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The Evaluation of Trigonometric Integrals Avoiding Spurious Discontinuities

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The $\tan(x/2)$ substitution, also called the Weierstrass substitution, is one method currently used by computer algebra systems for the evaluation of trigonometric integrals. The method needs to be improved, because the expressions obtained using it sometimes contain discontinuities, which unnecessarily limit the domains over which the expressions are correct. We show that the discontinuities are spurious in the following sense. Given an integrand and an expression for its antiderivative that was obtained by the Weierstrass substitution, a better expression can be found which is continuous on wider intervals than the first expression, and yet is still an antiderivative of the integrand. The origin of the discontinuities is identified, and an algorithm is presented for automatically finding the improved type of antiderivative. The new algorithm also enlarges the set of functions that can be used in the substitution. The algorithm works by first evaluating the given integral using the Weierstrass substitution in the usual way, and then removing any spurious discontinuities present in the antiderivative.

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1. INTRODUCTION.

Computer algebra systems integrate expressions that contain trigonometric functions using a variety of algorithms, including table look-up, the Risch algorithm, and substitutions such as those in section 2.50 of Gradshteyn and Ryzhik [1979]. One standard substitution used by all systems is $u = \tan\frac{1}{2}x$ (assuming the integrand is a function of x), first suggested by Weierstrass [Stewart 1989]. This substitution converts a rational function of $\sin x$ and $\cos x$ into a rational function of u, and the rational function so obtained can be integrated using algorithms such as those by Hermite, Horowitz and Rothstein—Trager [Geddes *et al.* 1992]. Even for integrands that are not rational functions of $\sin x$ and $\cos x$, the transformed integral in u can sometimes be evaluated by the system. Although the substitution is a well-established topic in calculus textbooks, all published accounts of it — and hence almost all implementations in computer algebra systems — are essentially incomplete, because they fail to address discontinuity problems in the antiderivatives obtained.

The unsatisfactory aspect of the existing theory can be quickly established by an example. Consider the problem of integrating $3/(5-4\cos x)$. This function is continuous and positive for all real x, and so its integral should be continuous and monotonically increasing. However, by letting $u=\tan\frac{1}{2}x$, we obtain

$$\int \frac{3\,\mathrm{d}x}{5-4\cos x} = \int \frac{6\,\mathrm{d}u}{1+9u^2} = 2\arctan(3\tan\tfrac{1}{2}x)\ , \tag{1}$$

the constant of integration being taken as understood. This result is unsatisfactory because the right-hand side of (1) is discontinuous at odd multiples of π, in spite of the fact that we just established that the integral should be continuous for all x. This is illustrated in Figure 1, where the right-hand side of (1) is plotted together with the integrand.

Expression (1) is a potential source of error for a computer-algebra system and for its users. For a start, the fact that (1) is discontinuous could easily be missed by a user who is not an experienced mathematician. Furthermore, if (1) were used in a routine unseen by the user, for example, in a differential-equation solver, there would be no warning of possible errors. It is also a source of inefficiency, because the only way to prove that

$$\int_0^{2\pi} \frac{3\,\mathrm{d}x}{5-4\cos x} = 2\pi \tag{2}$$

using (1) is to split the range of integration at $x=\pi$. Several systems, for example Maple V release 2, Mathematica 2.1 and Macsyma 417.125, have special checks in their *definite* integration routines to detect equations such as (2) and give them special treatment, but clearly a more efficient alternative would be to replace (1) with a continuous expression, making the extra code unnecessary.

The issue is not confined to computer algebra systems, because the substitution is the subject of misleading statements in all calculus books. Standard references that give the result (1), or something similar to it, include Gradshteyn and Ryzhik [1979, entry 2.553.3], Burkill [1962], Greenspan *et al.* [1986], Adams [1990], and Thomas and Finney [1984]. None of these books (nor any others known to the authors) points out that the right-hand side of (1) contains discontinuities that require special attention.

The problem addressed by this paper can be posed precisely by stating the algorithm for applying the Weierstrass $\tan(x/2)$ substitution, as it is universally

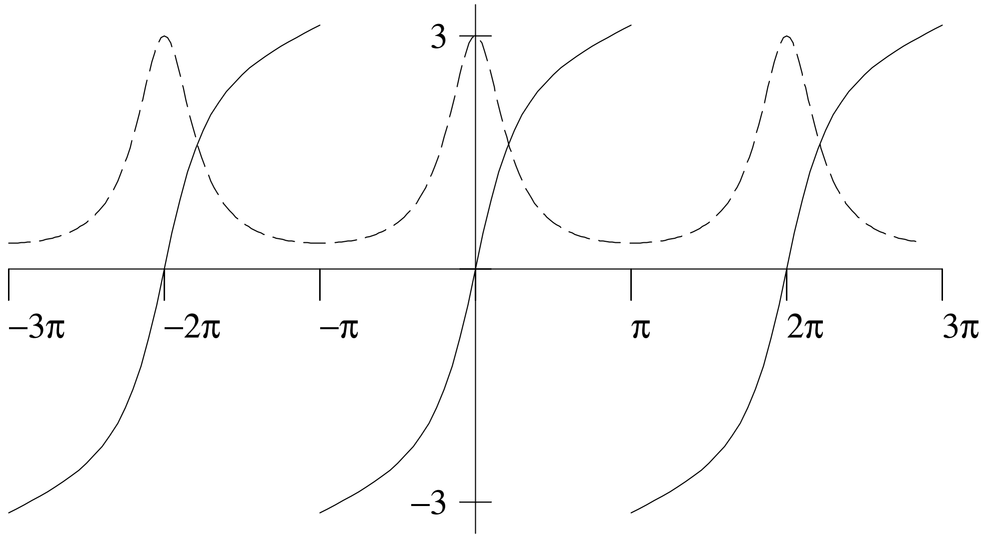

Fig 1. The integrand and integral given in eq. (1). — — —, integrand. ———, integral.

taught in calculus books and commonly implemented by computer algebra systems.

Standard algorithm. Given an integrable function $f(\sin x, \cos x)$ whose indefinite integral is required, make the transformation

$$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \frac{2}{1+u^2}\,\mathrm{d}u.$$

Evaluate the right-hand side using standard algorithms and substitute $u = \tan\frac{1}{2}x$ into the expression obtained to get the final result.

Here we modify this algorithm so that the result does not contain spurious discontinuities (spurious in the sense defined above).

2. AN EXAMPLE OF A CONTINUOUS ANTIDERIVATIVE.

In this section, we introduce the algorithm by showing that we can derive a replacement for (1) that is continuous everywhere on the real line. We first replace the indefinite integral in (1) with a definite integral, because indefinite integrals do not show the domain of the integration variables. Thus we consider

$$g(x) = \int_{-\pi}^{x} \frac{3\,\mathrm{d}\theta}{5 - 4\cos\theta}\ .$$

By rewriting Weierstrass's substitution for this integral in the form

$$\theta = 2\arctan u\ ,$$

and recalling that the range of arctan is the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we see that we must restrict θ to lie in the interval $(-\pi, \pi)$, and this requires that the integral be restricted to $|x| < \pi$. To evaluate the integral for $|x| > \pi$, we find an integer n such that $(2n-1)\pi < x < (2n+1)\pi$. Then

$$\int_{-\pi}^{x} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} = \int_{-\pi}^{(2n-1)\pi} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} + \int_{(2n-1)\pi}^{x} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta}\ .$$

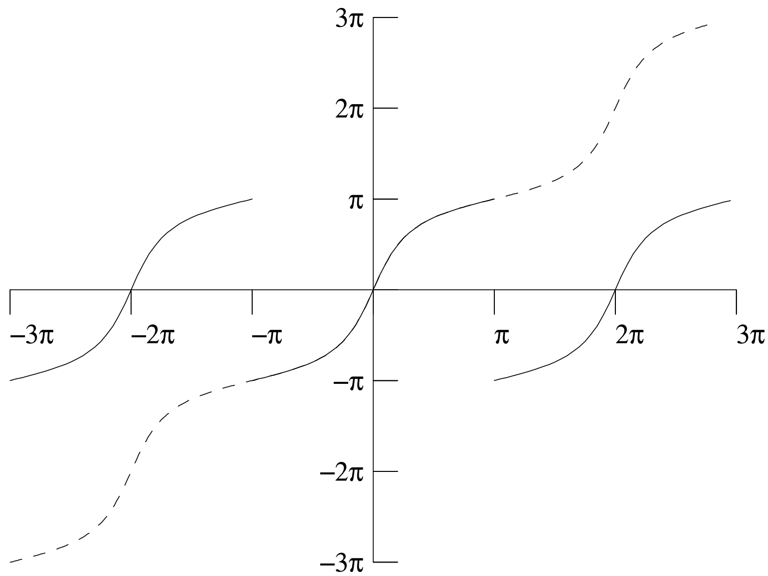

Fig 2. The discontinuous expression (1) and the continuous expression (3) for the integral in eq. (1). ———, integral (1). — — —, integral (3).

The first integral on the right-hand side can be simplified using the periodicity of the integrand and the second can be evaluated by modifying the Weierstrass substitution to $\theta = 2n\pi + 2\arctan u$. Then

$$\int_{-\pi}^{x} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} = n\int_{-\pi}^{\pi} \frac{3\,\mathrm{d}\theta}{5-4\cos\theta} + 2\arctan[3\tan(\tfrac{1}{2}x - n\pi)] + \pi$$
$$= 2n\pi + 2\arctan(3\tan\tfrac{1}{2}x) + \pi \ .$$

The intermediate quantity n can now be removed by using the floor function $\lfloor x \rfloor$, which returns the nearest integer less than or equal to x. By noting that $n = \lfloor (x+\pi)/2\pi \rfloor$, we can revert to indefinite-integral notation and replace (1) with a continuous expression. The result is

$$\int \frac{3\,\mathrm{d}x}{5-4\cos x} = 2\arctan(3\tan\tfrac{1}{2}x) + 2\pi \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor \ , \tag{3}$$

where the unnecessary constant π has been dropped. For the points $x = (2n+1)\pi$, this expression must be interpreted using a limit (from either side since the function is now continuous). In Figure 2, the discontinuous expression (1) and the continuous expression (3) are plotted on the same axes. A particular point of contrast between (3) and (1) is the fact that (1) is periodic in x whereas (3) contains a secular term.

3. EXAMPLES SHOWING THE SCOPE OF THE PROBLEM.

In the previous section, we showed that a continuous alternative could be found for our opening example, but we did it by reworking the entire integration. This approach is clumsy and difficult to automate in a computer algebra system, so here we look for an alternative. We start by considering some examples that establish the following four points. First, the Weierstrass substitution is better considered as a family of substitutions; secondly, the discontinuities that we need to remove are introduced by the *method* of integration, and are not a property of the integrand; thirdly, not all discontinuities are spurious; and lastly, the discontinuities can be present even in expressions that do not contain arctangents.

So that there is no possibility of the equations that follow being misunderstood, we shall adopt the following notation. When an integral is simplified to a form containing spurious discontinuities, we shall use the symbol $\doteq$ to remind the reader that the equality is unsatisfactory, and reserve the proper $=$ for equations in which the discontinuities have been removed.

Any algorithm must take into account the possibility of variations on the basic substitution. For example, although many systems evaluate the integral of $3/(5+4\sin x)$ using the standard $u = \tan\frac{1}{2}x$ to obtain

$$\int \frac{3\,\mathrm{d}x}{5+4\sin x} \doteq 2\arctan(\tfrac{5}{3}\tan\tfrac{1}{2}x + \tfrac{4}{3}) \doteq 2\arctan\frac{5\sin x + 4\cos x + 4}{3+3\cos x}\ , \tag{4}$$

some systems prefer the substitution $u = \tan(\frac{1}{2}x + \frac{1}{4}\pi)$ which gives

$$\int \frac{3\,\mathrm{d}x}{5+4\sin x} \doteq 2\arctan(3\tan[\tfrac{1}{2}x + \tfrac{1}{4}\pi]) \doteq 2\arctan\frac{3\cos x}{1-\sin x}\ . \tag{5}$$

Some systems consider (5) to be simpler than (4), and therefore any algorithm must support both choices. We note that the discontinuities present in both (4) and (5) are at different values of x. In the same way, $\sqrt{1-\cos x}$ integrates most easily using $u = \cot\frac{1}{2}x$, while $1/(1+\sin^2 x)$ is best integrated using $u = \tan x$.

The key to the automatic procedure is the fact that the discontinuities are consequences of the method of integration; they are not properties of the integrand. In the next section we give a theorem to this effect; here we give an example. Integrating $\sec^2 x/(1+\tan^2 x)$ using the substitution $u = \tan x$ gives the discontinuous result

$$\int \frac{\sec^2 x\,\mathrm{d}x}{1+\tan^2 x} \doteq \arctan(\tan x)\ , \tag{6}$$

instead of the obviously correct continuous result x. The two results are equivalent for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, but only the second is correct otherwise. It might appear that (6) could be corrected easily by transforming $\arctan(\tan x)$ to x. This, however, is just a fluke and cannot be generalized to other integrals: it is shown below in (8) that the same hopeful transformation can lead to completely incorrect results.

Not all discontinuities are spurious. If the integrand is not integrable at some points, then singularities will be present in the integral, and such singularities must be accepted and cannot be removed. An example that combines genuine discontinuities due to the integrand and spurious discontinuities due to the method of integration is

$$\int \frac{15\,\mathrm{d}x}{\cos x(5-4\cos x)} \doteq 3\ln\frac{\tan\frac{1}{2}x + 1}{\tan\frac{1}{2}x - 1} + 8\arctan(3\tan\tfrac{1}{2}x)\ . \tag{7}$$

The discontinuities at $x = (n + \frac{1}{2})\pi$ cannot be removed, while those at $x = (2n+1)\pi$ should be.

The final point we wish to make concerns the form of the integrals. The above examples give the impression that only integrals expressed using the arctangent function will be discontinuous, but the next example shows that algebraic simplification can remove the arctan functions from view.

$$\int \sqrt{1+\cos x}\,\mathrm{d}x \doteq 2\sqrt{2}\sin(\arctan(\tan\tfrac{1}{2}x)) \doteq \frac{2\sin x}{\sqrt{1+\cos x}}\ . \tag{8}$$

We mention in passing that the incorrect simplification of (8) to $2\sqrt{2}\sin\frac{1}{2}x$ is returned by some computer algebra programs.

4. BASIS OF THE ALGORITHM.

The basis of the algorithm is the following theorem. A similar theorem justifying integration by substitution can be found in most calculus textbooks, but they state it only for definite integrals, and here we need a statement for indefinite integrals.

Theorem. Given a function f that is continuous on an interval $[a,b]$ and a function ϕ that is differentiable and monotonically increasing on the interval $[\phi^{-1}(a),\phi^{-1}(b)]$, the function

$$g(x) = \int_{\phi^{-1}(a)}^{\phi^{-1}(x)} f(\phi(t))\phi'(t)\,\mathrm{d}t = \int_a^x f(s)\,\mathrm{d}s$$

is continuous for $x \in [a,b]$.

Proof. The function

$$g(x) = \int_a^x f(s)\,\mathrm{d}s$$

is continuous by the fundamental theorem of calculus. The two integral representations of g are equal by theorem 6.19 in Rudin (1976).

Our interest in this theorem lies in the conditions under which it fails. For the algorithm, we need to know that the expression

$$\int_{\phi^{-1}(a)}^{\phi^{-1}(x)} f(\phi(t))\phi'(t)\,\mathrm{d}t$$

can only be discontinuous at points where f is not integrable or at points where ϕ is not differentiable. There is nothing we can do about points where f is not integrable, so we focus on what happens when ϕ is singular. Suppose f is integrable on an interval $[a,c]$, and ϕ is differentiable and monotonically increasing at all points in $[a,c]$ except the isolated point b. On such an interval, the function we wish to find is g, defined by

$$g(x) = \int_a^x f(s)\,\mathrm{d}s$$

and it will be continuous by the fundamental theorem; the function we actually find, however, is $\hat{g}$.

$$\hat{g}(x) = \int f(\phi(x))\phi'(x)\,\mathrm{d}x \ .$$

So long as $x < b$, the function g can be expressed, by the above theorem, in terms of $\hat{g}$.

$$g(x) = \hat{g}(x) - \hat{g}(a) \ .$$

For $x > b$, the connection between g and $\hat{g}$ is obtained as follows.

$$\begin{aligned} g(x) = \int_a^x f(s)\,\mathrm{d}s &= \int_a^c f(s)\,\mathrm{d}s - \int_x^c f(s)\,\mathrm{d}s \\ &= \int_a^c f(s)\,\mathrm{d}s - \int_{\phi^{-1}(x)}^{\phi^{-1}(c)} f(\phi(t))\phi'(t)\,\mathrm{d}t \\ &= g(c) - \hat{g}(c) + \hat{g}(x) \ . \end{aligned}$$

To eliminate $g(c)$ from this equation, we calculate

$$\begin{aligned} \lim_{x\to b-} \hat{g}(x) - \lim_{x\to b+} \hat{g}(x) &= \lim_{x\to b-} (g(x) + \hat{g}(a)) - \lim_{x\to b+} (g(x) - g(c) + \hat{g}(c)) \\ &= g(c) - \hat{g}(c) + \hat{g}(a) \ , \end{aligned}$$

where the limits have been evaluated using the continuity of g at b. We can combine the expressions for g in the intervals $[a, b]$ and $[b, c]$ into a single equation using the Heaviside, or step, function H.

$$g(x) = \hat{g}(x) - \hat{g}(a) + H(x-b)\left[\lim_{x\to b-} \hat{g}(x) - \lim_{x\to b+} \hat{g}(x)\right] \ .$$

This gives a continuous expression for the desired function g in terms of the computable function $\hat{g}$. In graphical terms, the functions g and $\hat{g}$ will separate from each other by a series of jumps at isolated points. The continuous function g can be built from $\hat{g}$ by cancelling the jumps.

For the case of the substitution $\phi(x) = \tan\frac{1}{2}x$, the isolated points at which the substitution might introduce discontinuities are those where the tangent becomes unbounded, namely $x = (2n+1)\pi$, n being an integer. Thus integrals obtained using this substitution need to check the continuity only at $x = (2n+1)\pi$. In addition, since $\tan\frac{1}{2}x$ has period 2π, the substitution can only be applied to integrands having the same period, meaning that all jumps will be equal. Thus we need to examine only one point, say $x = \pi$, to obtain the size of the jump. Because of the periodicity, the jumps can be cancelled using the floor function $\lfloor x \rfloor$. If $\hat{g}(x)$ is a provisional antiderivative that has been obtained using the $\tan\frac{1}{2}x$ substitution, we calculate

$$K = \lim_{x\to\pi-} \hat{g}(x) - \lim_{x\to\pi+} \hat{g}(x) \ . \tag{9}$$

The function $\hat{g}(x) + K\lfloor (x+\pi)/2\pi \rfloor$ is then a corrected antiderivative of the given function that contains no spurious discontinuities and is discontinuous

Table 1. Functions $u = \phi(x)$ used in the Weierstrass algorithm and their corresponding substitutions.

Choice	$\phi(x)$	$\sin x$	$\cos x$	$\mathrm{d}x$	b	p
(a)	$\tan \frac{1}{2}x$	$\dfrac{2u}{1+u^2}$	$\dfrac{1-u^2}{1+u^2}$	$\dfrac{2\,\mathrm{d}u}{1+u^2}$	π	2π
(b)	$\tan(\frac{1}{2}x + \frac{1}{4}\pi)$	$\dfrac{u^2-1}{u^2+1}$	$\dfrac{2u}{u^2+1}$	$\dfrac{2\,\mathrm{d}u}{1+u^2}$	$\frac{1}{2}\pi$	2π
(c)	$\cot \frac{1}{2}x$	$\dfrac{2u}{1+u^2}$	$\dfrac{u^2-1}{1+u^2}$	$\dfrac{-2\,\mathrm{d}u}{1+u^2}$	0	2π
(d)	$\tan x$	$\dfrac{u}{\sqrt{1+u^2}}$	$\dfrac{1}{\sqrt{1+u^2}}$	$\dfrac{\mathrm{d}u}{1+u^2}$	$\frac{1}{2}\pi$	π

only when there is a singularity in the integrand. If either of the limits fail to exist, then a singularity of the integral coincides with the point, and no correction is needed.

5. COMPLETE STATEMENT OF THE ALGORITHM.

A complete statement of the modified algorithm, taking into account the points established in section 3 now follows. We start by defining a Weierstrass substitution to be one that uses a function appearing in table 1. There are other trigonometric substitutions used by computer algebra systems, sine and cosine being the obvious examples, but since these are never singular, they cannot lead to the problems addressed by this paper, and hence we have not included them in our definition.

Given an integrable function $f(\sin x, \cos x)$ whose indefinite integral is required, select one of the substitutions listed in table 1. The choice will be based on heuristics and since it affects only the form of the final result, it is not important to develop an extensive set of heuristics. However, we can note that choice (a) is good for integrands not containing $\sin x$, choice (b) is good for integrands not containing $\cos x$, (c) can be useful in cases in which (a) gives an integral that cannot be evaluated by the system, and (d) is good under conditions described in Gradshteyn and Ryzhik (1979, section 2.50). The integral is now transformed using the entries in the table. For example, for choice (c) we have

$$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right)\frac{-2\,\mathrm{d}u}{1+u^2} .$$

The integral in u is now evaluated using the standard routines of the system, and then u substituted for. Call the result $\hat{g}(x)$. Next we calculate

$$K = \lim_{x\to b-} \hat{g}(x) - \lim_{x\to b+} \hat{g}(x) ,$$

where the point b is obtained from the table. The correct integral is then

$$g(x) = \int f(\sin x, \cos x)\, \mathrm{d}x = \hat{g}(x) + K \left\lfloor \frac{x-b}{p} \right\rfloor ,$$

where the period p is taken from the table.

6. EXAMPLE APPLICATIONS.

We now rework the examples given earlier in the paper to show the algorithm in action. Starting with (1), we have seen that the integration gives $\hat{g}(x) = 2\arctan(3\tan\frac{1}{2}x)$ and therefore $K = 2\pi$. Hence we obtain

$$\int \frac{3\,\mathrm{d}x}{5 - 4\cos x} = 2\arctan(3\tan\tfrac{1}{2}x) + 2\pi \left\lfloor \frac{x-\pi}{2\pi} \right\rfloor , \tag{10}$$

which differs from (3) by the constant 2π. For (5), we use choice (b) and obtain

$$\int \frac{3\,\mathrm{d}x}{5 + 4\sin x} = 2\arctan(3\tan[\tfrac{1}{2}x + \tfrac{1}{4}\pi]) + 2\pi \left\lfloor \frac{x-\pi/2}{2\pi} \right\rfloor . \tag{11}$$

For (7), we use choice (a) and obtain $K = 8\pi$, giving

$$\int \frac{15\,\mathrm{d}x}{\cos x(5 - 4\cos x)} = 3\ln\frac{\tan\frac{1}{2}x + 1}{\tan\frac{1}{2}x - 1} + 8\arctan(3\tan\tfrac{1}{2}x) + 8\pi \left\lfloor \frac{x-\pi}{2\pi} \right\rfloor . \tag{12}$$

This expression is still singular at $x = (n + \frac{1}{2})\pi$, but, with the correction, any definite integral which is well defined, i.e. whose limits lie between adjacent singularities, can be evaluated by substitution in the antiderivative. Next, equation (8) is corrected using choice (a) to obtain

$$\int \sqrt{1+\cos x}\,\mathrm{d}x = \frac{2\sin x}{\sqrt{1+\cos x}} + 4\sqrt{2} \left\lfloor \frac{x-\pi}{2\pi} \right\rfloor . \tag{13}$$

The related integral of $\sqrt{1-\cos x}$ is interesting, because if choice (a) is used, we obtain

$$\int \sqrt{1-\cos x}\,\mathrm{d}x = \int \sqrt{\frac{2u^2}{1+u^2}}\frac{2\,\mathrm{d}u}{1+u^2} , \tag{14}$$

and most systems cannot integrate this correctly. On the other hand, using choice (c) we obtain

$$\int \sqrt{1-\cos x}\,\mathrm{d}x = \int \sqrt{\frac{2}{1+u^2}}\frac{-2\,\mathrm{d}u}{1+u^2} = -2\cot\tfrac{1}{2}x\sqrt{1-\cos x} + 4\sqrt{2} \left\lfloor \frac{x}{2\pi} \right\rfloor . \tag{15}$$

Finally, we do not need to correct (6), but if choice (d) were used, however unwisely, we would obtain $K = \pi$ and

$$\int \frac{\sec^2 x \, \mathrm{d}x}{1 + \tan^2 x} = \arctan(\tan x) + \pi \left\lfloor \frac{x - \pi/2}{\pi} \right\rfloor ,$$

which is equivalent to $x - \pi$, and this is the proper way to remove the discontinuity.

7. CONNECTION WITH ALGORITHMS BY RIOBOO AND OTHERS.

Many of the examples above contain arctangents, and since the integral with respect to u is often a rational function of u, the algorithm of Rioboo [Geddes *et al* 1992] comes naturally to mind. This algorithm is relevant to the present one in several respects. The problem it addresses is similar, namely spurious discontinuities in integrals expressed using the inverse tangent function, but the origins of the discontinuities are quite different. The discontinuities of concern to Rioboo are the result of converting complex logarithms to arctangents, whereas here the discontinuities are the result of unavoidable limitations in the substitution process. In addition, Rioboo's algorithm is based on certain properties of the inverse tangent function, but the Weierstrass substitution does not always lead to inverse tangents, as we saw in example (8). The present algorithm continues to work even when there are no arctangents in the final result. Therefore, although the two algorithms have similar aims, Rioboo's algorithm cannot be used in the present context.

There is another connection between this work and Rioboo's algorithm. After the present algorithm has transformed the initial integral in x to one in u, it relies on the system to evaluate the transformed integral with respect to u. The algorithm assumes that this will be done correctly, and that no spurious discontinuities will be introduced by the integration routine. If the algebra system returns a result that is not continuous, then the present algorithm fails. Here is an example. Using substitution (a), we see that

$$2\sqrt{2} \int \frac{1 + \cos x}{1 + \cos^2 x} dx = 4\sqrt{2} \int \frac{\mathrm{d}u}{1 + u^4} .$$

Before Rioboo's algorithm, many systems integrated the last integral incorrectly, returning the discontinuous

$$4\sqrt{2} \int \frac{\mathrm{d}u}{1 + u^4} \doteq \ln \frac{u^2 + \sqrt{u} + 1}{u^2 - \sqrt{u} + 1} + 2 \arctan \frac{u\sqrt{2}}{1 - u^2} .$$

(We revert to the $\doteq$ notation for this integral.) If we now substitute $u = \tan \frac{1}{2}x$ into the last expression, we find that the result has a spurious discontinuity at $x = \pi/2$ instead of at $x = \pi$ as the algorithm expects. With a correct, continuous result for the integral (using Rioboo's algorithm or another method), we obtain

$$2\sqrt{2}\int \frac{1+\cos x}{1+\cos^2 x}dx = \ln\frac{\tan^2\frac{1}{2}x+\sqrt{2}\tan\frac{1}{2}x+1}{\tan^2\frac{1}{2}x-\sqrt{2}\tan\frac{1}{2}x+1}$$
$$+2\arctan(\sqrt{2}\tan\tfrac{1}{2}x+1)+2\arctan(\sqrt{2}\tan\tfrac{1}{2}x-1)+4\pi\lfloor (x-\pi)/2\pi\rfloor\ .$$

This reminds us that the Weierstrass algorithm is not the only one that can introduce spurious discontinuities into expressions for indefinite integrals. All such algorithms should be corrected if computer algebra systems are going to deliver reliable results to users, with their varying levels of mathematical ability.

8. REMARKS UPON IMPLEMENTATION.

The algorithm described in this paper has been implemented by Derive (Rich and Stoutemyer 1988) to some extent, but not at present by any other system. There are no cases for which the algorithm does not work in theory, but the supporting functions of the algebra system may not be adequate to implement it in practice. One of the requirements for implementation is support for the floor function, or an equivalent such as the round function. Some systems do not have good treatments of these discontinuous functions. Another requirement is that all other routines return correct integrals, since this algorithm will fail if the integral in u is incorrect. Finally the limit packages must be able to handle the limit calculations required. If the integral in question contains symbolic entries, then it is possible that the limit package of the algebra system will fail. As an example, consider

$$\int \frac{dx}{a+\cos x}\ .$$

The first stage of the present algorithm gives the expression

$$\frac{2}{\sqrt{a^2-1}}\arctan\left(\frac{\sqrt{a^2-1}}{a+1}\tan\tfrac{1}{2}x\right)\ .$$

Therefore, the required limit calculation is

$$\lim_{x\to\pi-}\frac{2}{\sqrt{a^2-1}}\arctan\left(\frac{\sqrt{a^2-1}}{a+1}\tan\tfrac{1}{2}x\right)\ .$$

If $a>1$ then all quantities are real, and most systems can do this, but if $a<1$, then the limit requires the correct treatment of arctangents of complex arguments, which few systems are able to do. We emphasise, however, that the limits are indeed computable, and not difficult in spite of being beyond many systems. For example, the correct value of the above limit is

$$\frac{\pi}{\sqrt{a^2-1}}\operatorname{csgn}\left(\frac{\sqrt{a^2-1}}{a+1}\right)\ ,$$

provided the arctangent is defined according to Kahan, and

$$\operatorname{csgn}(z) = \begin{cases} 1\ , & \text{for } \Re(z) > 0 \text{ or } (\Re(z) = 0 \text{ and } \Im(z) > 0) \\ -1\ , & \text{for } \Re(z) < 0 \text{ or } (\Re(z) = 0 \text{ and } \Im(z) < 0) \\ 0\ , & \text{for } z = 0. \end{cases}$$

As a result of recent changes in textbooks on introductory calculus, which now omit the Weierstrass substitution, users will be less likely to know the substitution. This may not be such a bad thing, since the treatment in the books was always misleading. Not only did the question of the continuity of the integral pass unmentioned, the exercises always avoided situations which might alert the reader to the flaw in the treatment. Users of computer systems who check results obtained from their systems against the standard published tables of integrals should be aware that the tables continue to contain incorrect entries.

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