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The forces and couples acting on two nearly touching spheres in low-Reynolds-number flow

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1. Introduction

Two spheres, whose radii are a and b, are separated by a gap of width ε. Asymptotic expressions for the forces and couples acting on the spheres have been calculated by O'Neill & Majumdar [1] and by Majumdar [2] correct to order $\ln \varepsilon$ for all ratios $k = b/a$. In addition, O'Neill & Stewartson [3] and Cooley & O'Neill [4] have calculated results correct to order $\varepsilon \ln \varepsilon$ for the special case of a sphere near a plane ($k = \infty$). The object of this paper is to extend O'Neill & Majumdar's work to order $\varepsilon \ln \varepsilon$ for all k. It is important to do this because terms 0(1) appear in the asymptotic expressions between the $\ln \varepsilon$ and $\varepsilon \ln \varepsilon$ terms, and these 0(1) terms are the ones needed to estimate the mobility of unequal spheres acted on by specified forces [5]. During the calculations, some new results of physical interest were noticed and an error in [4] corrected. Otherwise the only new principle in the calculation is the handling of long algebraic expressions, which was accomplished by using the computer algebra systems Camal and Reduce [6, 7].

We follow the notation of [1] and consider a moving sphere S_A of radius a and a stationary sphere S_B of radius $-a/\lambda$. (Unlike [1] we shall consider only external spheres, so λ is always negative.) Three problems will be studied (see Fig. 1). As in [1], problems I and II are those in which sphere S_A rotates about or translates along an axis perpendicular to the line of centres, while problem III is that studied in [2] and requires S_A to rotated about the line of centres.

2. Problem I

The singular terms in the expressions for the forces and couples are determined by examining the flow in the gap between the spheres. We take cylindrical coordinates $(a r, \theta, a z)$ with the z-axis along the line of centres and the origin at

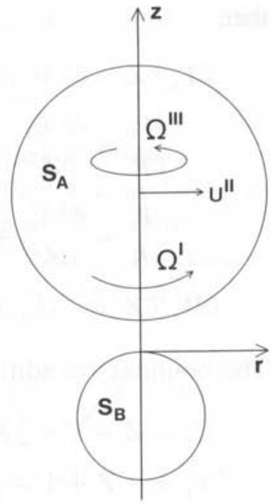

Figure 1
The three types of motion studied

the point of S_B that is nearest to S_A. Denoting the gap width by $a\,\varepsilon$, so that $\varepsilon \ll 1$, we stretch the coordinates according to

$$R = \varepsilon^{-1/2} r \quad \text{and} \quad Z = \varepsilon^{-1} z\,. \tag{2.1}$$

The surfaces of the spheres are then given by

$$Z_A = 1 + \tfrac{1}{2} R^2 + \varepsilon \tfrac{1}{8} R^4 + 0(\varepsilon^2), \tag{2.2}$$

$$Z_B = \tfrac{1}{2} \lambda R^2 + \varepsilon \tfrac{1}{8} \lambda^3 R^4 + 0(\varepsilon^2). \tag{2.3}$$

If sphere S_A rotates with angular velocity Ω^I, we can write the pressure field in the form $p = \mu\, \Omega^I \cos\theta\, P(r, z)$ and the cylindrical velocity components in the form $\Omega^I a\,[U \cos\theta,\ V \sin\theta,\ W \cos\theta]$. These quantities are now expanded in powers of ε:

$$P(r,z) = \varepsilon^{-3/2} P_0(R,Z) + \varepsilon^{-1/2} P_1(R,Z) + 0(\varepsilon^{1/2}), \tag{2.4}$$

$$U(r,z) = U_0(R,Z) + \varepsilon\, U_1(R,Z) + 0(\varepsilon^2), \tag{2.5}$$

$$V(r,z) = V_0(R,Z) + \varepsilon\, V_1(R,Z) + 0(\varepsilon^2), \tag{2.6}$$

$$W(r,z) = \varepsilon^{1/2} W_0(R,Z) + \varepsilon^{3/2} W_1(R,Z) + 0(\varepsilon^{5/2}). \tag{2.7}$$

The solutions for P_0, U_0, V_0 and W_0 were given in [1]; here we wish to find P_1, U_1, V_1 and W_1. The equations governing the next order are as follows. Define an operator

$$\Upsilon = \partial^2/\partial R^2 + R^{-1}\, \partial/\partial R\,, \tag{2.8}$$

then

$$\partial P_1/\partial Z = \partial^2 W_0/\partial Z^2, \tag{2.9}$$

$$\frac{\partial P_1}{\partial R} = \frac{\partial^2 U_1}{\partial Z^2} + \Upsilon U_0 - \frac{2}{R^2}(U_0 + V_0), \tag{2.10}$$

$$-\frac{P_1}{R} = \frac{\partial^2 V_1}{\partial Z^2} + \Upsilon V_0 - \frac{2}{R^2}(U_0 + V_0), \tag{2.11}$$

$$\partial W_1/\partial Z = -(U_1 + V_1)/R - \partial U_1/\partial R. \tag{2.12}$$

The boundary conditions imposed on the surface $Z = 1 + (1/2)R^2$ are

$$U_1 = Z - 1 - \tfrac{1}{8} R^4\, \partial U_0/\partial Z, \tag{2.13}$$

$$V_1 = -Z + 1 - \tfrac{1}{8} R^4\, \partial V_0/\partial Z, \tag{2.14}$$

$$W_1 = -\tfrac{1}{8} R^4\, \partial W_0/\partial Z, \tag{2.15}$$

while on $Z = (1/2)\lambda R^2$, the boundary conditions are

$$U_1 = -\tfrac{1}{8}\lambda^3 R^4\, \partial U_0/\partial Z, \tag{2.16}$$

$$V_1 = -\tfrac{1}{8}\lambda^3 R^4\, \partial V_0/\partial Z, \tag{2.17}$$

$$W_1 = -\tfrac{1}{8}\lambda^3 R^4\, \partial W_0/\partial Z. \tag{2.18}$$

The key step in solving these equations is the derivation of a differential equation for the pressure (the Reynolds equation). From (2.9), we find

$$P_1 = \partial W_0/\partial Z + Q(R), \tag{2.19}$$

where Q is to be determined.

The integration of the remaining equations can be easily performed by a computer, because integration is always with respect to Z and this variable appears in the equations only as Z^n, for some n. On the other hand the fact that the equations contain terms divided by the factor

$$H(R) = 1 + (1/2)(1 - \lambda)R^2$$

tended to produce ungainly expressions. It was found that the computer could be helped in its simplification of the expressions if λ was replaced by $q = (1 - \lambda)/2$, thus reducing H to the sum of two terms. In terms of q, the equation for Q was found to be

$$R^2 Q'' + [R + 6qR^3H^{-1}]Q' - Q$$
$$= [-\tfrac{72}{25}q^5R^9 - (40q^4 - \tfrac{224}{5}q^3 + \tfrac{102}{5}q^2)R^7 + \ldots]H^{-5}. \tag{2.20}$$

The lower powers of R will not be presented. A particular integral of this equation is

$$Q = [\tfrac{12}{25} q^4 R^7 + (\tfrac{596}{125} q^3 - \tfrac{112}{25} q^2 + \tfrac{51}{25} q) R^5 + \ldots] H^{-4}. \quad (2.21)$$

As in [3] and [4], the complementary functions need not be considered.

We now wish to obtain the singular contributions to the forces and couples following the method of [3], which is based on extracting the R^{-1} term in the integrands for the forces and couples. The integrands for sphere S_A are given in [3], and the integrands for sphere S_B are as follows. The force is $-6\pi\mu a^2 \Omega^I f_{12}$, and for this we consider

$$-\frac{1}{6} R\left(1 + \frac{1}{2}\varepsilon\lambda^2 R^2\right)\left[\lambda R(P_0 + \varepsilon P_1) + \left(1 - \frac{1}{2}\varepsilon\lambda^2 R^2\right)\frac{\partial}{\partial Z}(U - V)\right.$$
$$\left. - \varepsilon 2\lambda R\, \partial U/\partial R + \varepsilon\, \partial W_0/\partial R + \varepsilon\lambda R^2\, \partial(R^{-1} V)/\partial R - \varepsilon\lambda U + \varepsilon R^{-1} W_0\right], \quad (2.22)$$

while for the couple $-8\pi\mu a^3 \Omega^I g_{12}$ we consider

$$-\frac{1}{8}\lambda^{-1} R\left(1 + \frac{1}{2}\varepsilon\lambda^2 R^2\right)\left[(1 - \varepsilon\lambda^2 R^2)\left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z} + \varepsilon\frac{\partial W_0}{\partial R} + \varepsilon\frac{W_0}{R}\right)\right.$$
$$\left. - \varepsilon\lambda^2 R^2\frac{\partial U}{\partial Z} + \varepsilon\lambda R\left(2\frac{\partial W_0}{\partial Z} - 2\frac{\partial U}{\partial R} + R\frac{\partial(R^{-1} V)}{\partial R} - \frac{U}{R}\right)\right]. \quad (2.23)$$

The conclusions are that

$$f_{11} = \frac{2(1 - 4\lambda)}{15(1 - \lambda)^2}\ln\varepsilon + A_{11}(\lambda) + \frac{86 - 166\lambda - 66\lambda^2 - 64\lambda^3}{375(1 - \lambda)^3}\varepsilon\ln\varepsilon + 0(\varepsilon) \quad (2.24)$$

$$f_{12} = -\frac{2\lambda^2(1 - 4\lambda)}{15(1 - \lambda)^2}\ln\varepsilon + A_{12}(\lambda) - \frac{\lambda^2(86 - 166\lambda - 66\lambda^2 - 64\lambda^3)}{375(1 - \lambda)^3}\varepsilon\ln\varepsilon + 0(\varepsilon) \quad (2.25)$$

$$g_{11} = \frac{-2}{5(1 - \lambda)}\ln\varepsilon + B_{11}(\lambda) - \frac{66 - 12\lambda + 16\lambda^2}{125(1 - \lambda)^2}\varepsilon\ln\varepsilon + 0(\varepsilon), \quad (2.26)$$

$$g_{12} = \frac{\lambda^2}{10(1 - \lambda)}\ln\varepsilon + B_{12}(\lambda) + \frac{\lambda^2(43 + 24\lambda + 43\lambda^2)}{250(1 - \lambda)^2}\varepsilon\ln\varepsilon + 0(\varepsilon). \quad (2.27)$$

The functions A and B must be found from the numerical data in [8]. Comparing these results in the special case $\lambda = 0$ with those in [4], we see that there is a sign difference in the $\varepsilon\ln\varepsilon$ terms. However, not only do the present signs improve the agreement with numerical data, they also agree with the results of problem II as the reciprocal theorem requires.

3. Problem II

The governing equations for the previous section remain unchanged, provided we replace Ω^I with U^{II}/a. The boundary conditions (2.13)–(2.15) must be replaced by

$$U_1 = -\tfrac{1}{8} R^4\, \partial U_0/\partial Z\,, \tag{3.1}$$

$$V_1 = -\tfrac{1}{8} R^4\, \partial V_0/\partial Z\,, \tag{3.2}$$

$$W_1 = -\tfrac{1}{8} R^4\, \partial W_0/\partial Z\,. \tag{3.3}$$

The conditions (2.16)–(2.18) remain unchanged. The equation for $Q\,(R)$ is now

$$R^2 Q'' + [R + 6\,q\,R^3 H^{-1}]\, Q' - Q$$
$$= [\tfrac{72}{25}\, q^4 (q-1)\, R^9 + (40\, q^4 - \tfrac{464}{5}\, q^3 + \tfrac{396}{5}\, q^2 - \tfrac{132}{5}\, q)\, R^7 + \ldots]\, H^{-5}$$

with the particular integral

$$Q = [-\tfrac{12}{25}\, q^3 (q-1)\, R^7 - (\tfrac{596}{125}\, q^3 - \tfrac{1256}{125}\, q^2 + \tfrac{198}{25}\, q - \tfrac{66}{25})\, R^5 + \ldots]\, H^{-4} \tag{3.4}$$

Following the same steps as before, we obtain the forces and couples for this problem. They are

$$f_{21} = \frac{-4\,(2-\lambda+2\lambda^2)}{15\,(1-\lambda)^3} \ln \varepsilon + A_{21}(\lambda) - \frac{4\,(16+45\lambda+58\lambda^2+45\lambda^3+16\lambda^4)}{375\,(1-\lambda)^4}\, \varepsilon \ln \varepsilon + 0(\varepsilon). \tag{3.5}$$

$$f_{22} = \frac{-4\lambda\,(2-\lambda+2\lambda^2)}{15\,(1-\lambda)^3} \ln \varepsilon + A_{22}(\lambda) - \frac{4\lambda\,(16+45\lambda+58\lambda^2+45\lambda^3+16\lambda^4)}{375\,(1-\lambda)^4}\, \varepsilon \ln \varepsilon + 0(\varepsilon). \tag{3.6}$$

$$g_{21} = \frac{1-4\lambda}{10\,(1-\lambda)^2} \ln \varepsilon + B_{21}(\lambda) + \frac{43-83\lambda-33\lambda^2-32\lambda^3}{250\,(1-\lambda)^3}\, \varepsilon \ln \varepsilon + 0(\varepsilon), \tag{3.7}$$

$$g_{22} = \frac{\lambda(4-\lambda)}{10\,(1-\lambda)^2} \ln \varepsilon + B_{22}(\lambda) + \frac{\lambda(32+33\lambda+83\lambda^2-43\lambda^3)}{250\,(1-\lambda)^3}\, \varepsilon \ln \varepsilon + 0(\varepsilon). \tag{3.8}$$

These results agree with the special cases. The fact that $3\,f_{11} = 4\,g_{21}$ and $3\,f_{12} = 4\,\lambda^2\, g_{22}\,(\lambda^{-1})$ because of the reciprocal theorem has been pointed out in [8].

4. Physical aspects of the results

The results (2.24)–(2.27) and (3.5)–(3.8) imply results of physical significance, or, if one prefers it, there are physical requirements that prevent the results from being independent of each other. Singular forces and couples arise because of the lubricating effects of viscous fluid between close surfaces that are in relative motion. Thus if the spheres were to have velocities and rotations that resulted in no relative motion of their surfaces in the neighbourhood of the gap, then the forces and couples acting should be non-singular.

To discuss this further, we change to the notation of O'Neill & Majumdar [8] in which $k = -\lambda^{-1}$. We combine problems I and II by giving S_A angular velocity Ω_1, and velocity U_1, and S_B angular velocity Ω_2 and velocity U_2. The force and couple acting on sphere S_A are given by [8] as

$$F_A = -6\pi\mu a\,[\Omega_1 a f_{11}(k,\varepsilon) - \Omega_2 a f_{12}(k^{-1},\varepsilon k^{-1}) + U_1 f_{21}(k,\varepsilon) + U_2 f_{22}(k^{-1},\varepsilon k^{-1})], \tag{4.1}$$

$$G_A = -8\pi\mu a^2\,[\Omega_1 a g_{11}(k,\varepsilon) - \Omega_2 a g_{12}(k^{-1},\varepsilon k^{-1}) + U_1 g_{21}(k,\varepsilon) + U_2 g_{22}(k^{-1},\varepsilon k^{-1})]. \tag{4.2}$$

There are two motions that lead to no relative movement of the surfaces in the gap region. The first motion is that for which $U_1 = U_2$ and $\Omega_1 = \Omega_2 = 0$, i.e. side by side translation. The second motion is rotation as a rigid dumbbell, for which $\Omega_1 = \Omega_2$ and $U_2 = U_1 - a(1+k+\varepsilon)\,\Omega_1$. From (4.1) and (4.2) we therefore conclude that the following combinations should be independent of terms in $\ln\varepsilon$ and $\varepsilon\ln\varepsilon$:

$$f_{21}(k,\varepsilon) + f_{22}(k^{-1},\varepsilon k^{-1}),$$

$$g_{21}(k,\varepsilon) + g_{22}(k^{-1},\varepsilon k^{-1}),$$

$$f_{11}(k,\varepsilon) - f_{12}(k^{-1},\varepsilon k^{-1}) - (1+k+\varepsilon)\, f_{22}(k^{-1},\varepsilon k^{-1}),$$

$$g_{11}(k,\varepsilon) - g_{12}(k^{-1},\varepsilon k^{-1}) - (1+k+\varepsilon)\, g_{22}(k^{-1},\varepsilon k^{-1}),$$

as indeed they are. A motion which seems to offer the possibility of non-singular forces is that of rolling, in which $U_1 = U_2 = 0$ and $\Omega_1 = -k\Omega_2$. Such is not the case, however, because the couple remains singular.

5. Numerical results

The asymptotic results obtained above can be combined with the numerical results given in [8] to obtain values for the functions A_{ij} and B_{ij} appearing in (2.24)–(2.27) and (3.5)–(3.8). These functions have been used extensively in calculations of the mobilities of spheres [5] and it is important to provide an independent check of the accuracy of the tabulations. This was done by following the methods of [6]. From the numerical tabulations of [8], we subtract the singular terms and then a straight line is fitted to the non-singular data thus obtained and the intercept with $\varepsilon = 0$ taken as the value of the required function. The values thus obtained for the case $\lambda = -1$ (equal spheres) are $A_{11} = 0.1594$, $A_{12} = -0.0011$, $A_{21} = 0.9983$, $A_{22} = -0.2737$ and $B_{11} = 0.7029$, $B_{12} = 0.0274$. These agree to within 0.0001 with the corresponding results used in [5].

The sign error in [4] was corrected and the new values agree more closely with the numerical values. The terms $3.0\,\varepsilon$ and $7.2\,\varepsilon$ estimated by [4] now become $0.1\,\varepsilon$ and $0.3\,\varepsilon$.

6. Problem III

No previous asymptotic analysis exists for this problem, the work of Majumdar [2] being confined to the case of touching spheres. The velocity field when S_A rotates about the line of centres with angular velocity Ω^{III} is $a\,\Omega^{III}\,[o, V, o]$ and the pressure field is zero. In unstretched coordinates, V obeys

$$\partial^2 V/\partial z^2 + \partial^2 V/\partial r^2 + r^{-1}\,\partial V/\partial r - r^{-2}\,V = 0\,. \tag{6.1}$$

We look for a solution in the form

$$V = \varepsilon^{1/2}\,V_0(R, Z), \tag{6.2}$$

which is stretched coordinates obeys

$$\partial^2 V_0/\partial Z^2 = 0\,, \tag{6.3}$$

subject to the boundary conditions

$$V_0 = R \quad \text{on} \quad Z = 1 + \tfrac{1}{2} R^2\,, \tag{6.4}$$

and

$$V_0 = 0 \quad \text{on} \quad Z = \tfrac{1}{2}\lambda R^2\,. \tag{6.5}$$

The solution is

$$V_0 = (R\,Z + \tfrac{1}{2}\lambda R^3)\,H^{-1}\,. \tag{6.6}$$

The singular contribution to the couple is obtained from the R^{-1} term in the integrand $2\pi\mu a^3\,\Omega^{III}\,\varepsilon\,R^2\,\partial V_0/\partial Z$. In this case the 0(1) term has been found in closed form by Majumdar [2], so we can write the couple $-8\pi\mu a^3\,\Omega^{III}\,h_{11}$ acting on S_A as

$$h_{11} = \frac{\zeta(3, (1-\lambda)^{-1})}{(1-\lambda)^3} - \frac{1}{2(1-\lambda)^2}\,\varepsilon\ln\varepsilon + 0(\varepsilon)\,. \tag{6.7}$$

On S_B, the couple $-8\pi\mu\Omega^{III}b^3 h_{12}$ is

$$h_{12} = \frac{-\lambda^3\zeta(3,1)}{(1-\lambda)^3} - \frac{\lambda^3}{2(1-\lambda)^2}\varepsilon\ln\varepsilon + 0(\varepsilon). \tag{6.8}$$

The function $\zeta(z, a)$ is defined as

$$\zeta(z, a) = \sum_{k=0}^{\infty}(k+a)^{-z}. \tag{6.9}$$

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Abstract

When two unequal spheres are very close, the low-Reynolds-number flow in the narrow gap between them can be analysed using lubrication approximations, and asymptotic formulae for the forces and couples acting on the spheres deduced. The expressions for the forces and couples have previously been regarded as independent, but it is shown here that they are linked by simple physical considerations. The new formulae can be used to improve the accuracy of companion calculations which apply to cases in which the spheres are not close.

Zusammenfassung

Es wird die Strömung bei kleinen Reynoldszahlen untersucht in einem Spalt zwischen zwei ungleichen Kugeln, die sich nahezu berühren. Asymptotische Formeln für die Kräfte und Momente an den Kugeln werden hergeleitet. Es wird gezeigt daß die Ausdrücke für die Kräfte und Momente, die bis jetzt als unabhängig betrachtet wurden, tatsächlich durch einfache physikalische Betrachtungen verknüpft werden können. Die neuen Formeln können zur Verbesserung der Genauigkeit von Rechnungen für größere Kugelabstände benützt werden.

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