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Asymptotic analysis of interactions between highly conducting cylinders

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Abstract—We consider the conduction of heat or electricity between a pair of equal, touching cylinders embedded in a matrix. We solve exactly a problem in which each cylinder contains a heat source, and analyse the solution asymptotically for the case in which the conductivity of the fibres is much greater than that of the matrix. Using this analysis, we assess the accuracy of approximate schemes for calculating the interactions between the fibres of a composite material. We conclude that the approximate solutions are better at predicting global quantities, such as total flux, than they are at predicting pointwise quantities such as the temperature field.

INTRODUCTION

A fibre-reinforced composite material has an effective conductivity that depends both on the conductivity of the fibres and on their arrangement. The conductivity can be to heat or electricity; we use the language of heat conduction. We are interested in the case of highly conducting cylindrical inclusions embedded in a matrix. Two asymptotic limits have been investigated using approximate techniques: perfectly conducting cylinders that are nearly touching [1], and touching cylinders that are nearly perfectly conducting [2]. We are interested in the latter case. O'Brien calculated the conductivity of a composite material by concentrating his attention on the narrow gap separating two typical cylinders. He analysed the temperature field in the gap and in the parts of the cylinders adjacent to the gap and made assumptions about the behaviour of the field far from the gap, specifically, that the temperature in the cylinders would be asymptotic to a constant value. As a result of this procedure, he obtained the leading term for the total heat flux from a relatively simple calculation confined to the neighbourhood of the gap region, and his assumptions, and result, are what we test.

Some support for O'Brien's work has already been published. Perrins, McKenzie & McPhedran [3] calculated numerically the conductivities of square and hexagonal arrays of circular cylinders for cases in which α, the ratio of the conductivity of the cylinders to that of the matrix, lies in the range $2 \leq \alpha \leq 50$. The numerical calculations became difficult as α was increased, but reasonable agreement was obtained. McPhedran and Milton [4] studied the conduction between a pair of touching cylinders in a temperature gradient, and for that problem confirmed the asymptotic expression for the heat flux.

The new tests of the approximation described here clarify the scheme in several ways. First, we obtain the size of the terms neglected in the approximation, and thereby establish the speed with

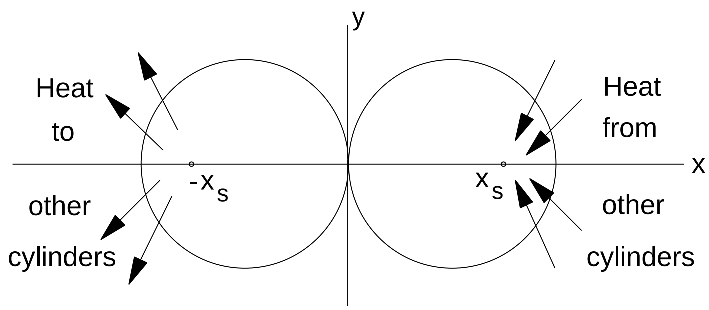

Figure 1. Two touching cylindrical fibres in a composite material.

which the problem approaches the asymptotic result. Secondly, we can test the temperature field, as well as the heat flux, and thus test not only an integrated quantity (the flux), but also a field quantity. Thirdly, because we are able to vary the conditions far from the gap in a simple way, we can demonstrate the degree to which the gap conditions dominate the results. Our method is to study a problem that both has an exact solution, and obeys the conditions necessary for O'Brien's assumptions to be valid. We then analyse the solution in detail and see how closely it obeys the assumptions of the approximate theory. The problem is constructed by focusing our attention on just two cylindrical fibres in a composite material. Figure 1 shows the situation. Heat enters the right-hand cylinder through some part of its surface that is far from the gap, then crosses the gap region to enter the left cylinder, and then leaves this cylinder through a part of its surface that is also far from the gap. We simplify this situation by considering only touching cylinders, and then we model the inward flux of heat by placing a line source at a point inside the cylinder, but far from the gap. Similarly, the outward flux of heat from the second cylinder is modelled by a line sink.

When interpreting the solution of the above problem, we must bear in mind that the variables are set up differently from the way in which O'Brien imagined the problem. In applications to composite media, the temperature difference between the cylinders is considered to be the given quantity, and the heat flux is what is calculated. In our problem, the heat flux is specified, and the temperature field is the unknown. This difference in point of view is easily adjusted for, however.

SPECIFICATION AND SOLUTION OF THE PROBLEM

We consider a pair of touching cylinders, each of thermal conductivity α and unit radius, placed in a medium of unit conductivity. We take a Cartesian rectangular coordinate system with the origin at the point of contact, the x-axis along the line of centres and the z-axis parallel to the axis of each cylinder. The first cylinder is then in the right half plane $x > 0$, and the second in $x < 0$. A line source of strength q is placed inside the cylinder at $x = x_s$ on the x-axis. Similarly, a line sink is placed at $x = -x_s$. The temperature T satisfies the usual Laplace's equation with the boundary conditions that the temperature and the normal heat flux are continuous at the surface of each cylinder. Since the problem clearly obeys

$$T(x,y) = T(x,-y) \qquad \text{and} \qquad T(x,y) = -T(-x,y) \ , \tag{1}$$

we need only discuss the solution for $x > 0$.

We solve this problem using tangent-circle coordinates, defined by

$$\xi + i\eta = \frac{2}{x+iy} \ . \tag{2}$$

In these coordinates, the surface of the (first) cylinder is given by $\xi = 1$. The boundary conditions become

$$T(\xi = 1-, \eta) = T(\xi = 1+, \eta) \ , \tag{3}$$

$$\left.\frac{\partial T}{\partial \xi}\right|_{\xi=1-} = \alpha^{-1} \left.\frac{\partial T}{\partial n}\right|_{\xi=1+} . \tag{4}$$

It is easily verified that inside the cylinder, $\xi \geq 1$, the solution is

$$T(\xi,\eta) = -\frac{q}{\alpha} \ln \left[\frac{(\xi - 2/x_s)^2 + \eta^2}{\xi^2 + \eta^2}\right] + q \int_0^\infty A(s) e^{-s\xi} \cos s\eta \, ds, \tag{5}$$

where

$$A(s) = \frac{e^{sz}}{s \cosh s + s\alpha \sinh s} \left(\cosh sz \sinh s - \alpha^{-1} \sinh sz \cosh s\right) \ , \tag{6}$$

and where $z = 1 - 1/x_s$. Outside the cylinder, $\xi \leq 1$, the solution is

$$T(\xi,\eta) = q \int_0^\infty B(s) \sinh s\xi \cos s\eta \, ds \ , \tag{7}$$

where

$$B(s) = \frac{e^{(2z-1)s}}{s \cosh s + s\alpha \sinh s}. \tag{8}$$

ASYMPTOTIC ANALYSIS OF THE SOLUTION

We approximate the integrals in (5) and (7) asymptotically as $\alpha \to \infty$, in order to test O'Brien's assumptions. The first test is to calculate the temperature along the x axis, the approximation predicting that it will be zero at the point of contact $x = 0$ and then rise over a distance of order α^{-1} to a constant value. There will be, in our calculation, a narrow peak around the line source, but that is incidental. Setting $\eta = 0$ on the x axis, we can write T in (5) as

$$T = -\frac{2q}{\alpha} \ln \left(1 - \frac{2}{\xi x_s}\right) + q(I_1 - \alpha^{-1} I_2) \ , \tag{9}$$

where

$$I_1 = \int_0^\infty \frac{e^{-s(\xi - z)} \cosh sz \sinh s}{s(\cosh s + \alpha \sinh s)} \, ds \, , \tag{10}$$

$$I_2 = \int_0^\infty \frac{e^{-s(\xi - z)} \sinh sz \cosh s}{s(\cosh s + \alpha \sinh s)} \, ds \, . \tag{11}$$

The qualitative behaviour of the integrands in (10) and (11) is determined by the denominator $\cosh s + \alpha \sinh s$. For $\tanh s << \alpha^{-1}$, the integrands are $O(1)$, while for $\tanh s >> \alpha^{-1}$ they are $O(\alpha^{-1})$. Therefore we break each integral at $\beta\alpha^{-1}$ where β is a number $O(1)$ to be determined.

$$I_1 = \int_0^{\beta\alpha^{-1}} \frac{e^{-s(\xi - z)} \cosh sz \sinh s}{s(\cosh s + \alpha \sinh s)} \, ds + \int_{\beta\alpha^{-1}}^\infty \frac{e^{-s(\xi - z)} \cosh sz}{s\alpha(1 + \alpha^{-1} \coth s)} \, ds \ . \tag{12}$$

For $0 \leq s \leq \beta\alpha^{-1}$, we expand the integrand as a power series in s around $s = 0$, and for $s \geq \beta\alpha^{-1}$ we expand as a series in powers of $\alpha^{-1} \coth s$. Integrating the two series term by term, we obtain

$$\begin{aligned} I_1 = \frac{1}{\alpha} \Bigg[& \ln \alpha - \gamma - \ln(\beta(\xi - z)) + (\beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \frac{\beta^4}{4} + \ \ldots) + (-\frac{1}{\beta} + \frac{1}{2\beta^2} \\ & - \frac{1}{3\beta^3} + \frac{1}{4\beta^4} + \ \ldots)\Bigg] + \frac{(\xi - z)}{\alpha^2} \Bigg[\ln \alpha - \ln \beta + (\beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \frac{\beta^4}{4} + \ \ldots) \\ & + (-\frac{1}{\beta} + \frac{1}{2\beta^2} - \frac{1}{3\beta^3} + \frac{1}{4\beta^4} + \ \ldots)\Bigg] + O(\alpha^{-3}) \, . \end{aligned} \tag{13}$$

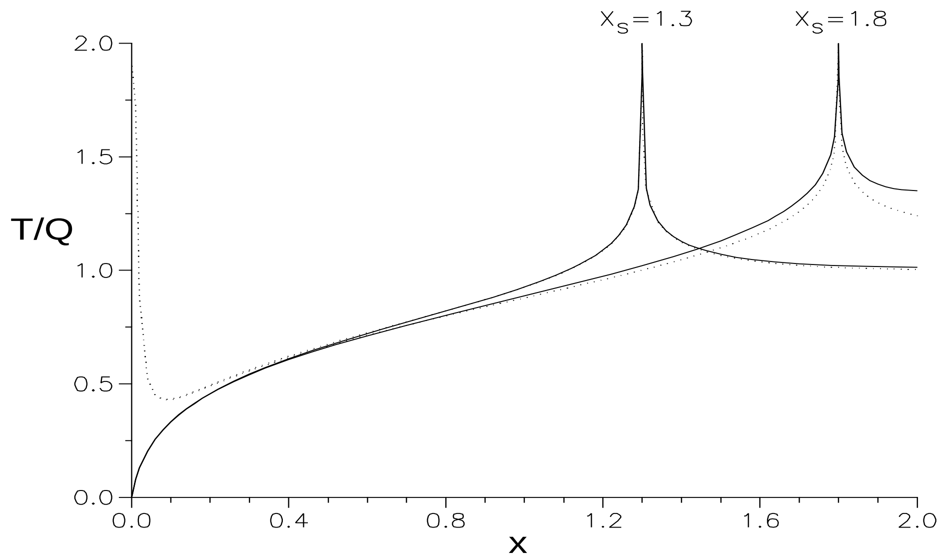

Figure 2. Comparison of asymptotic and exact solutions for the temperature distribution on the x-axis when $\alpha = 10^2$. The solid line is a numerical evaluation of the solution and the dotted line shows the asymptotic approximation.

Here γ is Euler's constant. The series cancel if $\beta = 1$, and so choosing this value and repeating for I_2, we obtain

$$I_1 - \alpha^{-1} I_2 = \frac{\ln\alpha}{\alpha} - \frac{\gamma + \ln(\xi - z)}{\alpha} + \frac{\ln\alpha}{\alpha^2}(\xi - 2z) + \frac{z}{\alpha^2}\left[\gamma + \ln(\xi - z)\right] + O(\alpha^{-3}) \ . \qquad (14)$$

The integral for T in (7) can likewise be approximated as

$$T = \frac{q\,\xi}{\alpha}\left[\ln\alpha - \gamma - \ln(1 - z)\right] + \frac{q\,\xi}{\alpha^2}(1 - 2z)\ln\alpha + O(\alpha^{-3}) \, . \qquad (15)$$

The expansion is valid outside a radius α^{-1} from the origin, the point of contact, because $\xi \to \infty$ as $x \to 0$, and in this limit the approximate treatment of the integrals breaks down.

We checked our asymptotic solution by evaluating the exact solution numerically. Figure 2 compares the asymptotic and numerical solutions for the temperature on the x-axis in the case $\alpha = 100$, when $x_s = 1.3$ and $x_s = 1.8$. The results agree well.

COMPARISONS

We are now in a position to look at the qualitative behaviour of our solution and compare it with the assumptions of O'Brien. He supposed that the temperature difference between the cylinders was fixed, and predicted that the flux between the cylinders would be $O(\alpha/\ln\alpha)$ as $\alpha \to \infty$. We can rescale the parameters in our problem to make it similar to O'Brien's by setting $q = \alpha Q/\ln\alpha$. Equation (9) becomes

$$T/Q = 1 - \frac{1}{\ln\alpha}\left[\gamma + \ln(\xi - z) + 2\ln\left(1 - \frac{2}{\xi x_s}\right)\right] + O(\alpha^{-1}) \, . \qquad (16)$$

This equation confirms O'Brien's assumption, and his main result. To leading order, the temperature in the body of the cylinders is constant when the flux is $O(\alpha/\ln\alpha)$. In addition, we can see that the temperature and flux are independent of x_s, which shows that the exact conditions

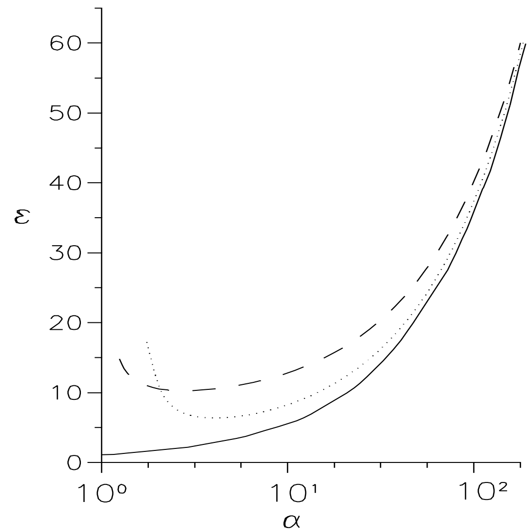

Figure 3. A comparison of results for the conductivity of a square array of cylinders. The solid line shows the numerical results of Perrins*et al.*, the broken line shows a formula due to Perrins*et al.* and the dotted line shows a formula from this work.

far from the gap are not important to leading order. The equation also shows that the error introduced by the assumptions is $O(1/\ln\alpha)$. Unless α is very large, the error will be numerically significant, and figure 2 shows that, for $\alpha = 100$, the temperature varies markedly from its asymptotic value of 1.

Figure 3 shows the conductivities ϵ computed by Perrins *et al* [3] for square arrays of cylinders in contact. The solid curve shows their results, and it can be compared with the broken curve that shows an asymptotic formula

$$\epsilon \sim \frac{1}{2}\pi\alpha/\ln\alpha + 6 \tag{17}$$

due to Perrins *et al* but based on O'Brien's work. The constant 6.0 was chosen to fit the numerical data. A better asymptotic formula is based on (15), which suggests that the simple result $\alpha/\ln\alpha$ should be modified by adding a constant to the logarithm. Thus the dotted line shows $\epsilon \sim \frac{1}{2}\pi\alpha/(\ln\alpha - 0.4)$, where the constant -0.4 was chosen to fit the data.

The conclusion of this study is that the assumptions of O'Brien's theory are correct to leading order in α, but because the errors are $O(1/\ln\alpha)$, the asymptotic state is approached slowly. However, in spite of the fact that at $\alpha = 100$ the temperature field still varies significantly in the cylinders, the heat flux is predicted quite accurately, even for α as low as 10.

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