j, \quad p_j^* = \frac{\overline{\alpha}_0}{b}q_j, \quad j \in N(1,n),$$

and

$$D = \frac{b}{2n\alpha_0(1+\beta\sigma_0 e^{-i\omega_0\sigma_0})}.$$

This gives

$$C_1(0) = 3D\sum_{j=1}^{n} e^{-i(2j\theta_0)}\left[\frac{e^{-i\omega_0\sigma_0}}{b}(\overline{\alpha}_0\alpha_j + \alpha_0\gamma_j) + d_j^* + d_{j-1}e^{i\theta_0}\right]$$

and

$$\mu_2 = -\mathrm{Re}(C_1(0)), \tag{46}$$

where $\gamma_j = s_{1,j}'''(0)/6,\ \alpha_j = s_{2,j}'''(0)/6,\ d_j^* = g_j'''(0)/6$ and $d_{j-1} = (f_{j-1}'''(0)/6),\ d_n = d_0$ for $j \in N(1,n)$.

Corollary 5.2. *Suppose that $b \neq 0$.*

(1) *If $\beta < 1 - |b|$, then the zero solution of (42) is asymptotically stable at $\sigma = 0$.*
(2) *If $\beta < -|1-|b||$, then the zero solution of (42) is asymptotically stable for $\sigma \in [0, \sigma_0)$ and unstable if $\sigma > \sigma_0$.*
(3) *Hopf bifurcation occurs at $\sigma = \sigma_0$ and its direction and stability are determined by μ_2 given by (46), namely, the Hopf bifurcation is supercritical (subcritical) and stable (unstable) if $\mu_2 > 0$ ($\mu_2 < 0$).*

For example, taking $k = 4$, $\tau = 0$, $s_j(x) = -2\tanh(x)$ and $h_j(x) = 2\tanh(x)$ in (42), then we have $\sigma_0 = 0.5612$, $\lambda_0 = -4$, $\omega_0 = 3.7321$, $\theta_0 = \pi$, and $\mu_2 > 0$. This shows that in the case $k = 4$, $\beta = -2$ and $b = 2$, Hopf bifurcation occurs at $\sigma = 0.5612$, which is supercritical and the bifurcated periodic solutions are asymptotically orbitally stable. The corresponding numerical simulations are presented in Figs. 7 and 8.

If $k = 6$ and $s_j(x)$ and $h_j(x)$ remain the same, then we can compute $\sigma_0 = 0.4209$, $\lambda_0 = -4(-(1/2) + (\sqrt{3}/2)i) = -4e^{i\theta_0}$, $\theta_0 = 2\pi/3$ and $\mu_2 > 0$. This shows supercritical Hopf bifurcation occurs at $\sigma = 0.4209$ and the bifurcating periodic solutions are asymptotically orbitally stable.

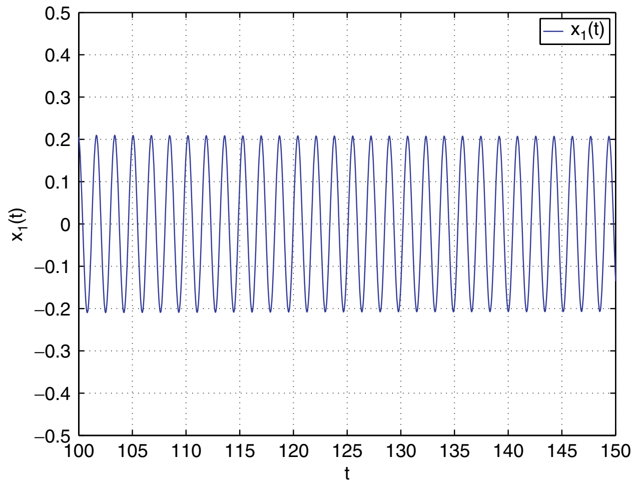

Fig. 7. A periodic solution of (42) bifurcates from zero solution at $\sigma = 0.57$. Here $b = 2$, $\beta = -2$, $k = 4$, the component $x_1(t)$ is shown and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

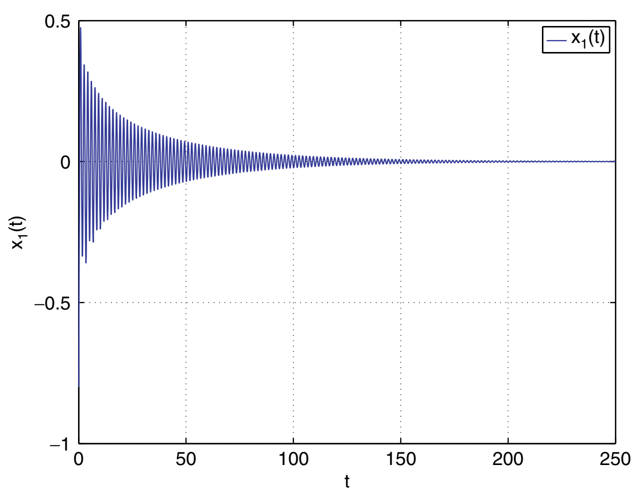

Fig. 8. The zero solution of (42) is locally stable when $\sigma = 0.55 < \sigma_0$. Here $b = 2$, $\beta = -2$, $k = 4$, the component $x_1(t)$ is shown and the behavior of $x_2(t)$, $y_1(t)$ and $y_2(t)$ are similar to that of $x_1(t)$.

Remark 5.1. Our result works for (42) whenever k is an even number.

6. Proofs of Theorems 2.1 and 2.2

6.1. *Proof of Theorem 2.1*

Let $(x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_n(t))^T$ be the solution of (4) with initial data $(\phi_1(s), \phi_2(s), \ldots, \phi_n(s), \psi_1(s), \psi_2(s), \ldots, \psi_n(s))^T, s \in [-r, 0]$, where r is the maximum of delays, i.e. $r = \max\{\tau_{ij}, r_{ij}, d_{ii}, m_{ii}, i, j \in N(1,n)\}$. It follows from (4) that

$$\frac{d|x_i(t)|}{dt} \le -|x_i(t)| + |c_{ii}|\mathrm{Lip}(g_i)|x_i(t-d_{ii})| + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)|y_j(t-\tau_{ij})| \tag{47}$$

$$\frac{d|y_i(t)|}{dt} \le -|y_i(t)| + |l_{ii}|\mathrm{Lip}(f_i)|y_i(t-m_{ii})| + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)|x_j(t-r_{ij})|. \tag{48}$$

By virtue of (5), we can choose a suitable real number $\delta > 0$ such that

$$p_i - p_i\delta - |c_{ii}|\mathrm{Lip}(g_i)p_i e^{\delta d_{ii}} - \mathrm{Lip}(g_i)\sum_{j=1}^{n}|b_{ji}|q_j e^{\delta r_{ij}} > 0 \tag{49}$$

and

$$q_i - q_i\delta - |l_{ii}|\mathrm{Lip}(f_i)q_i e^{\delta m_{ii}} - \mathrm{Lip}(f_i)\sum_{j=1}^{n}|a_{ji}|p_j e^{\delta\tau_{ij}} > 0 \tag{50}$$

for all $i, j \in N(1,n)$. Let

$$u_i(t) = e^{\delta t}|x_i(t)|, \quad v_i(t) = e^{\delta t}|y_i(t)|, \quad i \in N(1,n).$$

A direct calculation shows that

$$\begin{aligned}\frac{du_i(t)}{dt} &= e^{\delta t}\left(\delta|x_i(t)| + |x_i(t)|'\right)\\ &\le e^{\delta t}\Bigg(\delta|x_i(t)| - |x_i(t)| + |c_{ii}|\mathrm{Lip}(g_i)|x_i(t-d_{ii})| + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)|y_j(t-\tau_{ij})|\Bigg)\\ &= -(1-\delta)u_i(t) + |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}u_i(t-d_{ii}) + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)e^{\delta\tau_{ij}}v_j(t-\tau_{ij})\end{aligned} \tag{51}$$

and

$$\begin{aligned}\frac{dv_i(t)}{dt} &= e^{\delta t}(\delta|y_i(t)| + |y_i(t)|')\\ &\le e^{\delta t}\Bigg(\delta|y_i(t)| - |y_i(t)| + |l_{ii}|\mathrm{Lip}(f_i) \times |y_i(t-d_{ii})| + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)|x_j(t-r_{ij})|\Bigg)\\ &= -(1-\delta)v_i(t) + |l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}}v_i(t-m_{ii}) + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)e^{\delta r_{ij}}u_j(t-r_{ij}).\end{aligned} \tag{52}$$

Define

$$\begin{aligned}V(t) &= V(u,v)(t)\\ &= \sum_{i=1}^{n}p_i\Bigg(u_i(t) + |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}\int_{t-d_{ii}}^{t}|u_i(s)|ds + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)\int_{t-\tau_{ij}}^{t}e^{\delta\tau_{ij}}v_j(s)ds\Bigg)\\ &\quad + \sum_{i=1}^{n}q_i\Bigg(v_i(t) + |l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}}\int_{t-m_{ii}}^{t}|v_i(s)|ds + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)\int_{t-r_{ij}}^{t}e^{\delta r_{ij}}u_j(s)ds\Bigg)\end{aligned}$$

Then the upper right-hand derivative of $V(t)$ along the solution of (4) is

$$\begin{aligned}D^+V(t) &= \sum_{i=1}^{n}p_i\Bigg(\dot{u}_i(t) + |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}(u_i(t) - u_i(t-d_{ii})) + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)e^{\delta\tau_{ij}}(v_i(t) - v_j(t-\tau_{ij}))\Bigg)\\ &\quad + \sum_{i=1}^{n}q_i\Bigg(\dot{v}_i(t) + |l_{ii}|\mathrm{Lip}(f_i) \times e^{\delta m_{ii}}(v_i(t) - v_i(t-m_{ii})) + \sum_{j=1}^{n}|b_{ij}| \times \mathrm{Lip}(g_j)e^{\delta r_{ij}}(u_i(t) - u_j(t-r_{ij}))\Bigg).\end{aligned}$$

This, combined with (51) and (52), gives

$$
\begin{aligned}
D^+V(t) \le& \sum_{i=1}^{n} p_i(-(1-\delta-|c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}})u_i(t)) \\
&+ \sum_{i=1}^{n} p_i \sum_{j=1}^{n} |a_{ij}|\mathrm{Lip}(f_j)e^{\delta\tau_{ij}}v_j(t) \\
&+ \sum_{i=1}^{n} q_i(-(1-\delta-|l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}})v_i(t)) \\
&+ \sum_{i=1}^{n} q_i \sum_{j=1}^{n} |b_{ij}|\mathrm{Lip}(g_j)e^{\delta r_{ij}}u_j(t) \\
=& -\sum_{i=1}^{n}\left(p_i - p_i\delta - |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}p_i \right.\\
&\left. - \mathrm{Lip}(g_i)\sum_{j=1}^{n} q_j|b_{ji}|e^{\delta r_{ji}}\right)u_i(t) \\
&- \sum_{i=1}^{n}\left(q_i - q_i\delta - |l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}}q_i \right.\\
&\left. - \mathrm{Lip}(f_i)\sum_{j=1}^{n} p_j|a_{ji}|e^{\delta\tau_{ji}}\right)v_i(t) \\
\le& \, 0.
\end{aligned}
$$

This shows that $V(t) \le V(0)$ and $V(0)$ can be controlled by a finite positive number. In fact,

$$
\begin{aligned}
V(0) =& \sum_{i=1}^{n} p_i\left(u_i(0) + |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}\int_{-d_{ii}}^{o} u_i(s)ds \right.\\
&\left. + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)\int_{-\tau_{ij}}^{0} e^{\delta\tau_{ij}}v_j(s)ds\right) \\
&+ \sum_{i=1}^{n} q_i\left(v_i(0) + |l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}}\int_{-m_{ii}}^{o} v_i(s)ds \right.\\
&\left. + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)\int_{-r_{ij}}^{0} e^{\delta r_{ij}}u_j(s)ds\right) \\
\le& \sum_{i=1}^{n} p_i\left(|\phi_i(0) + |c_{ii}|\mathrm{Lip}(g_i)e^{\delta d_{ii}}d_{ii}\|\phi_i\| \right.\\
&\left. + \sum_{j=1}^{n}|a_{ij}|\mathrm{Lip}(f_j)e^{\delta\tau_{ij}}\tau_{ij}||\psi_j||\right) \\
&+ \sum_{i=1}^{n} q_i\left(|\psi_i(0) + |l_{ii}|\mathrm{Lip}(f_i)e^{\delta m_{ii}}m_{ii}\|\psi_i\| \right.\\
&\left. + \sum_{j=1}^{n}|b_{ij}|\mathrm{Lip}(g_j)e^{\delta r_{ij}}r_{ij}||\phi_j||\right) =: C_1 < \infty,
\end{aligned}
$$

where $\|\phi_i\| := \max\{|\phi_i(s)|,\ s \in [-r, 0]\}$ and $\|\psi_i\| := \max\{|\psi_i(s)|,\ s \in [-r, 0]\}$ for $i \in N(1, n)$. Then it follows from $V(t) \le V(0) \le C_1$ that

$$
\sum_{i=1}^{n} u_i(t) \le \frac{C_1}{\min\{p_i, i \in N(1,n)\}} =: C_2,
$$

$$
\sum_{i=1}^{n} v_i(t) \le \frac{C_1}{\min\{q_i, i \in N(1,n)\}} =: C_3
$$

and hence

$$
\sum_{i=1}^{n}|x_i(t)| \le C_2 e^{-\delta t}, \quad \sum_{i=1}^{n}|y_i(t)| \le C_3 e^{-\delta t}, \quad \text{for} \quad t \ge 0.
$$

This shows that any solution of (4) exponentially converges to zero and the proof is complete.

6.2. *Proof of Theorem 2.2*

Define a Liapunov functional as

$$
\begin{aligned}
V(t) =& \sum_{i=1}^{n} p_i\left(x_i^2(t) + |c_{ii}|\int_{t-d_{ii}}^{t} g_i^2(x_i(s))ds \right.\\
&\left. + \sum_{j=1}^{n}\frac{|a_{ij}|}{\xi_j}\int_{t-\tau_{ij}}^{t} f_j^2(y_j(s))ds\right) \\
&+ \sum_{i=1}^{n} q_i\left(y_i^2(t) + |l_{ii}|\int_{t-m_{ii}}^{t} f_i^2(y_i(s))ds \right.\\
&\left. + \sum_{j=1}^{n}\frac{|b_{ij}|}{\eta_j}\int_{t-r_{ij}}^{t} g_j^2(x_j(s))ds\right).
\end{aligned}
$$

Using a similar argument as in Theorem 2.1 of [van der Driessche & Zon, 1998], together with applying the following inequality

$$
2|a||b| \le a^2\eta + \frac{b^2}{\eta}, \quad \text{for } \eta > 0
$$

to estimate the upper right-hand derivative of $V(t)$ along the solution of (4), we can complete the proof and the details are omitted here.

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