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Rectifying Transformations for the Integration of Rational Trigonometric Functions

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The paper presents four rectifying transformations that can be applied to the integration of a real rational expression of trigonometric functions. Integration is with respect to a real variable. The transformations remove, from the real line, discontinuities and singularities that would otherwise appear. If the integration is with respect to a complex variable, the transformations remain valid. In that case, they move singularities from the real line to elsewhere in the complex plane.

1. Introduction

Let $\psi, \phi \in \mathbb{R}[x, y]$ be polynomials over $\mathbb{R}$, the field of real numbers. A rational trigonometric function over $\mathbb{R}$ is a function of the form

$$T(\sin z, \cos z) = \frac{\psi(\sin z, \cos z)}{\phi(\sin z, \cos z)} \ . \tag{1.1}$$

The problem considered here is the integration of such a function with respect to a real variable, in other words, to evaluate $\int T(\sin x, \cos x)\ \mathrm{d}x$ with $x \in \mathbb{R}$. The particular point of interest lies in the continuity properties of the expression obtained for the integral. General discussions of the existence of discontinuities in expressions for integrals have been given by Jeffrey (1993) and Jeffrey (1994). Those discussions were restricted to integration with respect to a real variable, but it is useful to extend them to the complex plane for the following reason. Given a function $f : \mathbb{R} \to \mathbb{R}$, and an integration problem $\int f(x) dx$ with $x \in \mathbb{R}$, almost all computer algebra systems (CAS) require that this problem actually be addressed as $\int f(z) dz$ for $z \in \mathbb{C}$ (the field of complex numbers), whether the user requires this generality or not. Therefore, we begin by reconsidering an example that has been previously discussed in the context of real variables (Jeffrey and Rich 1994), namely,

$$I_1(z) = \int \frac{3\ \mathrm{d}z}{5 - 4\cos z} = 2\arctan(3\tan\tfrac{1}{2}z) \ . \tag{1.2}$$

For $z \in \mathbb{C}$, consider the singularities and branch cuts possessed by the quantities appearing in (1.2). By periodicity, a discussion restricted to $-\pi \le \Re(z) \le \pi$ suffices, where

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$\Re(z)$ is the real part of z. The integrand has simple poles when $\cos z = \frac{5}{4}$, at the points $z = \pm i \ln 2$. Corresponding logarithmic singularities must therefore be present in the expression on the right-hand side of (1.2). This is verified by observing that arctangent has logarithmic singularities at $\pm i$ and that $3 \tan \frac{1}{2}(\pm i \ln 2) = \pm i$. Also, since it is standard for arctangent to have branch cuts from $\pm i$ along the lines $\{\pm iy, y \in [1, \infty)\}$, and since $3 \tan \frac{1}{2}(\pm iy) = \pm 3i \tanh \frac{1}{2} y$, the right side of (1.2) has branch cuts along the set of lines $L_1 = \{\pm iy, y \geq \ln 2\}$. These branch cuts, being associated with the logarithmic singularities, are unavoidable. However, there are other singularities and branch cuts present in the same expression. Since $3 \tan \frac{1}{2}(\pi + iy) = 3i \coth \frac{1}{2} y$, the right-hand side of (1.2) has singularities at $\pm\pi$ and branch cuts extending from these along the set of lines $L_2 = \{\pm\pi + iy, y \in R\}$. Figure 1 shows this information graphically; the singularities are indicated by triangles and diamonds, the branch cuts L_1 are shown using heavy wavy lines, and the branch cuts L_2 are shown using hatched lines. The figure uses periodicity to extend beyond the interval discussed above.

Now consider the equation

$$\hat{I}_1(z) = \int \frac{3\, \mathrm{d}z}{5 - 4\cos z} = z + 2 \arctan \frac{\sin z}{2 - \cos z} \ . \tag{1.3}$$

The right side of this equation has singularities at the points $z = \pm i \ln 2$ and branch cuts along the lines L_1, but it does not have singularities at $\pm\pi$, nor branch cuts along L_2. Therefore, if the integral were to be evaluated along a contour that cut one or more lines in L_2, the evaluation would be more efficient and reliable using (1.3). For contours along the real axis, (1.3) satisfies the definition of an integral on the real domain of maximum extent (Jeffrey 1993). The present paper gives an algorithm for obtaining (1.3) in place of (1.2).

Other examples show different forms of the problem. Consider

$$I_2(z) = \int \frac{(\cos z + 2\sin z + 1)\, \mathrm{d}z}{\cos^2 z - 2 \sin z \cos z + 2 \sin z + 3} = \arctan(\tan^2 \tfrac{1}{2} z + \tan \tfrac{1}{2} z) \ . \tag{1.4}$$

In the complex plane (again using periodicity to simplify the discussion), the integrand has poles at $z = 2 \arctan\left(-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 \pm 4i}\right)$. The right-hand side of (1.4) has logarithmic singularities at the same points, and in addition singularities at $z = \pm\pi$. The lines $\{2 \arctan\left(-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 \pm 4iy}\right)$, for $y \in [1, \infty)\}$ are branch cuts running between the singularities. For integrals along the real line the singularities appear as removable discontinuities. Thus any definite integral along the real line with an end point at $(2n+1)\pi$, for n an integer, must be evaluated using the fact that $\lim_{z \to (2n+1)\pi} I_2(z) = \frac{1}{2}\pi$.

Next, consider

$$\begin{aligned} I_3(z) &= \int \frac{(2 + 5 \sin z + \cos z)\, \mathrm{d}z}{4 \cos z + \sin z \cos z - 2 \sin z - 2 \sin^2 z} \\ &= -\ln(2 \tan \tfrac{1}{2} z - 1) - \ln(1 + \tan \tfrac{1}{2} z) + \ln(\tan^2 \tfrac{1}{2} z + \tan \tfrac{1}{2} z + 2) \ . \end{aligned} \tag{1.5}$$

The right side of (1.5) contains singularities at $z = -\frac{1}{2}\pi$ and $z = 2 \arctan \frac{1}{2}$, corresponding to singularities in the integrand. It also contains singularities at $z = \pm\pi$, which are avoidable. For integrals along the real line, these singularities appear as spurious discontinuities in the imaginary part of the integral at $\pm\pi$. For example, $I_3(\pi)$ is undefined, while $\lim_{z \to \pi+} I_3(z) = -\ln 2 - 2\pi i$ but $\lim_{z \to \pi-} I_3(z) = -\ln 2$. Algorithms are given here that remove the avoidable singularities.

An algorithm that avoids spurious discontinuities when integrating trigonometric func-

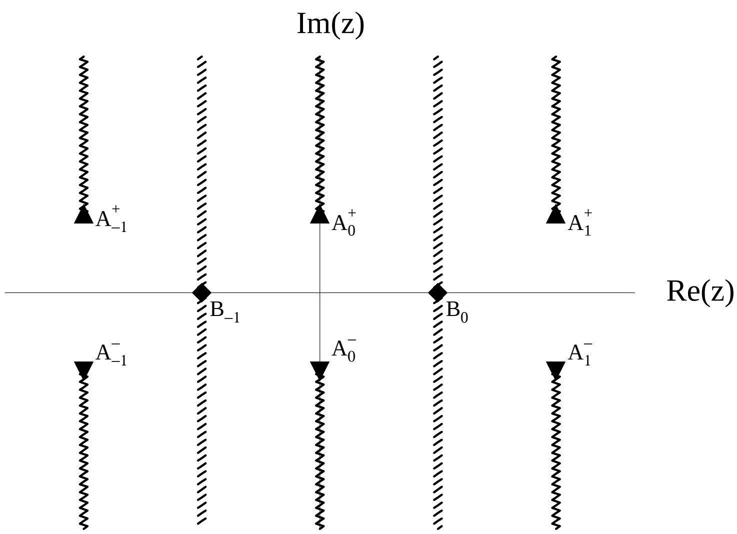

Figure 1. The branch cuts of (1.2) in the complex plane, indicated by the wavy and hatched lines. The points $A_n^{\pm}$ are at $2n\pi \pm i \ln 2$ and correspond to poles in the integrand. The branch cuts extending from them are labelled L_1 in the text. The points B_n are at $(2n+1)\pi$ and are singular points of (1.2) but have no corresponding singularity in the integrand. The branch cuts extending from them are called L_2 in the text.

tions was given by Jeffrey and Rich (1994), and applies to integrals with respect to a real variable. It uses the Weierstrass family of substitutions to convert a trigonometric integral into an integral of a rational function. The evaluation of this integral, followed by substitution, gives an antiderivative for the original integrand in a form that might contain a periodic discontinuity. The size of the possible discontinuity is obtained from a limit calculation and a final expression for the integral is then constructed by adding a suitable multiple of a floor function to the integral. For example, the antiderivative in (1.2) was shown above to contain discontinuities. The algorithm replaces it, for $x \in \mathbb{R}$, with

$$\int \frac{3\,\mathrm{d}x}{5 - 4\cos x} = 2\arctan(3\tan\tfrac{1}{2}x) + 2\pi\left\lfloor\frac{x+\pi}{2\pi}\right\rfloor . \tag{1.6}$$

The reasons for wishing to improve on this algorithm are as follows. First, the floor function is defined only for real arguments, and therefore (1.6) cannot be used in the complex plane. Second, there are removable discontinuities in (1.6) at $x = (2n+1)\pi$, and therefore a numerical evaluation at these points must use a limit. Third, obtaining a continuous function by combining discontinuous functions is less elegant than obtaining it by combining continuous functions. Fourth, the limit computations are time consuming and a method that avoids these will be more efficient.

2. Rectifying transformations and a new algorithm

The idea of a rectifying transformation was introduced by Jeffrey (1993), but a definition was not given.

Definition: Let $\mathbb{D} \subset \mathbb{C}$ be a domain in the complex plane. Let a function $f : \mathbb{D} \to \mathbb{C}$ be defined and integrable everywhere on $\mathbb{D}$ except on a set of isolated singular points $\mathbb{S}_f$. Let $F, G : \mathbb{D} \to \mathbb{C}$ be primitives of f, which is to say, their derivatives satisfy $F' = G' = f$, and further let them have sets of singular points $\mathbb{S}_F$ and $\mathbb{S}_G$ respectively. Necessarily, we have $\mathbb{S}_F, \mathbb{S}_G \supseteq \mathbb{S}_f$. A transformation $\mathcal{T}$ is a rectifying transformation on $\mathbb{D}$ if it has the properties that $\mathcal{T}(F) = G$ and $\mathbb{S}_G \subset \mathbb{S}_F$; if $\mathbb{S}_G = \mathbb{S}_F$ then the transformation is neutral, and if $\mathbb{S}_G \supset \mathbb{S}_F$ then the transformation is exacerbating.
Example: It was shown by Jeffrey (1993) that given functions $A, B : \mathbb{R} \to \mathbb{R}$, and the integrand $f = A'/A + B'/B$, the transformation $\mathcal{T}(\ln A + \ln B) = \ln(AB)$ can be rectifying or neutral on $\mathbb{R}$. In the particular case in which $A = \sin x$ and $B = \csc x - \cot x$, and thus $f = \cot \frac{1}{2}x$, it is rectifying. In the case $A = 1/x$ and $B = 1/(1+x)$, it is neutral. By reversing the roles of F and G, a transformation that is neutral or exacerbating is obtained.
Remark: Typically, the transformation $\mathcal{T}$ will be defined only on a specific and narrow class of functions.

The intention is that a rectifying transformation is used within an integration procedure after a primitive, or antiderivative, has been computed for a given integrand. The transformation then finds an equivalent antiderivative with better global properties. Therefore, before a description is given of the new transformations, the algorithm begins by reviewing an existing method for obtaining antiderivatives of trigonometric functions.

THEOREM 2.1. *Let $\phi, \psi \in \mathbb{Q}[x, y]$ be polynomials, where $\mathbb{Q}$ denotes the field of rational numbers, and let $T(\sin z, \cos z) = \phi(\sin z, \cos z)/\psi(\sin z, \cos z)$ be a rational trigonometric function over $\mathbb{Q}$. An integral of T can be computed in the general form*

$$\int T(\sin z, \cos z)\ \mathrm{d}z = R(u) + \sum_i \alpha_i \ln \nu_i(u) + \sum_j \beta_j \arctan P_j(u) \ , \tag{2.1}$$

where $u = \tan \frac{1}{2}z$, $R \in \mathbb{Q}^*(u)$ is a rational function, $\nu_i, P_j \in \mathbb{Q}^*[u]$, where $\mathbb{Q}^* \subset \mathbb{R}$ is an extension of $\mathbb{Q}$, and $\alpha_i, \beta_j \in \mathbb{Q}^*$. Moreover, the ν_i are monic, and the $P_j(u)$ have positive leading coefficients.

PROOF. The standard substitution $u = \tan \frac{1}{2}z$, sometimes attributed to Weierstrass, converts the integral of T into an integral of a rational function in u. Write this as

$$\int T(\sin z, \cos z)\ \mathrm{d}z = \int T\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \frac{2}{1+u^2}\ \mathrm{d}u = \int f(u)\ \mathrm{d}u \ .$$

Algorithms 11.1 and 11.2 (Hermite–Horowitz) in Geddes *et al.* (1992) reduce the integral to a rational part and a logarithmic part.

$$\int f(u) du = R(u) + \int \frac{a(u)}{b(u)}\ \mathrm{d}u \ ,$$

where a and b are polynomials. Algorithms 11.3 and 11.4 (Rothstein–Trager–Lazard) in Geddes *et al.* (1992) evaluate the logarithmic part as

$$\int \frac{a(u)}{b(u)} du = \sum_i \alpha_i \ln \nu_i \ , \tag{2.2}$$

where the ν_i are polynomials in u, possibly with complex coefficients. Let the functions ν_i be ordered so that those with purely real coefficients are numbered from 1 to m, for some m; these logarithms are then left unchanged. For $i > m$, the logarithms containing coefficients with nonzero imaginary parts are converted to arctangents using the rectifying transformation due to Rioboo (1991) that is described by Bronstein (1996) and also outlined by Geddes *et al.* (1992) in their exercise 11.18. By using the identity $\arctan(-f) = -\arctan f$, one can ensure that the leading coefficient of each P_j is positive, albeit, determining the sign of the leading coefficient might require considerable effort. □

Remark: The rectifying transformations that are now described are independent of theorem 2.1, but they are most usefully applied to expressions such as (2.1), whether obtained by the method given in the proof, or some other. In particular, some CAS may obtain (2.2) with a partial fraction computation. Also, the theorem is stated for functions over $\mathbb{Q}$, but in many cases, a CAS will be able to integrate functions defined over extensions of $\mathbb{Q}$.

LEMMA 2.1.1. *For a polynomial $P \in \mathbb{R}[u]$ and u as above, define the transformation $\mathcal{E}$ by*

$$\mathcal{E}P(z) = \mathcal{E}[P(u)] = (\cos \tfrac{1}{2}z)^{\deg P} P(\tan \tfrac{1}{2}z) \ . \tag{2.3}$$

Then $\mathcal{E}P(2n\pi + \pi)$ is finite and $\mathcal{E}P(z)$ has the same zeros on $\mathbb{C}$ as $P(\tan \frac{1}{2}z)$.

PROOF. By periodicity, only the domain $-\pi < \Re z \leq \pi$ need be considered. Let the polynomial $P(u) = \sum^m p_i u^i$, where $m = \deg P$. Then $\mathcal{E}P(\pi) = p_m \neq 0$. Further, $P(\tan \frac{1}{2}\pi) \neq 0$ because it is unbounded, and so neither P nor $\mathcal{E}P$ is zero at $z = \pi$. Since $z \neq \pi$ implies $\cos \frac{1}{2}z \neq 0$, then $P(\tan \frac{1}{2}z) = 0 \iff \mathcal{E}P(z) = 0$. □

In terms of this transformation, we now give the rectifying transformations that are the main results of this paper. First the logarithmic terms in (2.1) are considered.

THEOREM 2.2. *For monic polynomials $\nu_i \in \mathbb{R}[u]$, let a primitive $F(z) = \sum \alpha_i \ln \nu_i(u)$ with $u = \tan \frac{1}{2}z$ as before. Then the transformation*

$$\mathcal{T}_1 F(z) = \mathcal{T}_1 \left[\sum \alpha_i \ln \nu_i(u)\right] = \sum \alpha_i \ln \mathcal{E}\nu_i(z) - \sum \alpha_i \deg \nu_i \ln \cos \tfrac{1}{2}z \tag{2.4}$$

is rectifying on $\mathbb{C}$ if $\sum \alpha_i \deg \nu_i = 0$ and neutral otherwise.

PROOF. By periodicity, $\mathcal{T}_1$ is rectifying on $\mathbb{C}$ if it is rectifying when $-\pi < \Re z \leq \pi$. Straightforward computation shows that

$$\frac{d}{dz}F(z) = \frac{d}{dz}\mathcal{T}_1 F(z) \ .$$

For each i, as $z \to \pi$, $\nu_i(\tan \frac{1}{2}z) \to (\pi - z)^{-\deg \nu_i}$, because ν_i is monic. Therefore $F(z)$ is singular at $x = \pi$. However, if $\sum \alpha_i \deg \nu_i = 0$, then $\mathcal{T}_1 F(\pi)$ is finite, because by the lemma each $\mathcal{E}\nu_i(\pi)$ is finite. The zeros of $\nu_i(z)$ are the same as the zeros of $\mathcal{E}\nu_i(z)$, and so $\mathcal{T}_1 F$ and F have the same singular points in $\mathbb{C}$. Therefore, if $\sum \alpha_i \deg \nu_i = 0$, then $\mathbb{S}_{\mathcal{T}_1 F} = \mathbb{S}_F \backslash \{\pi\}$, and otherwise the two sets are equal. □

THEOREM 2.3. *Let $P(u)$ be an even-degree polynomial. Let $F(z) = \arctan P(u)$ with $u = \tan \frac{1}{2}z$. The transformation*

$$\mathcal{T}_2 F(z) = \mathcal{T}_2[\arctan P(u)] = \arctan \frac{k\mathcal{E}P(z) - (\cos\frac{1}{2}z)^{\deg P}}{k(\cos\frac{1}{2}z)^{\deg P} + \mathcal{E}P(z)} , \tag{2.5}$$

where k is chosen so that $(\forall u \in \mathbb{R})(P(u) + k) > 0$, is rectifying on $\mathbb{R}$.

PROOF. Periodicity is again understood. A lengthy, but straightforward, calculation shows

$$\frac{d}{dz}F(z) = \frac{d}{dz}\mathcal{T}_2 F(z) .$$

Further, since $\tan\frac{1}{2}\pi$ is undefined, $F(\pi)$ is undefined, but $\mathcal{T}_2 F(\pi) = \arctan k$. Since, by construction, $k + P(u) > 0$, then $\mathbb{S}_F = \{\pi\}$ and $\mathbb{S}_{\mathcal{T}_2 F} = \emptyset$.

A constructive proof is also given because it helps in understanding the nature of the transformation. Since a constant can always be added to an integral, the constant $-\arctan(1/k)$ is added to the expression $\arctan P$ obtained from the integration procedure:

$$\arctan P(u) - \arctan(1/k) = \arctan \frac{P - 1/k}{1 + P/k} .$$

The identity for adding arctangents is only true if the denominator remains positive; hence the need to choose k as shown. □

For odd degree polynomials, it is useful to have two transformations. First, the special case of a linear polynomial is given separate consideration, because it allows simplifications that cannot be used in the general case, but which are useful enough to warrant special attention.

THEOREM 2.4. *Let $F(z) = \arctan(au + b)$ with $u = \tan\frac{1}{2}z$, and $a, b \in \mathbb{R}$ and also $a > 0$. The transformation*

$$\mathcal{T}_3 F(z) = \mathcal{T}_3[\arctan(au+b)] = \frac{z}{2} + \arctan \frac{2ab\cos z - (1 + b^2 - a^2)\sin z}{(1+a)^2 + b^2 + (1 + b^2 - a^2)\cos z + 2ab\sin z} , \tag{2.6}$$

is rectifying in $\mathbb{R}$.

PROOF. The same approach as above shows that the derivatives before and after transformation are equal. Since $F(z)$ is singular at $z = \pi$, but $\mathcal{T}_3 F(\pi) = \frac{1}{2}\pi - \arctan(b/(a+1))$, a singularity is removed by $\mathcal{T}_3$. Further, since $[(1+a)^2 + b^2]^2 > (1 + b^2 - a^2)^2 + 4a^2b^2$, there are no other discontinuities.

More insight into the transformation is obtained if a constructive proof is considered. The expression $\frac{1}{2}z - \arctan u$ differentiates formally to zero, since formally $\arctan\tan\frac{1}{2}z = \frac{1}{2}z = \arctan u$. Therefore this is added to $\arctan(au + b)$ and the arctangents combined.

$$\arctan(au + b) - \arctan u + \tfrac{1}{2}z = \arctan \frac{(a-1)u + b}{1 + au^2 + bu} + \tfrac{1}{2}z ,$$

provided the denominator is positive. In general, though, the right side is not continuous because the denominator can change sign. Therefore a constant is subtracted from it.

$$\frac{z}{2} + \arctan\frac{(a-1)u+b}{1+au^2+bu} - \arctan K = \frac{z}{2} + \arctan\frac{b-K+(a-1-Kb)u-Kau^2}{1+Kb+(Ka-K+b)u+au^2} \; .$$

A value of K must now be found that ensures that the denominator is positive. The minimum value of the denominator is given by

$$\inf_{u\in\mathbb{R}}\left(1+Kb+(Ka-K+b)u+au^2\right) = 1+Kb-(Ka-K+b)^2/4a \; ,$$

and this must be greater than zero. The largest value of the infimum is obtained by setting $K = b(1+a)/(1-a)^2$, but using this value leads to a more complicated expression than the one in (2.6). The infimum is also positive for $K = b/(1+a)$ and using this leads to (2.6), after using $u = \sin z/(1+\cos z)$. □

We finally come to the general odd-degree case. The technique just used, of combining the arctangent with a constant, can be shown by counter example to fail. Thus the straightforward generalization would be to try to rectify an odd degree poynomial $P(u)$ by writing

$$\arctan P(u) - \arctan u - \arctan K = \arctan\frac{P-u-K(1+uP)}{1+uP+K(P-u)} \; .$$

Consider, however, the example $P = \frac{1}{10}u^3 - 4u$. The denominator then becomes equal to $\frac{1}{10}u^4 + \frac{1}{10}Ku^3 - 4u^2 - 5Ku + 1$. No value of K exists that makes this positive for all u. For example, the expression is always negative at $u = \frac{1}{10} - \frac{5}{8}K + \frac{1}{8}\sqrt{25K^2+16}$, if $K \geq 0$, and at a similar place if $K < 0$. Hence a more elaborate construction is needed.

THEOREM 2.5. *Let $P \in \mathbb{R}[u]$ be an odd degree polynomial with positive leading coefficient. If $F(z) = \arctan P(u)$ as before, then the transformation $\mathcal{T}_4$ defined by*

$$\mathcal{T}_4 F(z) = \frac{z}{2} + \arctan\frac{k\cos\frac{1}{2}z\mathcal{E}P - (\cos\frac{1}{2}z)^{\deg P}\mathcal{E}[u]}{k(\cos\frac{1}{2}z)^{1+\deg P} + \mathcal{E}[uP]} + \arctan\frac{(1-k)\cos\frac{1}{2}z\mathcal{E}u}{k(\cos\frac{1}{2}z)^2 + \mathcal{E}[u^2]} \; , \quad (2.7)$$

where $k > 0$ is chosen so that $(\forall u \in \mathbb{R})(k+uP) > 0$, is rectifying on $\mathbb{R}$.

PROOF. The formal proof follows the same path as those above and is not written out. The constructive proof starts by adding $\frac{1}{2}z - \arctan u$ to the arctangent, but simply combining the arctangents fails because

$$\arctan P - \arctan u + \tfrac{1}{2}z \neq \arctan\frac{P-u}{1+uP} + \tfrac{1}{2}z \; ,$$

since $1 + uP$ can change sign in general. Therefore, the transformation is written

$$\arctan P - \arctan u/k + \arctan u/k - \arctan u + \tfrac{1}{2}z \; .$$

Combining the terms pairwise gives

$$\arctan\frac{P-u/k}{1+uP/k} + \arctan\frac{u/k-u}{1+u^2/k} + \tfrac{1}{2}z \; ,$$

and provided k is chosen so that $uP + k > 0$, the expression in the theorem is obtained. □

3. Implementation and examples

The algorithm is now summarized.

1 The system identifies an integrand as rational trigonometric over $\mathbb{R}$ (in variable z let us say) and tries any preferred simpler evaluation strategies, for example, a sine or cosine substitution.
2 The system substitutes $u = \tan \frac{1}{2}z$ and passes the resulting rational function to its integrator. This step can be tried even if the coefficients are in $\mathbb{R}$ rather than $\mathbb{Q}$, because the integrator may succeed.
3 If an expression is returned in the form (2.1), then the algorithm can proceed. If it is not of this form (for example, it contains complex logarithms or arctangents of rational functions), then the algorithm fails.
4 For each term matching one of the four patterns, the rectifying transform is applied. Notice that the application of the transformation does not require any analysis or knowledge of the whole integral expression, but can be applied immediately upon recognising the pattern.
5 Any residual terms in u are returned to z.

This algorithm is now applied to the examples given in the introduction and to other examples. Starting with (1.2), the transformation $\mathcal{T}_3$ gives (1.3), by simple substitution in (2.6). To apply $\mathcal{T}_2$ to (1.4), a value $k > -\inf(u^2+u) = \frac{1}{4}$ must be chosen. Taking $k = 1$ gives

$$\mathcal{T}_2 I_2 = \mathcal{T}_2 \arctan(u^2+u) = \arctan \frac{\sin^2 \frac{1}{2}z + \sin \frac{1}{2}z \cos \frac{1}{2}z - \cos^2 \frac{1}{2}z}{\sin^2 \frac{1}{2}z + \sin \frac{1}{2}z \cos \frac{1}{2}z + \cos^2 \frac{1}{2}z} \ . \tag{3.1}$$

Combining half-angles simplifies this to

$$\mathcal{T}_2 I_2 = \arctan \frac{\sin z - 2\cos z}{2 + \sin z} \ ,$$

which is actually no more complicated than the original expression (1.4). Using this expression, we have $\mathcal{T}_2 I_2(\pi) = \frac{1}{4}\pi$, where before we had only the limit $\lim_{z\to\pi} I_2(z) = \frac{1}{2}\pi$. The difference between the two values is the constant that the transformation adds to the expression.

Applying $\mathcal{T}_1$ to (1.5) gives

$$\begin{aligned} \mathcal{T}_1 I_3 &= \mathcal{T}_1[\ln(\tan^2 \tfrac{1}{2}z + \tan \tfrac{1}{2}z + 2) - \ln(2\tan \tfrac{1}{2}z - 1) - \ln(1 + \tan \tfrac{1}{2}z)] \\ &= \ln(\sin^2 \tfrac{1}{2}z + \sin \tfrac{1}{2}z \cos \tfrac{1}{2}z + 2\cos^2 \tfrac{1}{2}z) \\ &\qquad - \ln(2\sin \tfrac{1}{2}z - \cos \tfrac{1}{2}z) - \ln(\cos \tfrac{1}{2}z + \sin \tfrac{1}{2}z) \ . \end{aligned} \tag{3.2}$$

In contrast to the properties of I_3 at π, which were described above, the new expression obeys $\mathcal{T}_1 I_3(\pi) = \lim_{z\to\pi} \mathcal{T}_1 I_3(z) = -\ln 2$. Therefore the discontinuity at $z = \pi$ has been removed, as has the one at $-\pi$. The logarithmic terms can also be combined if desired, and there will be no change in the continuity properties of the expression. The resulting expression can then be significantly simplified:

$$\mathcal{T}_1 I_3 = \ln \frac{3 + \sin z + \cos z}{1 + \sin z - 3\cos z} \ .$$

The opening examples do not cover all the points of interest, so now we continue with more examples. Applying $\mathcal{T}_1$ to the following integral results in a singularity being still present at $z = \pi$.

$$\int \frac{(7\cos z + 2\sin z + 3)\,\mathrm{d}z}{3\cos^2 z - \sin z\cos z + 4\cos z - 5\sin z + 1} = \ln\frac{3 + \sin z + \cos z}{\cos x - 2\sin x + 1}\ . \tag{3.3}$$

The singularity cannot be removed because it is induced by a pole in the integrand. Even so, the integral is in a form that is more convenient for further analysis than the original expression. Therefore, a CAS would do well to implement the transformation, independently of whether the transformation is rectifying. In this regard, it can be noticed that the transformation is linear in its argument, and even though theorem 2.2 was stated for a sum of logarithms, the transformation can be applied independently to each logarithm. Also, the polynomial arguments of the logarithms do not have to be monic, although again that is how the theorem was stated.

An example of the application of $\mathcal{T}_4$ is

$$\int \frac{(5\cos^2 z + 4\cos z - 1)\,\mathrm{d}z}{4\cos^3 z - 3\cos^2 z - 4\cos z - 1} = 2\arctan(\tan^3\tfrac{1}{2}z - 2\tan\tfrac{1}{2}z)\ .$$

A value of $k > -\inf(u^4 - 2u^2) = 1$ must be selected. Since k cannot equal 1, the obvious choice is 2. This gives

$$\int \frac{(5\cos^2 z + 4\cos z - 1)\,\mathrm{d}z}{4\cos^3 z - 3\cos^2 z - 4\cos z - 1} = z - 2\arctan\frac{7\cos z\sin z + 3\sin z}{5\cos^2 z + 2\cos z + 1} - 2\arctan\frac{\sin z}{3 + \cos z}.$$

If it is possible to choose $k = 1$, this is clearly the best choice, since one of the arctangent terms is then removed, as the next example shows. We have

$$\int \frac{(7\cos^2 z + 2\cos z - 5)\,\mathrm{d}z}{4\cos^3 z - 9\cos^2 z + 2\cos z - 1} = \arctan(2\arctan^3\tfrac{1}{2}z - \arctan\tfrac{1}{2}z)\ .$$

The minimum of $2u^4 - u^2$ is $-\frac{1}{8}$ and therefore $k = 1$ is possible, giving

$$\int \frac{(7\cos^2 z + 2\cos z - 5)\,\mathrm{d}z}{4\cos^3 z - 9\cos^2 z + 2\cos z - 1} = z - 2\arctan\frac{2\sin z\cos z}{2\cos^2 z - \cos z + 1}\ .$$

Jeffrey and Rich (1994) gave the example

$$\int \frac{3\,\mathrm{d}z}{5 + 4\sin z} = 2\arctan(3\tan[\tfrac{1}{2}z + \tfrac{1}{4}\pi]) \tag{3.4}$$

$$= 2\arctan(\tfrac{5}{3}\tan\tfrac{1}{2}z + \tfrac{4}{3}) \tag{3.5}$$

to show that sometimes the substitution $u = \tan(\frac{1}{2}z + \frac{1}{4}\pi)$ gives tidier results. This suggests that there is a need for additional transformations to handle such variations of the basic idea. If, however, one goes to the trouble of developing a rectifying transformation for (3.4), let us temporarily call it $\mathcal{T}_5$, one finds

$$\mathcal{T}_5\left[2\arctan\left(3\tan[\tfrac{1}{2}z + \tfrac{1}{4}\pi]\right)\right] = \mathcal{T}_3[2\arctan(\tfrac{5}{3}\tan\tfrac{1}{2}z + \tfrac{4}{3})] = z + 2\arctan\frac{\cos z}{2 + \sin z}\ .$$

Because of this, only the substitution $u = \tan\frac{1}{2}z$ was considered here, and at present there is no reason to pursue variants.

The question of appropriate substitution arises also in the example

$$\int \frac{2\,\mathrm{d}z}{1 + \cos^2 z} = \sqrt{2}z + \arctan\frac{\cos z}{\sin z + \sqrt{2} + 1} + \arctan\frac{\cos z}{\sin z - \sqrt{2} - 1}\ . \tag{3.6}$$

Here the $\mathcal{T}_3$ transformation has already been applied. If the substitution $u = \tan z$ is used instead of $u = \tan \frac{1}{2}z$, only a single arctangent is needed. Within the present scheme, the same effect is achieved by first making the preliminary transformation

$$\int \frac{2\,\mathrm{d}z}{1+\cos^2 z} = \int \frac{4\,\mathrm{d}z}{3+\cos 2z} = \int \frac{2\,\mathrm{d}\hat{z}}{3+\cos \hat{z}}\ , \tag{3.7}$$

where $\hat{z} = 2z$. The computation continues

$$\int \frac{2\,\mathrm{d}\hat{z}}{3+\cos \hat{z}} = \frac{\hat{z}}{\sqrt{2}} - \sqrt{2}\arctan \frac{\sin \hat{z}}{\cos \hat{z} + 2\sqrt{2} + 3}\ ,$$

so that finally the replacement for (3.6) is

$$\int \frac{2\,\mathrm{d}z}{1+\cos^2 z} = \sqrt{2}z - \sqrt{2}\arctan \frac{\sin 2z}{\cos 2z + 2\sqrt{2} + 3}\ .$$

The identification of the transformation (3.7), however, is not a subject of this paper, which focuses on the rectifying transforms $\mathcal{T}_i$ derived here. The point of the example is that the transformations given here are sufficient for obtaining the best known solution, provided the problem is correctly handled.

All of the transformations can be applied to integrals containing symbolic parameters. A generalization of the first example is

$$\int \frac{\mathrm{d}z}{p + q\cos z + r\sin z} \to \frac{2}{\Delta}\arctan \frac{(p-q)u + r}{\Delta}\ ,$$

where p, q, r are real and $\Delta = \sqrt{p^2 - q^2 - r^2}$ has been introduced for convenience. Since the transformation assumes a positive leading coefficient, the signum of $p - q$ must be taken to the front of the arctangent. Thus

$$\frac{2\mathrm{sgn}(p-q)}{\Delta}\mathcal{T}_3 \arctan \frac{(p-q)u + r}{\Delta \mathrm{sgn}(p-q)} =$$

$$\frac{x\mathrm{sgn}(p-q)}{\Delta} + \frac{2\mathrm{sgn}(p-q)}{\Delta}\arctan \frac{r\cos x - q\sin x}{p + q\cos x + r\sin x + \mathrm{sgn}(p-q)\Delta}\ . \tag{3.8}$$

The final expression for the integral was quoted without explanation by Jeffrey (1994). Also the signum factor was given as $\mathrm{sgn}p$ instead of $\mathrm{sgn}(p-q)$. This was correct, because of the proviso $p > q$ attached to the formula. The special case $r = 0$ and $q = 1$ was also discussed by Jeffrey and Rich (1994).

4. Concluding remarks

We return to the issue of the behaviour of the transformations in the complex plane. Although the transformations are rectifying on $\mathbb{R}$, they are not rectifying on $\mathbb{C}$. Referring to figure 1, the singular points B_i are not eliminated by transformation $\mathcal{T}_3$, they are moved to the branch cuts L_1, to the point $z = \arccos 2$, in fact. A more typical example of the linear case is obtained from a numerical example of the general formula (3.8). Consider $p = 13$, $q = 3$, $r = 4$. The untransformed and transformed expressions are

$$\int \frac{12\,\mathrm{d}z}{13 + 3\cos z + 4\sin z} = 2\arctan(\tfrac{5}{6}\tan \tfrac{1}{2}z + \tfrac{1}{3}) = z + 2\arctan \frac{4\cos z - 3\sin z}{25 + 3\cos z + 4\sin z}\ .$$

The untransformed integral has a branch cut running between the two singularities located at $B_{\pm} = \arctan(4/3) - \pi \pm i\ln 5$ through the singularity at $-\pi$. The transformed integral has branch cuts running parallel to the imaginary axis from the points $B_\pm$ to infinity, through the moved singularities now at $\arctan(4/3) - \pi \pm i\ln(2\sqrt{6}+5)$.

The branch cuts of (3.1) show similar properties. The branch cuts of (1.4) were described in the introduction, and it was because they met the real axis at the singularity that the integral on the real axis had removable discontinuities. The branch cuts of the transformed expression do not cross the axis, but join the singularity that is at $2\arctan\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4i}\right)$ to the singularity at $2\arctan\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1-4i}\right)$, passing through the moved singularity at $\frac{1}{2}\pi + i\ln(\sqrt{3}+2)$, all 3 points lying on the same side of the real axis. Similarly with the complex conjugates.

This paper has assumed that the basic integration of rational trigonometric functions is already a standard component of a CAS. The contribution of this paper has been the rectifying transformations, which can be thought of as being independent of the integration procedure used, although this is only partly true, since they cannot be applied unless the basic integration is performed correctly. In addition, the final expressions may not be in the simplest possible form unless the underlying integrator chooses the best approach.

The algorithm here does not completely replace that described by Jeffrey and Rich (1994). Several of the cases that can be handled by that algorithm are not covered here, for example the integral of $\sqrt{1+\cos x}$. Where there is an overlap, however, this method gives an expression that has better properties, particularly in the eyes of those who levelled the criticisms listed after (1.6) at the earlier method.

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