Capacity of Stable Periodic Solutions in Discrete-Time Bidirectional Associative Memory Neural Networks

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***Abstract*—The existence, stability, and even the number (capacity) of stable periodic solutions in discrete-time bidirectional associative memory neural networks are investigated in this paper. Some fundamental mathematical techniques instead of the bifurcation method, are employed to prove the existence of periodic solutions and establish the relation of the number of periodic solutions and the magnitude of delays. Our results show that we can simply design a delayed neural network model to store memories or patterns as stable periodic solutions.**

***Index Terms*—Bidirectional associative memory (BAM), delay, equilibrium, neural networks, periodic solution, stability.**

I. Introduction

One of the main tasks that artificial neural networks can fulfill is associate memory. In an associative memory neural network, the addressable memories or patterns are stored as stable equilibria or stable periodic solutions. Thus, for the purpose of the associate memories, it is desirable for the network to have as large a capacity as possible for retrievable memories. In terms of the terminology of dynamical systems, this requires that the network admits as many stable equilibria or stable periodic solutions as possible.

In continuous-time models, the series papers [1]–[4] established the co-existence of multiple periodic solutions and described their domains of attraction. However, all these periodic solutions, except one, are unstable and they have large domains of attraction only in some submanifolds. On the contrary, for the discrete-time models, a large number of stable periodic solutions can possibly coexist. In this context, Zhou and Wu [17], [18] proved the existence of two stable periodic solutions with special periods for a class of discrete-time neural network model with two identical neurons. For this model, Wu and Zhang [15] recently explored the existence of periodic orbits with all possible periods and even provided a formula to compute the number of all possible stable periodic orbits. More recently, Wu *et al.* [16] extended the idea in [15] to a model with ring structure and showed that the number of neurons and the delays all have impacts on the periodic solutions capacity of the neural network model under certain conditions. One naturally wonders what would happen if the network has other types of connection structure. For a general connection topology, it is very difficult, if not impossible, to answer this question. In this paper, we will further consider a class of discrete-time neural network models with more trainable parameters and with another special connection topology: bidirectional associative memory (BAM) models. As is seen in [13], a ring network with an even number of neurons is a special case of BAM networks. More precisely, we study the delayed discrete-time BAM neural network model described by

$$\begin{cases} x_i(n) = \beta_i x_i(n-1) + \sum_{j=1}^{m} a_{ij} f_j(y_j(n-k_j)) \\ y_i(n) = \alpha_i y_i(n-1) + \sum_{j=1}^{m} b_{ij} g_j(x_j(n-l_j)) \end{cases} \quad (1.1)$$

where $\beta_i, \alpha_i \in (0,1), i \in (1,2,\dots,m) =: N(1,m)$ are decay rates; $a_{ij}, b_{ij}, i,j \in N(1,m)$ are the connection weights between the neurons in two layers, the X layer with neurons whose states denoted by $x_i, i \in N(1,m)$ and the Y layer with neurons whose states denoted by $y_i, i \in N(1,m)$; and the positive integers $k_i, l_i, i \in N(1,m)$ are the associated delays due to the finite transmission speed among neurons in different layers in the network. The activation functions $f_i, g_i, i \in N(1,m)$ are of class $\mathrm{CL}^{\epsilon}_{(r,R]}$, where

$$\mathrm{CL}^{\epsilon}_{(r,R]} := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, \begin{array}{ll} |f(x)-1| \le \epsilon, & x \in (r,R] \\ |f(x)+1| \le \epsilon, & x \in [-R,-r) \end{array} \right\}$$

and the constants $\epsilon > 0, 0 \le r \le R$ as well as $\beta_i, \alpha_i \in (0,1), i \in N(1,m)$ will be specified later. We will show that for this network, the delays, together with the size of the network, also have advantageous impact on the capacity of stable periodic solutions.

Note that it is Kosko (see [8]–[10]) who first proposed the continuous-time BAM neural network model (which can be regarded as a generalization of the well-known Hopfield neural networks [6]) and discussed its applications. Networks with the bidirectional structure have practical applications in storing paired patterns or memories and the ability to search the desired patterns via both directions: forward and backward. See [5], [8]–[10], and [11], where, in [5] and [11], the delayed continuous-time BAM neural networks were discussed.

The delays in (1.1) do not change the number of its equilibria; however, as we will show, they are related to the number of periodic solutions of (1.1) under certain assumptions and indeed the delayed discrete-time BAM neural networks can have large periodic solution capacity to store the paired patterns or memories.

The rest of this paper is organized as follows. In Section II, we give some preliminaries. Section III is devoted to the study of capacity for periodic solutions in (1.1). Some discussions are given in Section IV.

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II. Preliminaries

As usual, a solution of (1.1) is a sequence

$$\{(x_1(n), x_2(n), \ldots, x_m(n), y_1(n), y_2(n), \ldots, y_m(n))\}$$

of points in $\mathbb{R}^{2m}$, which is defined for every integer $n \geq -\max\{k_i, l_i, i \in N(1,m)\}$ and satisfies (1.1) for $n \geq 1$. In what follows, we denote

$$A = (a_{ij})_{m\times m}$$
$$B = (b_{ij})_{m\times m}$$
$$K = \sum_{i=1}^{m} k_i$$
$$L = \sum_{i=1}^{m} l_i$$

and suppose that A is *strongly diagonally dominant*, that is

$$a_{ii} > \sum_{j\neq i} |a_{ij}| =: \bar{A}_i, i \in N(1,m)$$

and B is *strongly quasi-diagonally dominant*, i.e.,

$$b_{ii+1} > \sum_{j\neq i+1} |b_{ij}| =: \bar{B}_i, i \in N(1,m)$$

where $b_{mm+1} := b_{m1}$. Let

$$u_{i,j}(n) = x_i(n - l_i + j - 1), j = 1, 2, \ldots, l_i$$
$$v_{i,j}(n) = y_i(n - k_i + j - 1), j = 1, 2, \ldots, k_i$$
$$u_i(n) = (u_{i,1}(n), u_{i,2}(n), \ldots, u_{i,l_i}(n))^T \in \mathbb{R}^{l_i}$$
$$v_i(n) = (v_{i,1}(n), v_{i,2}(n), \ldots, v_{i,k_i}(n))^T \in \mathbb{R}^{k_i},$$
$$\text{for} \quad i \in N(1,m).$$

Denoting

$$\omega(n) := (u_1(n), v_1(n), u_2(n), v_2(n), \ldots, u_m(n), v_m(n))$$

by

$$\omega(n) = (\omega_1(n), \omega_2(n), \ldots, \omega_{K+L}(n)) \in \mathbb{R}^{K+L}$$

and letting

$$\bar{k}_i := \sum_{j=1}^{i} k_j, \quad \bar{k}_0 := 0, \quad \bar{l}_i := \sum_{j=1}^{i} l_j, \quad \bar{l}_0 := 0$$

we may rewrite (1.1) as

$$\omega(n+1) = F(\omega(n)) \tag{2.2}$$

where $F : \mathbb{R}^{K+L} \to \mathbb{R}^{K+L}$ is defined in (2.3), shown at the bottom of the page, with

$$F_{\bar{l}_i+\bar{k}_{i-1}}(\omega) := \beta_i \omega_{\bar{l}_i+\bar{k}_{i-1}} + \sum_{j=1}^{m} a_{ij} f_j(\omega_{\bar{l}_j+\bar{k}_{j-1}+1})$$

and

$$F_{\bar{l}_i+\bar{k}_i}(\omega) := \alpha_i \omega_{\bar{l}_i+\bar{k}_i} + \sum_{j=1}^{m} b_{ij} g_j(\omega_{\bar{l}_{j-1}+\bar{k}_{j-1}+1}).$$

We denote the solution of (2.2) with initial value $w(0)$ by $\omega(n, w(0)), n = 1, 2, \ldots$. For $\omega = (\omega_1, \ldots, \omega_{K+L}) \in \mathbb{R}^{K+L}$, its norm is defined by

$$\|\omega\| = \max\{|\omega_j|, j \in N(1, K+L)\}.$$

Let

$$d = \max\left\{\frac{\sum_{j=1}^{m} |a_{ij}|}{1-\beta_i}, \frac{\sum_{j=1}^{m} |b_{ij}|}{1-\alpha_i}, i \in N(1,m)\right\}.$$

We assume that the second equation shown at the bottom of the page, holds. Let $r_* := \min\{R - b^*, a^* - r\}$ and define

$$a_c := a^* - c \quad b_c := b^* + c, \qquad \text{for } c \in [0, r_*).$$

In the sequel, we will use the following notations:

$$\operatorname{sgn}(x) := \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases} \quad \text{for } x \in \mathbb{R}.$$

$$\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \ldots, \operatorname{sgn}(x_m))$$
$$\text{for } x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$
$$\Sigma := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{K+L}) \in \mathbb{R}^{K+L},$$
$$\sigma_j \in \{-1, 1\}, j \in N(1, K+L)\}$$
$$\mathrm{CL}_{(r,R]}^{Lip} = \{f : \mathbb{R} \to \mathbb{R}; |f(x) - f(y)|$$
$$\leq \mathrm{Lip}\,(f)|x-y|, x, y \in [-R, -r) \cup (r, R]\}$$
$$\Omega := \{\omega \in \mathbb{R}^{K+L}; \|\omega\| \in (r, R)\}$$
$$\Omega(\sigma, c) := \{\omega \in \mathbb{R}^{K+L}; \|\omega\| \in [a_c, b_c]$$
$$\operatorname{sgn}(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}$$
$$\Omega(\sigma) := \{\omega \in \mathbb{R}^{K+L}; \|\omega\| \in (r, R)$$
$$\operatorname{sgn}(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}$$
$$\Omega^*(\sigma) := \{\omega \in \mathbb{R}^{K+L}; \|\omega\| \in (r, b^*)$$
$$\operatorname{sgn}(\omega_i) = \sigma_i, \sigma = (\sigma_1, \ldots, \sigma_{K+L}) \in \Sigma\}.$$

We point out that $\mathrm{CL}_{(r,R]}^{\epsilon}$ and $\mathrm{CL}_{(r,R]}^{\mathrm{Lip}}$ include those frequently used sigmoid functions when r and R are properly chosen.

$$F_s(\omega) := \begin{cases} F_s(\omega), & s \in S := \{\bar{l}_i + \bar{k}_j, j = i-1, i, \quad \text{and} \quad i \in N(1,m)\} \\ \omega_{s+1}, & s \in N(1, K+L)\backslash S \end{cases} \tag{2.3}$$

$$(\mathrm{DH}_1) : \begin{cases} 0 < \beta_i < \frac{1}{2}\left(1 - \frac{\bar{A}_i}{a_{ii}}\right), \quad 0 < \alpha_i < \frac{1}{2}\left(1 - \frac{\bar{B}_i}{b_{ii+1}}\right) \\ \epsilon < \min_{i\in N(1,m)} \left\{ \begin{array}{c} \frac{1-2\beta_i}{1-\beta_i}\frac{a_{ii}}{\bar{A}_i}\left[1 - (1-2\beta_i)\frac{\bar{A}_i}{a_{ii}}\right] \\ \frac{1-2\alpha_i}{1-\alpha_i}\frac{b_{ii+1}}{\bar{B}_i}\left[1 - (1-2\alpha_i)\frac{\bar{B}_i}{b_{ii+1}}\right] \end{array} \right\} \\ R > d(1+\epsilon) =: b^* \\ r < \min_{i\in N(1,m)} \left\{ \begin{array}{c} a_{ii} - \bar{A}_i - \beta_i b^* - \sum_{j=1}^{m} |a_{ij}|\epsilon \\ b_{ii+1} - \bar{B}_i - \alpha_i b^* - \sum_{j=1}^{m} |b_{ij}|\epsilon \end{array} \right\} =: a^* \end{cases}$$

III. Multiplicity of Stable Periodic Solutions

Define a mapping $\pi : \Sigma \to \Sigma$ by $\sigma \in \Sigma$

$$(\pi\sigma)_j = \begin{cases} \sigma_{j+1}, & \text{for } j \in N(1, K+L-1) \\ \sigma_1, & \text{for } j = K+L. \end{cases} \tag{3.4}$$

For $p \geq 2$, the mapping $\pi^p : \Sigma \to \Sigma$ is given by

$$\pi^p\sigma = \pi(\pi^{p-1}\sigma)$$

and it follows that

$$(\pi^p\sigma)_j = \begin{cases} \sigma_{j+p}, & \text{for } j \in N(1, K+L-p) \\ \sigma_{j-K-L+p}, & \text{for } j = K+L-p+1, \ldots, K+L \end{cases}$$

and

$$\pi^{K+L}\sigma = \sigma \qquad \forall \sigma \in \Sigma.$$

We denote by

$$\Sigma_p := \{\sigma \in \Sigma : \pi^p\sigma = \sigma, \pi^q\sigma \neq \sigma, q \in \{1, 2, \ldots, p-1\}\}$$

the set of all p periodic points of π in Σ for $p = 1, 2, \ldots$. Thus, Σ_1 is the set of all fixed points of π in Σ.

The following Lemma is needed in the proofs of our main results.

Lemma 3.1: Assume that K and L are positive integers. Then the following hold.

1) For each $p \in N(1, K+L), \Sigma_p \neq \emptyset \Longleftrightarrow p \mid K+L$.
2) $\Sigma = \bigcup_{p \mid K+L} \Sigma_p$.
3) For each $p \in N(1, K+L)$, the number of elements in Σ_p, denoted by $N(\Sigma_p)$, is given by

$$N(\Sigma_p) = \begin{cases} 2, & p = 1 \\ 2^p - 2, & p \text{ is prime} \\ 2^p - \sum_{q \mid p, q<p} N(\Sigma_q), & \text{otherwise.} \end{cases}$$

Proof: See [15] or [16]. □

We next give an existence result for periodic solutions of (2.2).

Theorem 3.1: Assume that (DH_1) is satisfied and $f_i, g_i \in \mathrm{CL}^{\epsilon}_{(r,R]}$ for $i \in N(1, m)$. Then, for any p and σ with $p|K+L$ and $\sigma \in \Sigma_p$, (2.2) has a p-periodic solution $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$.

Proof: We first show that for any $\sigma \in \Sigma$ and $c \in [0, r_*)$

$$F : \Omega(\sigma, c) \to \Omega(\pi\sigma, c).$$

Define

$$h_1(z) := \beta_1 z_0 + \sum_{j=1}^{m} a_{1j} f_j(z_j)$$

where

$$z = (z_0, z_1, \ldots, z_m) \in \mathbb{R}^{m+1}, \quad \text{with } |z_j| \in [a_c, b_c], \quad j = 0, 1, \ldots, m.$$

We claim that

$$\mathrm{sgn}\,(h_1(z)) = \mathrm{sgn}\,(z_1) \quad \text{and} \quad |h_1(z)| \in [a_c, b_c].$$

To this end, we have two cases: 1) $z_1 \geq 0$ and 2)$z_1 < 0$, to be considered. If $z_1 \geq 0$, we then have

$$h_1(z) \leq \beta_1 b_c + a_{11}(1+\epsilon) + \sum_{j\neq 1} |a_{1j}|(1+\epsilon) \leq b_c$$

and

$$h_1(z) \geq -\beta_1 b_c + a_{11}(1-\epsilon) - \sum_{j\neq 1} |a_{1j}|(1+\epsilon) \geq a_c$$

which are due to

$$\begin{aligned}(1-\beta_1)b_c &\geq (1-\beta_1)b^* \\ &= (1-\beta_1)d(1+\epsilon^*) \\ &\geq \sum_{j=1}^{m} |a_{1j}|(1+\epsilon)\end{aligned}$$

and

$$\begin{aligned} a_c &+ \beta_1 b_c \\ &= a^* - c + \beta_1(b^* + c) \\ &= a^* + \beta_1 b^* - (1-\beta_1)c \\ &\leq a^* + \beta_1 b^* \leq a_{11} - \sum_{j\neq 1} |a_{1j}| - \sum_{j=1}^{m} |a_{1j}|\epsilon \\ &= a_{11}(1-\epsilon) - \sum_{j\neq 1} |a_{1j}|(1+\epsilon).\end{aligned}$$

Similarly, for case 2), we can show that

$$-b_c \leq h_1(z) \leq -a_c.$$

Therefore, our claim is true. Using this argument and the definition of F, we can show that

$$|F_j(\omega)| \in [a_c, b_c], \qquad \text{for } j = 1, 2, \ldots, K+L$$
$$\mathrm{sgn}\,(F_j(\omega)) = \mathrm{sgn}\,(\omega_{j+1}) = \sigma_{j+1}, \qquad j = 1, 2, \ldots, K+L-1$$

and

$$\mathrm{sgn}\,(F_{K+L}(\omega)) = \mathrm{sgn}\,(\omega_1) = \sigma_1.$$

This shows that for any $\omega \in \Omega(\sigma, c), F(\omega) \in \Omega(\pi\sigma, c)$. Notice that $\Omega(\sigma, c) \in \mathbb{R}^{K+L}$ is convex and closed. Also, for any $p \mid K+L$ and $\sigma \in \Sigma_p, F^p(\Omega(\sigma, c)) \subset \Omega(\pi^p\sigma, c) = \Omega(\sigma, c)$. Now, by the well-known Brouwer's fixed point theorem, the continuous mapping F^p admits a fixed point in $\Omega(\sigma, c)$, which is exactly a p-periodic solution, denoted by $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$, of (2.2) with initial value in $\Omega(\sigma, c)$. The proof is complete. □

Theorem 3.2: In addition to the conditions in Theorem 3.1, assume that $f_i, g_i \in \mathrm{CL}^{\mathrm{Lip}}_{(r,R]}$ with

$$J := \max_{i\in N(1,m)} \left\{ \beta_i + \sum_{j=1}^{m} |a_{ij}| \,\mathrm{Lip}\,(f_j), \alpha_i + \sum_{j=1}^{m} |b_{ij}| \,\mathrm{Lip}\,(g_j) \right\} < 1.$$

Then, the following hold.

i) For any $p \mid K+L$ and $\sigma \in \Sigma_p$, (2.2) has a unique p-periodic solution $\{\omega(n, \omega^\sigma)\}_{n\in\mathbb{N}}$ with $\omega^\sigma \in \Omega(\sigma, 0)$ and this solution is exponential stable in the sense that for any $\bar{\omega}^\sigma$ with $\|\bar{\omega}^\sigma - \omega^\sigma\| < r(\sigma)$, we have

$$\|\omega(n, \bar{\omega}^\sigma) - \omega(n, \omega^\sigma)\| \leq C\xi^n \|\bar{\omega}^\sigma - \omega^\sigma\|$$

where

$$\xi := J^{\frac{1}{K+L}} < 1, \qquad C := \xi^{1-(K+L)} > 0$$

and

$$r(\sigma) := \min\{\|\omega^\sigma\| - a^* + r_*, b^* + r_* - \|\omega^\sigma\|\} > 0.$$

ii) If $\{\omega(n)\}_{n\in\mathbb{N}}$ is a p-periodic solution of (2.2) in Ω; then, $p \mid K+L$ and there exists a unique $\sigma \in \Sigma_p$ and some $\omega^\sigma \in \Omega(\sigma, 0)$ such that $\omega(n) = \omega(n, \omega^\sigma)$.

iii) For any solution $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ of (2.2) with $\|\omega(0)\| \in (a^*-r_*, b^*+r_*)$, there exist a unique $p \in \mathbb{N}$ with $p\,|\,K+L$ and a unique $\sigma \in \Sigma_p$ such that

$$|\omega(n,\omega(0)) - \omega(n,\omega^\sigma)\| \le C\xi^n \|\omega(0) - \omega^\sigma\|, n \in \mathbb{N}.$$

iv) For any $p \in \mathbb{N}$ with $p\,|\,K+L$, (2.2) has $N(\Sigma_p)$ $p-$periodic solutions in Ω, which are all exponentially stable. If $p \nmid K+L$, (2.2) has no p-periodic solution in Ω.

To prove this theorem, we first establish the following useful lemmas under the same assumptions.

Lemma 3.2: For $\omega', \omega'' \in \Omega(\sigma,c)(\Omega, \Omega^*(\sigma))$, we have

$$\|F^{K+L}(\omega') - F^{K+L}(\omega'')\| \le J\|\omega' - \omega''\|. \tag{3.5}$$

Proof: This can be easily proved by the fact that $J \in (0,1)$ and the definition of F. □

Lemma 3.3: If $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ is a p-periodic solution of (2.2) in Ω, then $\|\omega(n,\omega(0))\| \le b^*$.

Proof: Since $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ is a p-periodic solution of (2.2) in Ω, we can then obtain a p-periodic solution $\{(x_1(n),\ldots,x_m(n),y_1(n),\ldots,y_m(n))\}_{n\in\mathbb{N}}$ for (1.1). We will show that

$$|x_i(n)| \le d_i(1+\epsilon) \quad |y_i(n)| \le d_{i+m}(1+\epsilon)$$

where

$$d_i := \frac{\sum_{j=1}^m |a_{ij}|}{1-\beta_i}, \quad d_{i+m} := \frac{\sum_{j=1}^m |b_{ij}|}{1-\alpha_i}, \qquad i =\in N(1,m).$$

By way of contradiction, suppose that for some i, there exists n_0 such that $|x_i(n_0)| > d_i(1+\epsilon)$, say, $x_i(n_0) = d_i(1+\epsilon)+\delta_0$ (the proof for the case $x_i(n_0) < -d_i(1+\epsilon)$ is similar) for some $\delta_0 > 0$. Then, from (1.1), we have

$$\begin{aligned} x_i(n_0-1) &= \frac{1}{\beta_i}\left(x_i(n_0) - \sum_{j=1}^m a_{ij} f_j(y_j(n-k_j))\right) \\ &\ge \frac{1}{\beta_i}\left(d_i(1+\epsilon) + \delta_0 - \sum_{j=1}^m |a_{ij}|(1+\epsilon)\right) \\ &= \frac{1}{\beta_i}\delta_0 + d_i(1+\epsilon) \\ &> d_i(1+\epsilon) + \delta_0 (\text{since } \beta_i < 1) \\ &= x_i(n_0). \end{aligned}$$

Repeating this procedure, we can show $x_i(n_0-p) > x_i(n_0)$, which is a contradiction. Thus, we have shown that for all $n \in \mathbb{N}$

$$|x_i(n)| \le d_i(1+\epsilon) \quad |y_i(n)| \le d_{i+m}(1+\epsilon)$$

which implies that

$$\|\omega(n,\omega(0))\| \le b^* := \max\{d_i(1+\epsilon), i \in \{1,2,\ldots,2m\}\}$$

and the proof is complete. □

Lemma 3.4: If $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ is a p-periodic solution of (2.2) in Ω, then $p\,|\,K+L$ and $\omega(n,\omega(0)) = \omega(n,\omega^\sigma)$ for some $\sigma \in \Sigma_p$, and $\omega^\sigma \in \Omega(\sigma,c)$.

Proof: Note that

$$\Omega = \bigcup_{q\,|\,K+L}\ \bigcup_{\sigma\in\Sigma_q} \Omega(\sigma). \tag{3.6}$$

Then, there exist q and σ with $q\,|\,K+L$ and $\sigma \in \Sigma_q$ such that $\omega(0) \in \Omega(\sigma)$. From Lemma 3.3, we further know $\omega(n,\omega(0)) \in \Omega^*(\sigma)$. Moreover, for such q and σ, it follows from Theorem 3.1 and Lemma 3.3 that (2.2) has a q-periodic solution denoted by $\{\omega(n,\omega^\sigma))\}$ with $\omega(n,\omega^\sigma)) \in \Omega(\sigma,c)$ for $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|\omega(n,\omega(0)) - \omega(n,\omega^\sigma)\| &= \|\omega(n+pq(K+L),\omega(0)) \\ &\quad - \omega(n+pq(K+L),\omega^\sigma)\| \\ &= \|(F^{K+L})^{pq}(\omega(n,\omega(0))) \\ &\quad - (F^{K+L})^{pq}(\omega(n,\omega^\sigma))\| \\ &\le J^{pq}\|\omega(n,\omega(0)) - \omega(n,\omega^\sigma)\| \end{aligned}$$

which shows that $\omega(n,\omega(0)) = \omega(n,\omega^\sigma)$ for $n \in \mathbb{N}$ and $q = p$ and hence $p\,|\,K+L$. □

Now we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2:

i) The existence and the uniqueness follow from Theorem 3.1 and Lemma 3.4. We just need to show the exponential stability. For any $n \in \mathbb{N}$, we write $n = s(K+L)+q$ with $q \in \{1,2,\ldots,K+L-1\}$, and then for any $\bar{\omega}^\sigma$ with $\|\bar{\omega}^\sigma - \omega^\sigma\| < r(\sigma)$, it follows from Lemma 3.2 that

$$\begin{aligned} \|\omega(n,\bar{\omega}^\sigma) - \omega(n,\omega^\sigma)\| &= \|F^{s(K+L)+q}(\bar{\omega}^\sigma) \\ &\quad - F^{s(K+L)+q}(\omega^\sigma)\| \\ &\le |F^{s(K+L)}(\bar{\omega}^\sigma) - F^{s(K+L)}(\omega^\sigma)\| \\ &\le J^s\|\bar{\omega}^\sigma - \omega^\sigma\| \\ &= C\xi^{(s(K+L)+K+L-1}\|\bar{\omega}^\sigma - \omega^\sigma\| \\ &\le C\xi^n\|\bar{\omega}^\sigma - \omega^\sigma\|. \end{aligned}$$

ii) The proof follows from Lemma 3.4.

iii) We may find a $c \in [0, r_*)$ such that $\|\omega(0)\| \in [a_c, b_c]$. Now, let $\sigma \in \Sigma$ with $\sigma = \operatorname{sgn}(\omega(0))$, that is, $\omega^\sigma := \omega(0) \in \Omega(\sigma,c)$. Since $\Sigma = \bigcup_{p\,|\,K+L}\Sigma_p$, there must exist a unique $p\,|\,K+L$ such that $\sigma \in \Sigma_p$. For such σ and ω^σ, there exists a p-periodic solution $\{\omega(n,\omega^\sigma)\}_{n\in\mathbb{N}}$. The rest of the proof follows from Lemma 3.2 and (I).

iv) This follows from the definition of $N(\Sigma_p)$, (I), and Lemma 3.4.

Remark 3.1: (I) gives a domain of attraction for each stable periodic solution of (2.2).

It is possible for two periodic solutions to have the same orbit. To distinct orbits, we give a definition for equivalent periodic solutions:

Definition 3.1: Two p-periodic solutions $\{\omega(n,\omega(0))\}_{n\in\mathbb{N}}$ and $\{\omega(n,\bar{\omega}(0))\}_{n\in\mathbb{N}}$ are said to be equivalent, denoted by

$$\omega(n,\omega(0)) \sim \omega(n,\bar{\omega}(0))$$

if there exists $q \in \{1,2,\ldots,p-1\}$ such that

$$\omega(n,\omega(0)) = \omega(n+q,\bar{\omega}(0)).$$

In other words, two p-periodic solutions are equivalent if they generate the same orbit.

Lemma 3.5: For any $p\,|\,K+L$ and any $\sigma,\bar{\sigma} \in \Sigma_p$ with $\sigma \ne \bar{\sigma}$; then, the two p-periodic solutions $\{\omega(n,\omega^\sigma)\}_{n\in\mathbb{N}}$ and $\{\omega(n,\omega^{\bar{\sigma}})\}_{n\in\mathbb{N}}$ generated by ω^σ and $\omega^{\bar{\sigma}}$ are equivalent if and only if there exists a $q \in \{1,2,\ldots,p-1\}$ such that $\bar{\sigma} = \pi^q\sigma$, or $\sigma = \pi^q\bar{\sigma}$.

TABLE I
$N(\Sigma_p)$ AND $n(p)$ FOR SOME p

p	1	2	3	4	5	10	15	20
$N(\Sigma_p)$	2	2	6	12	30	990	32730	1047540
$n(p)$	2	1	2	3	6	99	2182	252377

TABLE II
$n(K+L)$ FOR SOME $K+L$

$K+L$	2	3	4	5	10	15	20
$p{:}p \mid K+L$	$1,2$	$1,3$	$1,2,4$	$1,5$	$1,2,5,10$	$1,3,5,15$	$1,2,4,5,10,20$
$n(K+L)$	3	4	6	11	108	2192	52488

Proof: Suppose $\sigma, \bar{\sigma} \in \Sigma_p$ and $\bar{\sigma} = \pi^q \sigma$ for some $q \in \{1, 2, \ldots, p-1\}$. Note that $F^q : \Omega(\sigma, 0) \to \Omega(\pi^q \sigma, 0)$ and $\omega(n+q, \omega^\sigma) = \omega(n, F^q(\omega^\sigma))$, which implies that $\omega(n, F^q(\omega^\sigma))$ is a p-periodic solution with initial value $F^q(\omega^\sigma) \in \Omega(\pi^q\sigma, 0) = \Omega(\bar{\sigma}, 0)$. On the other hand, we know that $\omega(n, \omega^{\bar{\sigma}})$ is a p-periodic solution with initial value $\omega^{\bar{\sigma}} \in \Omega(\bar{\sigma}, 0)$ too. Therefore, we have

$$\omega(n, \omega^{\bar{\sigma}}) = \omega(n, F^q(\omega^\sigma)) = \omega(n+q, \omega^\sigma), \qquad n = 0, 1, \ldots, .$$

That is, $\{\omega(n, \omega^\sigma)\}_{n \in \mathbb{N}}$ and $\{\omega(n, \omega^{\bar{\sigma}})\}_{n \in \mathbb{N}}$ are equivalent. Next, suppose that $\omega(n, \omega^\sigma) \sim \omega(n, \omega^{\bar{\sigma}})$, Then, there exists $q \in \{1, 2, \ldots, p-1\}$ such that $\omega(0, \omega^\sigma) = \omega(q, \omega^{\bar{\sigma}})$. It follows that

$$\sigma = \operatorname{sgn}(\omega(0, \omega^\sigma)) = \operatorname{sgn}(\omega(q, \omega^{\bar{\sigma}})) = \operatorname{sgn}(F^q(\omega^{\bar{\sigma}})) = \pi^q \bar{\sigma}.$$

This completes the proof. □

Consequently, we have

Corollary 3.1: For any $p \mid K+L$ and any $\sigma \in \Sigma_p$, we have

$$\omega(n, \omega^\sigma) \sim \omega(n, \omega^{\pi\sigma}) \sim \cdots \sim \omega\left(n, \omega^{\pi^{p-1}\sigma}\right).$$

If we use $n(p)$ to denote the number of all p-periodic orbits of (2.2) (and thus that of (1.1)), then we have

Theorem 3.3: $\forall p \in \mathbb{N}$ with $p \mid K+L$,

$$n(p) = \frac{N(\Sigma_p)}{p}.$$

Proof: The proof follows immediately from the definition of $N(\Sigma_p)$ and Corollary 3.1.

Remark 3.2: The number of all periodic orbits of (1.1) is

$$n(K+L) = \sum_{p \mid K+L} n(p).$$

The related numbers for $N(\Sigma_p)$, $n(p)$ and $n(K+L)$ are given in Tables I and II.

IV. Discussions

We have shown that the delayed discrete-time BAM neural network (1.1) can admit $\sum_{p \mid K+L} n(p)$ stable periodic solutions. We have also investigated the relation between the number of periodic solutions and the sum of all delays $(K+L)$ and discussed the multistability of those periodic solutions. This shows that (1.1) is a network model admitting large capacity of stable periodic solutions and thus, has great potential for applications in associative memories of periodic patterns.

Note that for a simple two-neuron discrete-time neural network with delayed feedback, [15], [17] and [18] discussed the existence and stability of periodic solutions, and [15] also showed the large capacity of periodic solutions. However, in their models, there are just a few parameters, and thus as pointed out in [15], it is hard to train the network to store the large number of stable periodic solutions. In contrast, many parameters are adopted in (1.1), which can be used to train the network to have the ability to generate a large number of stable periodic solutions so that the network can serve the purpose storing large number of content-addressable memories or patterns.

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