English translation of *Sur l'inversion de $y^\alpha e^y$ au moyen de nombres de Stirling associés* published in Comptes Rendus de L'académie des Sciences, Paris, 320 (1995) 1449-1452.

On the inversion of $y^\alpha e^y$ in terms of associated Stirling numbers

BY D. J. Jeffrey[1], R. M. Corless[1], D. E. G. Hare[2], and D. E. Knuth[3]

[1] *Department of Applied Mathematics, The University of Western Ontario*
London, CANADA, N6A 5B7

[2] *Symbolic Computation Group, University of Waterloo*
Waterloo, CANADA, N2L 3G1

[3] *Department of Computer Science, Stanford University*
Stanford, USA, 94305-2140

The function $y = \Phi_\alpha(x)$, the solution of $y^\alpha e^y = x$ for x and y large enough, has a series expansion in terms of $\ln x$ and $\ln\ln x$, with coefficients given in terms of Stirling cycle numbers. It is shown that this expansion converges for $x > (\alpha e)^\alpha$ for $\alpha \geq 1$. It is also shown that new expansions can be obtained for Φ_α in terms of associated Stirling numbers. The new expansions converge more rapidly and on a larger domain.

1. STIRLING NUMBERS — Stirling cycle numbers $\left[{n \atop m}\right]$ are defined [1] by

$$\ln^m(1+z) = m!\sum_n (-1)^{n+m}\left[{n \atop m}\right]\frac{z^n}{n!}\ . \tag{1a}$$

The numbers $(-1)^{n+m}\left[{n \atop m}\right]$ are also called Stirling numbers of the first kind [8]. Stirling partition numbers $\left\{{n \atop m}\right\}$, also called Stirling numbers of the second kind, are defined by

$$(e^z-1)^m = m!\sum_n \left\{{n \atop m}\right\}\frac{z^n}{n!}\ , \tag{1b}$$

and 2-associated Stirling partition numbers $\left\{{n \atop m}\right\}_{\geq 2}$ are defined by [2, exercise 5.7; 7, p. 296; 9, §4.5]

$$(e^z-1-z)^m = m!\sum_n \left\{{n \atop m}\right\}_{\geq 2}\frac{z^n}{n!}\ . \tag{1c}$$

2. SOLUTION BY COMTET OF $y^\alpha e^y = x$. — For fixed real α, we let $\Phi_\alpha(x)$ be the value of y that is the unique positive solution of the equation $y^\alpha e^y = x$. If α is negative, then $y > -\alpha$ and $x > e^{-\alpha}(-\alpha)^\alpha$. An asymptotic expansion for $\Phi_\alpha(x)$, in terms of Stirling cycle numbers and the quantities $L_1 = \ln x$ and $L_2 = \ln\ln x$, is given in the following theorem [3,5].

Theorem 1. — *With the preceding notation, the function $\Phi_\alpha(x)$ has the following series development, convergent if x is large enough.*

$$\Phi_\alpha(x) = L_1 - \alpha L_2 + \alpha\sum_{n\geq 1}\frac{\alpha^n}{L_1^n}\sum_{m=1}^{n}(-1)^{n+m}\left[{n \atop n-m+1}\right]\frac{L_2^m}{m!}\ . \tag{2a}$$

Proof. – We recall some details of the proof given in [5] for use below. We introduce a function $w(x)$ defined by

$$y = \Phi_\alpha(x) = L_1 - \alpha L_2 + \alpha w\ , \tag{2b}$$

which satisfies

$$1 - e^{-w} + \sigma w - \tau = 0, \qquad \sigma = \frac{\alpha}{L_1}, \qquad \tau = \alpha\frac{L_2}{L_1} = \sigma\ln\left(\frac{\alpha}{\sigma}\right)\ . \tag{2c}$$

By the Lagrange Inversion Theorem [4], w has the expansion

$$w = \sum_{m\ge 1} \frac{\tau^m}{m!} \sum_{l\ge 0} (-1)^l \left[{l+m \atop l+1} \right] \sigma^l \ . \tag{2d}$$

One converts from σ and τ back to L_1 and L_2 to complete the theorem.

Since the domain of convergence of (2a) is described only as 'x large enough' by de Bruijn and Comtet, we give a stronger statement in the next theorem.

Theorem 2. — *For $\alpha \ge 1$, the series (2a) is convergent for $x > (\alpha e)^\alpha$, while for $\alpha < 1$ it is convergent for $x > e$.*

Abbreviated Proof. – We let $f(w) = \sigma w - \tau$ and $g(w) = 1 - e^{-w}$. For $\alpha > 1$ and $x > (\alpha e)^\alpha$, we define $\delta > 0$ by $\delta = 1 - \ln(\alpha e)^\alpha / \ln x$, and then $\sigma = (1-\delta)/(1+\ln\alpha)$. We also set $w_0 = \ln(1+\ln\alpha)$. On the contour consisting of the lines $\Re(w) = w_0 + \delta$, $\Im(w) = \pm 2\delta^{1/2}$, and $\Re(w) = -2$, one can show that $|g| > |f|$, and therefore $f+g$ has only one root within the contour, by Rouché's theorem. Using Cauchy's theorem to express this root as an integral around the contour, we establish the convergence of (2a) by expanding the integrand as a series in f/g and integrating term by term [3]. For $\alpha < 1$, the contour must remain the same as that for $\alpha = 1$.

3. A New Expansion. — In view of the relation

$$\Phi_\alpha(x) = \alpha \Phi_1\left(\frac{x^{1/\alpha}}{\alpha}\right) = \alpha W\left(\frac{x^{1/\alpha}}{\alpha}\right) \ ,$$

where W is the Lambert W function [6], we shall simplify our equations by considering only the case $\alpha = 1$ from now on. By changing to the variable $\zeta = 1/(1+\sigma)$, we obtain a new series for $W = \Phi_1$ that converges on a wider domain than does (2a).

Theorem 3. — *With the preceding notation, W has the series development*

$$W(x) = L_1 - L_2 + \sum_{m\ge 1} \frac{\tau^m}{m!} \sum_{p=0}^{m-1} (-1)^{p+m-1} \zeta^{p+m} \left\{ {p+m-1 \atop p} \right\}_{\ge 2} \ , \tag{3a}$$

and this is convergent for $x \ge 2$.

Proof. — Into (2c), we substitute $\sigma = 1/\zeta - 1$ and obtain

$$\tau + e^{-w} - 1 + w - w/\zeta = 0 \ . \tag{3b}$$

To invert this using the Lagrange Inversion Theorem, we introduce the operator $[w^p]$ to represent the coefficient of w^p in a series expansion in w, and obtain

$$\begin{aligned} w &= \sum_{n\ge 1} \frac{\zeta^n}{n} \left[w^{n-1}\right] \left(\tau + e^{-w} - 1 + w\right)^n \ , \\ &= \sum_{n\ge 1} \frac{\zeta^n}{n} \left[w^{n-1}\right] \sum_m \binom{n}{m} \tau^m \left(e^{-w} - 1 + w\right)^{n-m} \\ &= \sum_{n\ge 1} (-1)^{n-1} \zeta^n \sum_m \frac{\tau^m}{m!} \left\{ {n-1 \atop n-m} \right\}_{\ge 2} \end{aligned}$$

which can be rearranged to obtain the theorem.

To prove convergence, we let $f(w) = \zeta(e^{-w} - 1 + w) + \tau\zeta$ and $g(w) = -w$. On the rectangular contour bounded by the four lines $\Re(w) = 2$, $\Im(w) = \pm 2$ and $\Re(w) = -1$, it is simple to show that $|f| < |g|$ for all $x \in [2, e]$. Hence the series converges there. Since (3a) is equivalent to (2a) for $x > e$ because of the relation

$$\left[{l \atop m} \right] = \sum_{p=0}^{l-m} (-1)^{p+l-m} \left\{ {p+l-m \atop p} \right\}_{\ge 2} \binom{p+l-1}{p+l-m} \ , \tag{3c}$$

the theorem follows.

4. Expansions using new variables. — Two further series developments can be obtained by introducing the variables $L_\tau = \ln(1-\tau)$ and $\eta = \sigma/(1-\tau)$.

Theorem 4.—*With the preceding notation, W has the series development*

$$W(x) = L_1 - L_2 - L_\tau - \sum_{n\ge 1}(-\eta)^n \sum_{m=1}^{n}(-1)^{m+1}\left[{n \atop n-m+1}\right]\frac{L_\tau^m}{m!} \ . \tag{4a}$$

Proof.— We set $w = v - L_\tau$ in (2c) and obtain, after rearranging,

$$1 - e^{-v} + \frac{\sigma}{1-\tau}v = \frac{\sigma}{1-\tau}L_\tau \ . \tag{4b}$$

This equation has exactly the form of (2c) itself, and therefore the expansion for v can be obtained from (2d) by replacing σ with $\sigma/(1-\tau)$ and τ with $\sigma L_\tau/(1-\tau)$. The theorem then follows by rearrangement.

The expansion (4a) converges more slowly than (2a), but when we transform it using the methods of theorem 3, we obtain a very rapidly convergent expansion, as we show in section 5.

Theorem 5.—*With the above notation, $W(x)$ has the development*

$$W(x) = L_1 - L_2 - L_\tau + \sum_{m\ge 1}\frac{1}{m!}L_\tau^m \eta^m \sum_{p=0}^{m-1}(-1)^{p+m-1}\left\{{p+m-1 \atop p}\right\}_{\ge 2}\frac{1}{(1+\eta)^{p+m}} \ , \tag{4c}$$

Proof. — The proof follows exactly that of Theorem 3.

The process of generating series in new variables can be continued. If $w(\sigma,\tau)$ satisfies (2c), then Theorem 4 is equivalent to the identity

$$w(\sigma,\tau) = -\ln(1-\tau) + w\left(\frac{\sigma}{1-\tau}, \frac{\sigma\ln(1-\tau)}{1-\tau}\right) \ , \tag{4d}$$

which clearly can be applied repeatedly.

5. Rate of convergence. — We consider the accuracy obtained by truncating each of the series (2a), (3a) and (4c) at $N-1$ terms. Since the series are asymptotic series, the error terms for x large are respectively $O(L_2^N/L_1^N)$ for (2a) and (3a) and $O(L_2^N/L_1^{2N})$ for (4c), so (4c) is clearly better. In addition to being asymptotic, however, the series are absolutely convergent, and can be used for relatively small values of x. We observe that $\tau = L_\tau = 0$ at $x = e$, and hence the infinite sums in (2a), (3a) and (4c) are zero there. Thus any truncated series will be exact at $x = e$ and asymptotically correct as $x \to \infty$, implying that the error will have a maximum at some $x > e$.

However, although (2a) is correct at $x = e$, its derivative does not converge there. In contrast, (3a) and (4c) give finite sums at $x = e$ for all derivatives. To put it another way, taking N terms of (3a) or (4c) and expanding about $x = e$ gives N terms of the Taylor series for $\Phi_1(x)$ about $x = e$. Both (3a) and (4c) are much more accurate than (2a) near this point. Numerical experiments confirm these results.

We conjecture that (2a) and (4c) converge for all $x > 1$.

References

[1] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[2] L. Comtet, *Analyse Combinatoire*,2, Presses Univ. de France, 1970.

[3] N. G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, 1961.

[4] C. Carathéodory, *Theory of Functions of a Complex Variable*, Chelsea, 1954.

[5] L. Comtet, *C. R. Acad. Sc. Paris*, 270, 1970, p. 1085–1088.

[6] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, "The Lambert W Function", Advances in Computational Mathematics, to appear.

[7] F.N. David and D.E. Barton, *Combinatorial Chance*, Hafner, 1962.

[8] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Teubner, 1906.

[9] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, 1958.