ACM Communications in Computer Algebra, Vol. 48, No. 2, Issue 188, June 2014

Some definite integrals containing the Tree T function

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Abstract

We consider, from a symbolic point of view, a pair of definite integrals containing Lambert W, recently considered from a numerical point of view by Walter Gautschi. We transform the integrals to a shape that can be integrated in special cases by a computer-algebra system or by using tables of integrals, such as Prudnikov *et al.*

1 Introduction

The paper [4] examines, in effect, numerical schemes for the evaluation of the integrals[1]

$$I_0(\alpha,\beta) = \int_1^\infty T_0\left(xe^{-x}\right)^\alpha x^{-\beta}\,dx \tag{1}$$

and

$$I_1(\alpha,\beta) = \int_0^1 T_1\left(xe^{-x}\right)^\alpha x^{-\beta}\,dx\ , \tag{2}$$

where α and β are restricted to values ensuring convergence[2]. Here, T_k is the Tree T function, satisfying $T(z)\exp(-T(z)) = z$. It is a cognate of the Lambert W function through

$$T_k(z) = -W_{-k}(-z)\ ;$$

see [1, 8, 2] for more discussion. The notation in this paper makes a new convention for the signs of the branches: we realized with this work that the indices of the branches of the Tree T function should also be negated, relative to W, as in the equation above. This means that while for Lambert W the only real-valued branches have indices $k = 0$ and $k = -1$, the corresponding indices for the Tree T function are $k = 0$ and $k = 1$. The advantage is that for real-valued arguments the larger index implies larger function values: $W_0 \geq W_{-1}$ and $T_1 \geq T_0$. See figure 1.

One should note that, in (1), $T_0\left(xe^{-x}\right) \neq x$. This follows from the relations

$$x = \begin{cases} T_0\left(xe^{-x}\right)\ , & x \leq 1\ , \\ T_1\left(xe^{-x}\right)\ , & x \geq 1\ . \end{cases} \tag{3}$$

It might seem natural to define one branch of T so that $x = T\left(xe^{-x}\right)$ for all x, but this cannot be done. The equation $ye^{-y} = xe^{-x}$ has two real solutions for y given any positive x, for example $x = 1/2$ gives either $y = 1/2$ or $y = 1.7564\ldots$ and thus, since $e^{-1/2}/2 = 0.303\ldots$, then $T(0.303\ldots)$ could be either 0.5 or $1.7564\ldots$, and only by having different branches can the correct choice be specified. The branch definitions impose a non-differentiable corner at $x = 1$ for each branch of $T_k(xe^{-x})$. See figure 2. As we shall see, the nontrivial branch can be described parametrically by $y = v\exp(v)/(\exp(v)-1)$ and $x = v/(\exp(v)-1)$ where the branch difference $v = y - x$ runs from $-\infty$ to ∞.

[1] We make two changes in notation from Gautschi [4]. Gautschi used the Lambert W function while we use the Tree T function; we omit a subscript specifying the range of integration.

[2] See §1.1.

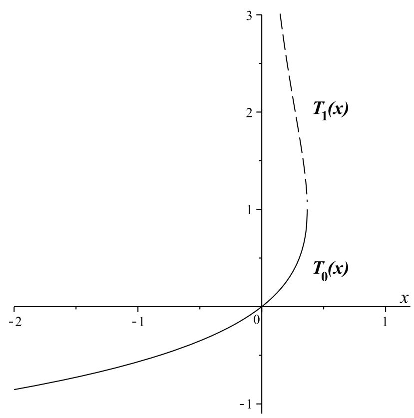

Figure 1: The Tree T function $T_k(z) = -W_{-k}(-z)$. The principal branch $T_0(z)$ satisfies $T_0(z) \le 1$, while the only other branch with real values has $T_1(z) \ge 1$. The branch point is at $z = 1/e$, where $T(z) = 1$. Note that the branch indices change sign from those of Lambert W.

1.1 Properties

We have

$$T_0(z) = \sum_{n \ge 1} \frac{n^{n-1}}{n!} z^n \tag{4}$$

for $-1/e \le z < 1/e$. The Tree T function is the generating function for the number of rooted trees with n nodes. The series for $T_1(z)$ near $z = 0$ is also of interest:

$$T_1(z) = \ln \frac{1}{z} + \ln \ln \frac{1}{z} + \frac{\ln \ln \frac{1}{z}}{\ln \frac{1}{z}} + \frac{\ln \ln \frac{1}{z} - \frac{1}{2} \ln^2 \ln \frac{1}{z}}{\ln^2 \frac{1}{z}} + \cdots , \tag{5}$$

and the higher-order terms can be expressed in terms of Stirling numbers. This series converges for small enough $z > 0$, as proved first by Comtet in a different context.

These results allow us to write down convergence requirements for the integrals. For I_0, we use (4) to see that as $x \to \infty$ the integrand behaves as $(xe^{-x})^\alpha x^{-\beta}$, whence convergence requires $\alpha > 0$ or $\alpha = 0$ and $\beta > 1$. For I_1, the behaviour at $x = 0$ is critical, and in that case, we use (5) to see $T_1(xe^{-x}) = O\left(\ln(1/(xe^{-x}))\right) = O(-\ln x + x)$, and hence the integrand is $O(x^{-\beta} \ln^\alpha x)$, so $\beta < 1$.

1.2 Change of variables relating the integrals

In view of (3), the integrands in (1) and (2) can be made to look symmetrical. Thus (1) can be written

$$I_0(\alpha, \beta) = \int_1^\infty T_0^\alpha \left(xe^{-x}\right) T_1^{-\beta} \left(xe^{-x}\right) \, dx \, . \tag{6}$$

and the integral (2) can be written

$$I_1(\alpha, \beta) = \int_0^1 T_1^\alpha \left(xe^{-x}\right) T_0^{-\beta} \left(xe^{-x}\right) \, dx \, . \tag{7}$$

It should be noted that this symmetry could introduce a numerical difficulty in (6) because the argument of T_1 would underflow for large x. In IEEE double precision, this would happen for $x > x^* = 714.968$, which satisfies $x^* \exp(-x^*) = 2^{-1022}$, which is the minimum positive real in IEEE. Because we are working symbolically, this is not a problem here.

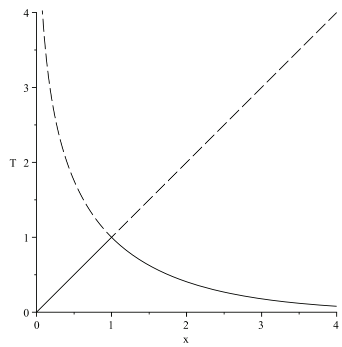

Figure 2: The equation $y\exp(-y) = x\exp(-x)$ has two solutions for y, which we may write as $y = T_k(x\exp(-x))$ for $k = 0$ (solid line) and $k = 1$ (dashed line). Notice that each curve has a corner at $x = 1$; at that point the trivial solution $y = x$ crosses the nontrivial solution, and the descriptions of each solution in terms of the Tree T function changes. A parametric description of the nontrivial solution is given by $y = v\exp(v)/(\exp(v) - 1)$ and $x = v/(\exp(v) - 1)$ where v runs from $-\infty$ to ∞. If $v \geq 0$, we recover the branch with $x \leq 1$, whereas if $v \leq 0$ then $x \geq 1$.

2 Branch Differences

In other contexts, the branch difference $q(x) = T_1(x) - T_0(x)$ has proved useful. It obeys, or solves, the equation

$$x = \frac{q}{e^q - 1}\exp\left(\frac{q}{1 - e^q}\right) .$$

Here we use a related branch difference,

$$v = T_1\left(xe^{-x}\right) - T_0\left(xe^{-x}\right) , \tag{8}$$

to simplify the integrals. A similar transformation has been used by [7] for an integral not containing the Tree T function. They used the transformation without describing it as a branch difference.

In detail, we note that for $x > 0$, we have $v \geq 0$, with $v = 0$ only at $x = 1$. Then

$$xe^{-x} = T_0\left(xe^{-x}\right)\exp\left(-T_0\left(xe^{-x}\right)\right) = (T_0 + v)e^{-T_0 - v} .$$

So

$$T_0 e^{-T_0} = (T_0 + v)e^{-T_0 - v} ,$$

or $T_0 = (T_0 + v)e^{-v}$. This linear equation can be solved if $v \neq 0$ to get

$$T_0 = \frac{ve^{-v}}{1 - e^{-v}} = \frac{v}{e^v - 1} ,$$

and this recovers the correct limit $T_0 \to 1$ as $v \to 0$. Then

$$T_1 = T_0 + v = \frac{v}{e^v - 1} + v = \frac{ve^v}{e^v - 1} = \frac{v}{1 - e^{-v}} .$$

Finally, $x = T_0(xe^{-x})$ if $x \leq 1$ gives

$$x = \frac{v}{e^v - 1} \qquad \text{if} \quad x \leq 1 \tag{9}$$

and $x = T_1(xe^{-x})$ if $x \geq 1$ gives

$$x = \frac{v}{1-e^{-v}} \qquad \text{if} \quad x \geq 1 \ . \tag{10}$$

Replacing v by $-v$ interchanges (9) and (10), which is the basis of the remark at the end of §1.

For the integral I_0 we use the substitution (9). Here $x = 1$ corresponds to $v = 0$ and $x = \infty$ to $v = \infty$. Thus, since

$$\mathrm{d}x = \frac{1-(1+v)e^{-v}}{(1-e^{-v})^2}\mathrm{d}v \ ,$$

with a harmless removable singularity at $v = 0$,

$$I_0 = \int_{v=0}^{\infty} \left(\frac{v}{e^v - 1}\right)^{\alpha} \left(\frac{v}{1-e^{-v}}\right)^{-\beta} \left(\frac{1-(1+v)e^{-v}}{(1-e^{-v})^2}\right) \mathrm{d}v \ ,$$

and this integral contains only elementary functions in its integrand. A separate analysis of convergence shows it requires $\alpha > 0$ or $\alpha = 0$ and $\beta > 1$ as above. Observe that all singularities at $v = 0$ are harmless; $e^v - 1 = O(v)$ and $1-(1+v)e^{-v} = O(v^2)$, and thus each bracketed term is $O(1)$ as $v \to 0$.

At this point, it is tempting to split the integrand into 3 terms

$$\frac{v^{\alpha-\beta}e^{(2-\beta)v}}{(e^v-1)^{\alpha-\beta+2}} - \frac{v^{\alpha-\beta}e^{(1-\beta)v}}{(e^v-1)^{\alpha-\beta+2}} + \frac{v^{\alpha-\beta+1}e^{(1-\beta)v}}{(e^v-1)^{\alpha-\beta+2}}$$

and search in tables for the common pattern $\int v^a e^{-bv}(e^v-1)^{-c}dv$, e.g., [9, §2.3.13-6]. Unfortunately, the separate integrals do not converge.

For I_1, we use (10), whence $\mathrm{d}x = ((1-v)e^v - 1)(e^v-1)^{-2}\mathrm{d}v$ and $x = 1$ becomes $v = 0$ and $x = 1$ is $v \to \infty$. Thus

$$I_1 = \int_{v=0}^{\infty} \left(\frac{v}{1-e^{-v}}\right)^{\alpha} \left(\frac{v}{e^v-1}\right)^{-\beta} \left(\frac{1-(1-v)e^{v}}{(e^v-1)^2}\right) \mathrm{d}v \ .$$

Again, all singularities at $v = 0$ are removable. This integral converges for $\beta < 1$, as above.

These definite integrals no longer contain the Tree T function and will be amenable to direct methods or contour integration methods. Note that although several definite integrals for I_0 and I_1 were given in [4], the expressions given here appear to be new.

Because the integrals for I_0 and I_1 are so similar, we can easily establish that

$$I(\alpha,\beta) = I_0(\alpha,\beta) + I_1(1-\beta, 1-\alpha) = \int_0^{\infty} e^{-\alpha v}\left(\frac{v}{1-e^{-v}}\right)^{\alpha-\beta+1} dv \ . \tag{11}$$

In the case $\alpha = \beta$ this can be shown to be $\Psi_1(\alpha)$, the trigamma function. One method to see this is to write $1/(1-\exp(-v))$ as a geometric series and then integrate term by term to get $1/(\alpha+k)^2$; summing over k gives the trigamma function. Similarly, one can see with one integration by parts that $I_0(\alpha,\alpha) = \alpha\Psi_1(\alpha)$, and from thence that $I_1(\alpha,\alpha) = \alpha\Psi_1(1-\alpha)$.

3 Special Cases

As just discussed, the paper [4] notes that the integral is a trigamma function if $\alpha = \beta$. Neither Maple nor Wolfram Alpha can identify the trigamma function from the integrals presented here. However, both systems can evaluate the integrals for a variety of integer values of α and β. For instance, Maple easily evaluates

$$I_0(1,2) = \int_0^{\infty} \frac{\mathrm{e}^{-v}\left(1-(1+v)\,\mathrm{e}^{-v}\right)}{v\left(1-\mathrm{e}^{-v}\right)} dv = 1-\gamma \ , \tag{12}$$

$$I_0(2,3) = \int_0^\infty \frac{\mathrm{e}^{-2\,v}\,(1-(1+v)\,\mathrm{e}^{-v})}{v\,(1-\mathrm{e}^{-v})} dv = \tfrac{3}{2} - \gamma - \ln 2\,, \tag{13}$$

and

$$I_0(3,2) = \int_0^\infty \frac{\mathrm{e}^{-3\,v} v\,(1-(1+v)\,\mathrm{e}^{-v})}{(1-\mathrm{e}^{-v})^3} dv = \tfrac{5}{2} - 6\zeta(3) + \tfrac{1}{2}\pi^2\,. \tag{14}$$

Similarly, I_1 is easily evaluated in Maple if α and β are integer values (or are equal).

$$I_1(2,0) = -\int_0^\infty \frac{v^2 \mathrm{e}^{-v}\,(1-v-\mathrm{e}^{-v})}{(1-\mathrm{e}^{-v})^4} dv = \tfrac{1}{3} + 2\zeta(3) + \tfrac{1}{3}\pi^2\,. \tag{15}$$

An interesting variation is that Maple can sometimes evaluate the integrals if α and β differ by an integer but are not themselves integers:

$$I_0\left(\tfrac{3}{2},\tfrac{1}{2}\right) = \int_0^\infty e^{-3v/2} \frac{(1-e^{-v}-ve^{-v})}{(1-e^{-v})^3}\, dv = -\tfrac{1}{2} + \tfrac{3}{4}\pi^2 - \tfrac{21}{4}\zeta(3)\,. \tag{16}$$

This does not always work, however. If $\alpha = 1/4$ and $\beta = -3/4$, then the difference is an integer but neither Maple nor Mathematica is able to evaluate the integral. For the simpler integral $I = I_0(\alpha,\beta) + I_1(1-\beta, 1-\alpha)$ this also happens.

At the time of writing, we do not know if any computer algebra system can evaluate these integrals for values of α and β that have non-integer differences, or even merely for arbitrary α and β whose difference is an integer.

4 Series

We pointed out earlier that when $\alpha = \beta$ the integrals for I_0 and I_1 could be identified as containing Ψ_1, the trigamma function, by using a series expansion. One is tempted to try the same thing for $\alpha \neq \beta$. We are successful in writing $I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha)$ as a series, as follows.

The integrand in equation (11) can be expanded in a convergent series if $v > 0$.

$$\begin{aligned} e^{-\alpha v}\left(\frac{v}{1-\exp(-v)}\right)^{\alpha-\beta+1} &= v^{\alpha-\beta+1} e^{-\alpha v}\left(1-e^{-v}\right)^{\beta-1-\alpha} \\ &= v^{\alpha-\beta+1}\sum_{\ell\ge 0}\binom{\beta-1-\alpha}{\ell}(-1)^\ell e^{-(\alpha+\ell)v}\,. \end{aligned} \tag{17}$$

Now

$$\int_0^\infty v^{\alpha-\beta+1} e^{-(\alpha+\ell)v}\, dv = \frac{\Gamma(\alpha-\beta+2)}{(\alpha+\ell)^{\alpha-\beta+2}} \tag{18}$$

so the series for the sum $I_0(\alpha,\beta) + I_1(1-\beta,1-\alpha)$ is

$$\begin{aligned} I &= \sum_{\ell\ge 0}(-1)^\ell \binom{\beta-1-\alpha}{\ell}\frac{\Gamma(\alpha-\beta+2)}{(\alpha+\ell)^{\alpha-\beta+2}} \\ &= \Gamma(\alpha-\beta+2)\sum_{\ell\ge 0}\binom{\alpha-\beta+\ell}{\ell}\frac{1}{(\alpha+\ell)^{\alpha-\beta+2}}\,. \end{aligned} \tag{19}$$

Notice that we need $\beta > -2$ for the integral (18) to converge and for the series to be valid. This is a new restriction, additional to the $\alpha > 0$ needed for convergence of the original integrals $I_0(\alpha,\beta)$ and $I_1(1-\beta,1-\alpha)$.

When $\alpha = \beta$ this reduces, as claimed, to

$$\Psi_1(\alpha) = \sum_{\ell \geq 0} \frac{1}{(\alpha+\ell)^2} \,. \tag{20}$$

If we introduce a new variable m with the definition $\beta = \alpha - m$, the sum in (19) becomes

$$S(m,\alpha) = \sum_{\ell \geq 0} \binom{m+\ell}{\ell} \frac{1}{(\alpha+\ell)^{m+2}} \,. \tag{21}$$

For explicit integers m, Maple can evaluate this sum in terms of known special functions such as the polygamma functions $\Psi_j(\alpha)$ for $j \leq m+1$. For example,

$$\begin{aligned} S(4,\alpha) =& \tfrac{1}{24}\,\Psi_1(\alpha) + \left(\tfrac{1}{12}\,\alpha - \tfrac{5}{24}\right)\Psi_2(\alpha) + \left(-\tfrac{5}{24}\,\alpha + \tfrac{1}{24}\alpha^2 + \tfrac{35}{144}\right)\Psi_3(\alpha) \\ &+ \tfrac{1}{288}\,(2\,\alpha - 5)\left(\alpha^2 - 5\,\alpha + 5\right)\Psi_4(\alpha) \\ &+ \tfrac{1}{2880}\,(\alpha-1)\,(\alpha-2)\,(\alpha-3)\,(\alpha-4)\,\Psi_5(\alpha) \,. \end{aligned} \tag{22}$$

Indeed, for explicit integers $m \geq 0$, this sum is proportional to the value of a hypergeometric function:

$$S(m,\alpha) = \frac{1}{\alpha^{m+2}} \cdot {}_{(m+3)}F_{(m+2)}\left(\left.\begin{matrix} \alpha, \alpha, \ldots, \alpha, m+1 \\ \alpha+1, \alpha+1, \ldots, \alpha+1 \end{matrix}\right| 1\right) \,. \tag{23}$$

The notation follows [6, 5.5], except that we have retained the subscripts, owing to their being symbolic[3]. To show this, we recall that

$${}_pF_q\left(\left.\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{\alpha_1^{\overline{k}} \alpha_2^{\overline{k}} \ldots \alpha_p^{\overline{k}}}{\beta_1^{\overline{k}} \beta_2^{\overline{k}} \ldots \beta_q^{\overline{k}}} \cdot \frac{z^k}{k!}$$

Then we can write

$$\binom{m+\ell}{\ell} \frac{1}{(\alpha+\ell)^{m+2}} = \frac{(m+1)^{\overline{\ell}}}{\ell!} \left(\frac{\alpha^{\overline{\ell}}}{\alpha(\alpha+1)^{\overline{\ell}}}\right)^{m+2} ,$$

and the result (23) follows. In summary, we have the following expression for $I(\alpha)$:

$$I(\alpha, \alpha - m) = \frac{\Gamma(m+2)}{\alpha^{m+2}} \cdot {}_{(m+3)}F_{(m+2)}\left(\left.\begin{matrix} \alpha, \alpha, \ldots, \alpha, m+1 \\ \alpha+1, \alpha+1, \ldots, \alpha+1 \end{matrix}\right| 1\right) \,.$$

Finally, at least one of the integrals with $\alpha = 1/4$ and $\beta = -3/4$ that Maple was unable to integrate explicitly before can be found using these sums, namely

$$\begin{aligned} I\left(\frac{1}{4}, -\frac{3}{4}\right) &= \Gamma\left(\frac{1}{4} - \left(-\frac{3}{4}\right) + 2\right) S(1, \tfrac{1}{4}) \\ &= 2S(1, \tfrac{1}{4}) = 24\,i\,(P_3(-i) - P_3(i)) + 2\pi^2 + 16C + 42\zeta(3) \end{aligned} \tag{24}$$

where $P_a(z) = \sum_{n \geq 1} z^n/n^a$ is the polylogarithm function and

$$C = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2}$$

is Catalan's constant. This last example, of which we are sure there are many more, points the way to improve some of Maple's evaluation of definite integrals.

[3]From [6]: "Standard reference books often use '${}_mF_n$' instead of 'F' as the name of a hypergeometric with m upper parameters and n lower parameters. But the extra subscripts tend to clutter up the formulas and waste our time, if we are compelled to write them over and over. We can count how many parameters there are, so we don't need extra additional unnecessary redundancy." In the present case the number of parameters cannot be counted.

5 Comparison with Examples from [4]

We have seen here a reference solution for some examples in which α and β differ by an integer. For those integrals for which we cannot derive a symbolic solution, we could use numerical methods such as those in Maple's `evalf/Int`, but this is less interesting. Tables 1 and 2 show that Gautschi's results are as accurate as he claimed.

Numerical integration of I_0 and I_1 in the forms containing T is not challenging in Maple; Gautschi's task was to do it outside a system that had easy and accurate evaluation of T, or W. Numerical evaluation of the exponential forms given here is also not challenging: the singularities at $v = 0$ are removable and standard tricks for the accurate evaluation of $e^v - 1$, or $\ln(1 - y)$ in an equivalent logarithmic form of the integral, make it easy. Moreover, Maple's `evalf/Int` is more powerful yet. It uses singularity detection and generalized series to eliminate most difficulties [5], and has no trouble here.

6 Concluding Remarks

One aim of this paper is to provide reference expressions for the integrals (1) and (2) in terms of quantities such as γ, the Euler-Mascheroni Constant, and evaluations of functions such as the Riemann ζ function, which we consider to be known and partially understood.[4]

Since the discovery of these special forms was the result of examining the properties of the two real branches of the Tree T function, as well as the properties of their difference, the exploration of branch relations in other Lambert W integrals may lead to further development of solutions to special cases.

This paper has shown a transformation that takes some integrals from forms containing T, or W, to an elementary form which can then be attacked by computer algebra systems, or perhaps by making further transformations by hand to match table entries such as [9, §2.3.13-6]

$$\int_0^\infty \frac{x^n e^{-px}}{(e^x - 1)^\rho} \mathrm{d}x = (-1)^n \frac{\partial^n}{\partial p^n} B(1 - \rho, p - \rho) \ , \tag{25}$$

where B is the beta-function. Our expression for I in terms of $S(m, \alpha)$ seems neither known to CAS nor to be in the tables, and we have identified a weakness in some CAS. Of course that is not surprising, since definite integrals are hard. It is interesting that asking Maple to evaluate the nth derivative in (25) for a symbolic n reproduced our hypergeometric formula (23). The Maple session is

```
> diff(Beta(1-rho, p+rho), p$n) assuming n::integer;
```

$$\frac{(-1)^n n!\, \Gamma(p+\rho)^{n+1} \mathrm{hypergeom}([\rho, p+\rho\$n+1],\ [p+\rho+1\$n+1],\ 1)}{\Gamma\,(p+\rho+1)^{n+1}}$$

The output uses the curious \$ function, which means 'sequence' and is difficult to read, but Maple got it right! One has to read $p + \rho\$n + 1$ as $(p + \rho)\$(n + 1)$, or $n + 1$ occurrences of $p + \rho$.

Symbolic expressions can shed light on questions such as the asymptotic behaviour of the integrals for large α, or small α. They can also contribute to questions regarding the conditioning or ill-conditioning of the integrals. Such questions will be pursued in future work.

We thank the referee, who brought the results in Prudnikov *et al.* [9] and the reference [7] to our attention.

[4] Finch [3] points out that 'only' several hundred million digits of γ are known.

Table 1: The relative error of Gautschi's approximations $G(\alpha,\beta)$ when compared with exact symbolic values $I_0^{(s)}(\alpha,\beta)$. Note that a dash indicates integrals for which we do not have symbolic expressions. We also include the differences between Gautschi's approximations and Maple's `evalf/Int` using `Digits:=35`, denoted $I_0^{(m)}$.

α	β	$\left\lvert\frac{I_0^{(s)}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)}\right\rvert$	$\left\lvert\frac{I_0^{(m)}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)}\right\rvert$
2	2	3.58×10^{-32}	3.60×10^{-32}
	0	2.04×10^{-31}	2.04×10^{-31}
	-2	1.40×10^{-32}	1.40×10^{-32}
1	1	7.46×10^{-33}	7.44×10^{-33}
	0	2.20×10^{-32}	2.20×10^{-32}
	-1	2.31×10^{-32}	2.31×10^{-32}
$\frac{1}{2}$	2	-	3.35×10^{-33}
	0	-	2.24×10^{-32}
	-2	-	6.43×10^{-33}

Table 2: The relative error of Gautschi's approximations $G(\alpha,\beta)$ when compared with exact symbolic values for $I_1^{(s)}(\alpha,\beta)$. Note that a dash indicates integrals for which we do not have symbolic expressions. We also include the differences between Gautschi's approximations and Maple's `evalf/Int` using `Digits:=35`, denoted $I_1^{(m)}$.

α	β	$\left\lvert\frac{I_1^{(s)}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)}\right\rvert$	$\left\lvert\frac{I_1^{(m)}(\alpha,\beta)-G(\alpha,\beta)}{G(\alpha,\beta)}\right\rvert$
2	$\frac{1}{2}$	-	5.60×10^{-35}
	0	2.71×10^{-33}	2.70×10^{-33}
	-2	1.31×10^{-32}	1.29×10^{-32}
$\frac{1}{2}$	$\frac{1}{2}$	1.09×10^{-32}	1.09×10^{-32}
	0	-	5.00×10^{-33}
	-1	-	2.10×10^{-33}
$-\frac{1}{2}$	0	-	3.41×10^{-33}
	-1	-	7.72×10^{-33}
	-2	-	1.78×10^{-32}

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