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Stieltjes and other integral representations for functions of Lambert *W*

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We show that many functions containing the Lambert W function are Stieltjes functions. We extend the known properties of the set of Stieltjes functions and also prove a generalization of a conjecture of Jackson, Procacci & Sokal. In addition, we consider the relationship of functions of W to the class of completely monotonic functions and show that W is a complete Bernstein function.

Keywords: Lambert W function; Stieltjes functions; completely monotonic functions; Bernstein functions; complete Bernstein functions

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1. Introduction

The Lambert W function is the multivalued inverse of the mapping $W \mapsto We^W$. The branches, denoted by W_k $(k \in \mathbb{Z})$, are defined through the equations [11]

$$\forall z \in \mathbb{C}, \qquad W_k(z)\exp(W_k(z)) = z \ , \tag{1.1}$$

$$W_k(z) \sim \ln_k z \text{ as } \Re z \to \infty \ , \tag{1.2}$$

where $\ln_k z = \ln z + 2\pi i k$, and $\ln z$ is the principal branch of natural logarithm [14]. This paper considers only the principal branch $k = 0$, which is the branch that maps the positive real axis onto itself, and therefore we shall usually abbreviate W_0 as W herein.

Many functions of W are members of a number of function classes, specifically, the classes of Stieltjes functions, Pick functions and Bernstein functions, including subclasses Thorin-Bernstein functions and complete Bernstein functions. This is mainly due to the fact that W is a real symmetric function, in the terminology of [5, p. 160] (see also [23, p. 155]), with positive values on the positive real line. The mentioned classes are of particular interest because they admit certain integral representations. A description of the classes can be found in a review paper [8] and a recently published book [19]. In this paper we show that many functions containing W are Stieltjes functions. Also, we extend the properties of the set of Stieltjes functions in Sections 1.2 and 4. In addition, we give one more proof of the fact [15] that W function is Bernstein. Moreover, we show that W is a complete Bernstein function.

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The classes of Stieltjes functions and Bernstein functions are intimately connected with the class of completely monotonic functions, which have many applications in different fields of science; a list of appropriate references is given in [2]. Therefore we shall also study the complete monotonicity of some functions containing W.

The properties and integral representations mentioned above have interesting computational implications. For example, the fact that $W(z)/z$ is a Stieltjes function means that the poles of successive Padé approximants interlace and all lie on the negative real axis [5, p. 186] (here in the interval $-\infty < z < -1/e$). In addition, some of the integral representations permit spectrally convergent quadratures for numerical evaluation.

1.1. *Properties of W*

For convenience, we recall from [11] some properties of W that are used below. The function is continuous from above on its branch cut $\mathbb{B} \subset \mathbb{R}$, defined to be the interval $\mathbb{B} = (-\infty, -1/e]$. On the cut plane $\mathbb{C}\backslash\mathbb{B}$, the function is holomorphic. Its real values obey $-1 \le W(x) < 0$ for $x \in [-1/e, 0)$, $W(0) = 0$ and $W(x) > 0$ for $x > 0$. The imaginary part of $W(t)$ has the following range of values for real t

$$\Im W(t) \in (0, \pi) \text{ for } t \in (-\infty, -1/e) \text{ and } \Im W(t) = 0 \text{ otherwise.} \tag{1.3}$$

$\Im W(t) \to \pi$ as $t \to -\infty$. Also, $\Im W(t)$ is continuously differentiable for $t \neq -1/e$. $\Im W(z)$ and $\Im z$ have the same sign in the cut plane $\mathbb{C}\backslash\mathbb{R}$, or equivalently

$$\Im W(z) \Im z > 0 \ . \tag{1.4}$$

W has near conjugate symmetry, meaning $W(\overline{z}) = \overline{W(z)}$, except on the branch cut $\mathbb{B}$. The Taylor series near $z = 0$ is

$$W(z) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{z^n}{n!} \tag{1.5}$$

with the radius of convergence $1/e$, while the asymptotic behaviour of $W(z)$ near its branch point is given by

$$W(z) \sim -1 + \sqrt{2(ez+1)} \quad z \to -1/e \ . \tag{1.6}$$

It follows from (1.5) and (1.2) that

$$W(z)/z \to 1 \quad \text{as} \quad z \to 0 \ , \tag{1.7}$$

$$W(z)/z \to 0 \quad \text{as} \quad z \to \infty \ . \tag{1.8}$$

If $z = t + is$ and $W(z) = u + iv$, then

$$e^u(u \cos v - v \sin v) = t, \quad e^u(u \sin v + v \cos v) = s \ .$$

For the case of real z, i.e. $s = 0$, the functions $u = u(t)$ and $v = v(t)$ are defined by

$$u = -v \cot v, \tag{1.9}$$

$$t = t(v) = -v \csc(v) e^{-v \cot v} \ . \tag{1.10}$$

For the case of purely imaginary z, i.e. $t = 0$, the functions $u = u(s)$ and $v = v(s)$ obey

$$u = v \tan v, \tag{1.11}$$

$$s = s(v) = v \sec(v) e^{v \tan v} \ . \tag{1.12}$$

The derivative of $W(z)$ is given by

$$W'(z) = \frac{W(z)}{z(1 + W(z))} \ . \tag{1.13}$$

Further, the following lemma will be used below.

LEMMA 1.1 *Function $\Im W(-t)$ is nonnegative and bounded on the real line and continuously differentiable for $t \neq 1/e$. Specifically, it is zero for $t \in (-\infty, 1/e]$ and a monotone increasing function for $t \in (1/e, \infty)$ so that $\Im W(-t) \to \pi$ as $t \to \infty$. Correspondingly, the derivative $d\Im W(-t)/dt$ is zero for $t < 1/e$ and positive for $t > 1/e$. In addition, $d\Im W(-t)/dt = o(1/t)$ as $t \to \infty$.*

Proof Owing to the above properties of function $\Im W(t)$ (see (1.3)), the function $\Im W(-t)$ is nonnegative and bounded for real t and $\Im W(-t) \to \pi$ as $t \to \infty$. The function is also continuously differentiable everywhere except $t = 1/e$. We set $v(t) = \Im W(t)$ and compute the derivative $v'(t)$; it is conveniently found by taking the imaginary part of (1.13) and using (1.9)

$$v'(t) = \frac{A(v(t))}{t}, \quad A(v) = \frac{v}{v^2 + (1 - v \cot v)^2} \ . \tag{1.14}$$

Then the derivative $d\Im W(-t)/dt = A(v(-t))/t$, which implies that it is zero for $t < 1/e$ and positive for $t > 1/e$ as $v(t) = 0$ for $t > -1/e$ and $v(t) > 0$ for $t < -1/e$. It remains to justify the estimation of the derivative $d\Im W(-t)/dt$ at large t but it immediately follows from the two facts that $v(-t) \to \pi$ as $t \to \infty$ and that $A(v) \to 0$ as $v \to \pi$. ■

1.2. *Stieltjes functions*

We now review the properties of Stieltjes functions, again concentrating on results that will be used in this paper. We must note at once that there exist several different definitions of Stieltjes functions in the literature, and here we follow the definition of Berg [8].

DEFINITION 1.2 *A function $f : (0, \infty) \to \mathbb{R}$ is called a* Stieltjes function *if it admits a representation*

$$f(x) = a + \int_0^\infty \frac{d\sigma(t)}{x + t} \quad (x > 0), \tag{1.15}$$

where a is a non-negative constant and σ is a positive measure on $[0, \infty)$ such that $\int_0^\infty (1 + t)^{-1} d\sigma(t) < \infty$.

A Stieltjes function is also called a *Stieltjes transform* [9, p. 127]. Except in Section 2.3 below, the term Stieltjes function will here always refer to definition (1.15).

Theorem 1.3 *The set $\mathcal{S}$ of all Stieltjes functions forms a convex cone [9, p. 127] and possesses the following properties.*

(i) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f(1/x)} \in \mathcal{S}$
(ii) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{xf(x)} \in \mathcal{S}$
(iii) $f \in \mathcal{S} \Rightarrow \frac{f}{cf+1} \in \mathcal{S} \quad (c \geq 0)$
(iv) $f, g \in \mathcal{S} \setminus \{0\} \Rightarrow f \circ \frac{1}{g} \in \mathcal{S}$
(v) $f, g \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f \circ g} \in \mathcal{S}$
(vi) $f, g \in \mathcal{S} \Rightarrow f^{\alpha} g^{1-\alpha} \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$
(vii) $f \in \mathcal{S} \Rightarrow f^{\alpha} \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$
(viii) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{x}\left(\frac{f(0)}{f(x)} - 1\right) \in \mathcal{S}$
(ix) $f \in \mathcal{S} \setminus \{0\}, \lim_{x \to 0+} xf(x) = c \geq 0 \Rightarrow f(x) - c/x \in \mathcal{S}$
(x) $f \in \mathcal{S} \Rightarrow f^{\alpha}(0) - f^{\alpha}(1/x) \in \mathcal{S} \quad (0 \leq \alpha \leq 1)$
(xi) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{x}\left(1 - \frac{f(x)}{f(0)}\right) \in \mathcal{S}$
(xii) $f \in \mathcal{S}, \lim_{x \to \infty} f(x) = c > 0 \Rightarrow (c^{\beta} - f^{\beta}) \in \mathcal{S} \quad (-1 \leq \beta \leq 0)$

In the above statements constants c and $f(0) = \lim_{x \to 0+} f(x)$ are assumed to be finite.

Proof Properties (i)-(vii) are listed in [8]; property (vi) is due to the fact that the Stieltjes cone is logarithmically convex [7] and property (vii) is its immediate consequence. Property (viii) is taken from [6, p. 406]. Property (ix) follows from properties (ii) and (viii) in the following way: $f \in \mathcal{S} \setminus \{0\} \Rightarrow g(x) = 1/(xf(x)) \in \mathcal{S} \Rightarrow (g(0)/g(x) - 1)/x = (xf(x)/c - 1)/x \in \mathcal{S} \Rightarrow f(x) - c/x \in \mathcal{S}$. The last three properties (x)–(xii) will be proved in Section 4. ■

A Stieltjes function f has a holomorphic extension to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying $f(\bar{z}) = \overline{f(z)}$ (see [7], [3] and [19, p. 11-12])

$$f(z) = a + \int_0^{\infty} \frac{d\sigma(t)}{z+t} \quad (|\arg(z)| < \pi). \tag{1.16}$$

In addition, a Stieltjes function $f(z)$ in the cut plane $\mathbb{C}\backslash(-\infty, 0]$ can be represented in the integral form [5, p.158]

$$f(z) = \int_0^{\infty} \frac{d\Phi(u)}{1+uz} \quad (|\arg(z)| < \pi)\ , \tag{1.17}$$

where $\Phi(u)$ is a bounded and non-decreasing function with finite real-valued moments $\int_0^{\infty} t^n \, d\Phi(t)$ $(n = 0, 1, 2, \dots)$. The integral (1.17) is used in [5, Ch. 5] for a study of Padé approximants to the Stieltjes functions; it is equivalent to the representation (1.16) by virtue of the following observation. According to properties (i) and (ii), if a function $f \in \mathcal{S}$ then $f(1/x)/x \in \mathcal{S}$ as well and hence the latter admits representation (1.15)

$$\frac{1}{x} f\left(\frac{1}{x}\right) = a + \int_0^{\infty} \frac{d\sigma(t)}{x+t}\ ,$$

which after replacing x with $1/x$ gives

$$f(x) = \frac{a}{x} + \int_0^{\infty} \frac{d\sigma(t)}{1+xt}\ ,$$

where the first term can be included into the integral since $a \geq 0$ and $1/x$ is a Stieltjes function (see e.g. [8]). Finally, one considers the holomorphic extension of the last integral to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ in a way similar to the obtaining of (1.16).

There are various kinds of necessary and sufficient conditions implying that a function f is a Stieltjes function. Some of them are based on the classical results established by R. Nevanlinna, F. Riesz, and Herglotz. Here we quote two such theorems taken from [1, p. 93] and [8, Theorem 3.2].

THEOREM 1.4 *A function $g(z)$ admits an integral representation in the upper half-plane in the form*

$$g(z) = \int_{\mathbb{R}} \frac{d\Phi(u)}{u - z} \quad (\Im z > 0) \ , \tag{1.18}$$

with a non-decreasing function $\Phi(u)$ of bounded variation on $\mathbb{R}$ (i.e. $\int_{\mathbb{R}} d\Phi(u) < \infty$ for smooth $\Phi(u)$), if and only if $g(z)$ is holomorphic in the upper half-plane and

$$\Im g(z) \geq 0 \quad \textit{and} \quad \mathop{\textit{sup}}_{1<y<\infty} |yg(iy)| < \infty \ . \tag{1.19}$$

To apply Theorem 1.4 to the integral (1.17) one should set $g(z) = -f(-1/z)/z$ (cf. [5, (6.12) on p. 215]), then conditions (1.19) read as

$$\Im f(-1/z)/z \leq 0 \quad \text{and} \quad \sup_{1<y<\infty} |f(i/y)| < \infty \ . \tag{1.20}$$

THEOREM 1.5 *A function $f : (0, \infty) \to \mathbb{R}$ is a Stieltjes function if and only if $f(x) \geq 0$ for $x > 0$ and there is a holomorphic extension $f(z)$, $z = x + iy$, to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying*

$$\Im f(z) \leq 0 \textit{ for } \Im z > 0. \tag{1.21}$$

Remark 1 The inequalities (1.21) alone express a necessary condition for f to be a Stieltjes function. In the terminology of [6, p. 358], a holomorphic function $f(z)$ is called a Herglotz function if $\Im f > 0$ when $\Im z > 0$, $\Im f = 0$ when $\Im z = 0$ and $\Im f < 0$ when $\Im z < 0$. Thus, for f to be a Stieltjes function it is necessary that f be an anti-Herglotz function (cf. [6, p. 406]).

2. Stieltjes functions containing $W(z)$

In this section we consider a number of functions containing $W(z)$ and prove that they are Stieltjes functions. We begin with the function $W(z)/z$.

2.1. *The function $W(z)/z$*

Although the fact that $W(z)/z$ is a Stieltjes function could be established conveniently by applying one of the criteria stated in Section 1.2, we nonetheless first present a direct proof which is of great importance for further investigations. Moreover, compared with using the criteria above, the present way allows us to make additional useful observations, which will be given in the remarks following the proof and then used in further discussion.

Theorem 2.1 *$W(z)/z$ is a Stieltjes function.*

Proof From (1.7), the function $W(z)/z$ is single-valued and holomorphic in the same domain as $W(z)$, namely $D = \{z \in \mathbb{C} \mid z \notin \mathbb{B}\}$, and can be represented by the Cauchy integral formula

$$\frac{W(z)}{z} = \frac{1}{2\pi i} \int_C \frac{W(t)}{t(t-z)} dt \ , \qquad (2.1)$$

where C is the standard 'keyhole' contour which consists of a small circle around the branch point $t = -1/e$ of radius, say r, and a large circle around the origin of radius, say R; the circles being connected through the upper and lower edges of the cut along the negative real axis. Then for sufficiently small r and large R the interior of the contour C encloses any point in D.

Let us consider the integral (2.1) in the limit in which $r \to 0$ and $R \to \infty$. Using asymptotic estimations (1.6) and (1.2), it is easily seen that the contributions of each circle to the integral (2.1) go to zero. As a result, in accordance with the assignment of values of W function on the branch cut, the integral becomes

$$\frac{W(z)}{z} = \frac{1}{2\pi i} \int_{-\infty}^{-1/e} \frac{W(t)}{t(t-z)} dt + \frac{1}{2\pi i} \int_{-1/e}^{-\infty} \frac{\overline{W(t)}}{t(t-z)} dt \ ,$$

which reduces to

$$\frac{W(z)}{z} = \frac{1}{\pi} \int_{-\infty}^{-1/e} \frac{\Im W(t)}{t(t-z)} \, dt \ , \qquad (2.2)$$

where $|\arg(z)| < \pi$. Changing t to $-t$ transforms the integral (2.2) to the form (1.16)

$$\frac{W(z)}{z} = \int_{1/e}^{\infty} \frac{1}{z+t} \frac{\mu(t)}{t} dt \ , \qquad (2.3)$$

where

$$\mu(t) = \frac{1}{\pi} \Im W(-t) \ . \qquad (2.4)$$

According to Lemma 1.1, $\mu(t) \in (0,1)$ for $t \in (1/e, \infty)$, and therefore we have $\int_{1/e}^{\infty} \mu(t)dt/[t(1+t)] < \infty$ and the conditions in Definition 1.2 are satisfied. Thus the integral (2.3) is a Stieltjes function. ∎

Remark 1 The function $W(z)/z$ is a real symmetric function as is any Stieltjes function (this immediately follows from Definition 1.2), which just corresponds to the near conjugate symmetry property.

Remark 2 Representation (2.3)-(2.4) is also obtained in [17].

Remark 3 The representation of $W(z)/z$ in the form (1.17) equivalent to (2.3) is

$$\frac{W(z)}{z} = \int_0^e \frac{d\Phi(t)}{1+tz} \ , \qquad (2.5)$$

where $d\Phi(t) = \mu(1/t)dt$. Since $\mu(1/t) \in (0,1)$ for $t \in (0,e)$ by Lemma 1.1, $\Phi'(t) \geq 0$ and thus $\Phi(t)$ is a bounded and non-decreasing function. In addition, all the moment integrals $\int_0^e t^n \, d\Phi(t)$ $(n = 0, 1, 2, \dots)$ exist. This remark is useful for justifying the use of Padé approximants for the evaluation of $W(z)$ based on the theory in [5, Ch. 5].

Remark 4 An existence of representation (2.5) also follows from Theorem 1.4. Indeed, for function $f(z) = W(z)/z$ conditions (1.20) read as

$$\Im W(-1/z) \geq 0 \quad \text{and} \quad \sup_{1<y<\infty} |yW(i/y)| < \infty \, .$$

The first condition is satisfied by (1.4) because $\Im(-1/z)$ and $\Im z$ are of the same sign. To verify the second condition we set $W(i/y) = u + iv$ and put $s = 1/y$ in (1.11) and (1.12). As a result, since $0 < v < \pi/2$ for $y > 0$, we obtain

$$|yW(i/y)|^2 = y^2(u^2 + v^2) = y^2v^2(1 + \tan^2 v) = y^2v^2/\cos^2 v = e^{-2v\tan v} \leq 1.$$

To extend the result to the lower half-plane $\Im z < 0$ it is enough to take the complex conjugate of both sides of the representation (1.17) and use the near conjugate symmetry of W. Thus Theorem 1.4 gives us one more way to prove that $W(z)/z$ is a Stieltjes function.

2.2. *Other functions containing W*

By Theorem 2.1, $W(x)/x \in \mathcal{S}$. Using this result and the properties of the set $\mathcal{S}$ listed in Section 1.2 we now give some classes of functions that are members of $\mathcal{S}$.

Theorem 2.2 *The following functions belong to the set $\mathcal{S}$, for $x > 0$.*

(a) $1/(c + W(x))$, $c \geq 0$
(b) $W^\alpha(1/x), 0 \leq \alpha \leq 1$
(c) $x^\beta W^\beta(1/x), -1 \leq \beta \leq 0$
(d) $W(x)/[x(c + W(x))]$, $c \geq 0$
(e) $1/W(x) - 1/x$
(f) $c + W(x^\beta)$, $c \geq 0$, $-1 \leq \beta \leq 0$
(g) $1/(c + W(x^\alpha))$, $c \geq 0$, $0 \leq \alpha \leq 1$
(h) $x^{\alpha\beta\gamma} W^{-\alpha\gamma}(x^\beta)[1 + W(x^\beta)]^{1-\gamma}$, $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 0$, $0 \leq \gamma \leq 1$
(i) $1 - x^\alpha W^\alpha(1/x)$, $0 \leq \alpha \leq 1$
(j) $1 - x^{-\alpha\beta} W^\alpha(x^\beta)[1 + W(x^\beta)]^{-\alpha}$, $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 0$

Proof We use the properties listed in Theorem 1.3.

(a) We apply property (ii) to $W(x)/x$ to find that $1/W(x) \in \mathcal{S}$ and then apply (iii) to $1/W(x)$.
(b) We first apply (i) to $f(x) = 1/W(x)$ that is in $\mathcal{S}$ by statement (a) and find $W(1/x) \in \mathcal{S}$. Then we apply (vii) to $W(1/x)$.
(c) Apply (i) to $W(x)/x$ and apply then (vii) to the result.
(d) Apply (xi) to the function in the statement (a) using $W(0) = 0$.
(e) Apply (viii) to $W(x)/x$ using (1.7) or apply (ix) to the function in the statement (a) with $c = 0$.
(f) Apply (v) to the function in the statement (a) and $g(x) = x^\beta$ $(-1 \leq \beta \leq 0)$ that is in $\mathcal{S}$ [8, 9].
(g) Apply (iv) to the function in the statement (a) and $g(x) = x^{-\alpha} \in \mathcal{S}$ for $0 \leq \alpha \leq 1$.
(h) Apply (v) to functions $f(x) = W(x)/x$ and $g(x) = x^\beta$ $(-1 \leq \beta \leq 0)$ and find $x^\beta W^{-1}(x^\beta) \in \mathcal{S}$. Hence by (vii) $a(x) = x^{\alpha\beta} W^{-\alpha}(x^\beta) \in \mathcal{S}$ for

$0 \le \alpha \le 1$. Then apply (v) to the function in the statement (a) with $c = 1$ and $g(x) = x^\beta$ to get $b(x) = 1 + W(x^\beta) \in \mathcal{S}$. Finally apply (vi) to $a(x)$ and $b(x)$.

(i) Apply (xii) to the function in the statement (c) with $\beta = -1$ using (1.7) (or apply (x) to $W(x)/x$).

(j) Apply (x) (or (xii)) to the result of application of (iv) (respectively (v)) to the function in the statement (d) with $c = 1$ and $g(x) = x^\beta$ $(-1 \le \beta \le 0)$.

■

Corollary 2.3 *The derivative $W'(x)$ is a Stieltjes function.*

Proof The proof follows from statement (d) of Theorem 2.2, taken with $c = 1$, together with formula (1.13). ■

The next theorem proves and generalizes a conjecture in [13].

Theorem 2.4 *The following functions are Stieltjes functions for each fixed real $a \in (0, e]$:*

$$F_0(z) = \frac{z}{1+z} W(a(1+z)) / \left[W(a(1+z)) - W(a)\right]^2 \ , \tag{2.6}$$

$$F_1(z) = zW\left(\frac{a}{1+z}\right) \Big/ \left[W(a) - W\left(\frac{a}{1+z}\right)\right]^2 \ . \tag{2.7}$$

Proof We first apply Theorem 1.5 to the function $F_0(z)$. To do so we note that $F_0(z) \ge 0$ for real $z > 0$ $(a \in (0, e])$ and $F_0(z)$ is a holomorphic function in the cut plane $\mathbb{C} \backslash (-\infty, 0]$ (cf. the branch cut $\mathbb{B}$). For convenience, we define a function $V(z) = \Im F_0(z)$, then it remains to show that $V(z) \le 0$ in the upper half-plane. Since $V(z)$ is a harmonic function in the domain $\Im z > 0$, it is subharmonic there. Thus we can apply either the maximum principle for harmonic functions in the form of [4, Corollary 1.10] or the maximum principle for subharmonic functions [12, p. 19–20]. In both cases, to get the desired result it is sufficient to ascertain that the superior limit of $V(z)$ at all boundary points including infinity is less than or equal to 0 [2]. In other words, $V(z) \le 0$ for $\Im z > 0$ if (cf. [16, p. 27])

$$\lim_{|z| \to \infty} V(z) \le 0 \quad (\Im z > 0)$$

and

$$\limsup_{y \to 0+} V(x + iy) \le 0 \text{ for all } x \in \mathbb{R} \ . \tag{2.8}$$

Since $F_0(z) \sim 1/\ln z$ for large z due to (1.2), $V(z) \to 0$ as $|z| \to \infty$ and the first condition is satisfied.

To verify the second condition we introduce variables $t = a(1 + x)$ and $s = ay$ and set $W(t + is) = u + iv$ where $u = u(t, s), v = v(t, s)$. We also introduce a constant $b = W(a) \in (0, 1]$. Then the condition (2.8) becomes $H(t) \ge 0$ for all $t \in \mathbb{R}$, where

$$H(t) = \limsup_{s \to 0+} \frac{v[t(t-a) + s^2](u^2 + v^2 - b^2) + as\left[(u^2 + v^2)(2b - u) - b^2 u\right]}{(t^2 + s^2)[v^2 + (b - u)^2]} \tag{2.9}$$

For analysis of function $H(t)$, it is convenient to consider the following five cases: (i) $-\infty < t < -1/e$, (ii) $-1/e \le t < 0$, (iii) $t = 0$, (iv) $(0 < t < a) \cup (a < t < \infty)$, and (v) $t = a$. We start with the case (i). Since $V(z)$ is continuous (from above) on the real line $z = x \in \mathbb{R}$, the expression under the limit sign in (2.9) is continuous in domain $\{(t,s)|t \in \mathbb{R}, s > 0\}$. Then using relation (1.9) we obtain

$$H(t) = \frac{v}{[(b + v\cot v)^2 + v^2]^2}\left(\frac{v^2}{\sin^2 v} - b^2\right)\left(1 - \frac{a}{t}\right) .$$

We have $v \in (0, \pi)$ for $t \in (-\infty, -1/e)$, hence $v^2/\sin^2 v > 1$. Since $0 < b \le 1$, we conclude that in case (i) $H(t) > 0$. Taking into account that $v = 0$ in cases (ii), (iv) and (v) and relations (1.11) and (1.12) in case (iii) it is not difficult to show that in all of these cases $H(t) = 0$. Thus $H(t) \ge 0$ for all real t, i.e. the condition (2.8) is satisfied and $F_0(z)$ is a Stieltjes function.

The theorem for the function $F_1(z)$ follows from the relation

$$F_1(z) = -F_0\left(-\frac{z}{1+z}\right) \tag{2.10}$$

because in terms of the conditions of Theorem 1.5 the transformation in the right-hand side of (2.10) retains the properties of $F_0(z)$. In particular, $\Im F_1(z) \le 0$ for $\Im z > 0$ because, first, $\Im z$ and $\Im(-z/(1+z))$ are of the opposite signs and secondly, $\Im F_0(z) \ge 0$ for $\Im z < 0$ which follows from $F_0(\bar{z}) = \overline{F_0(z)}$ due to near conjugate symmetry and the established above non-positivity of $\Im F_0(z)$ in the upper half-plane. Thus $F_1(z)$ is also a Stieltjes function. ∎

Remark 5 We can note the behaviour of functions (2.6) and (2.7) for large and small z. Specifically, using (1.2) and (1.7) one can obtain respectively $F_0(z) \to 0$ and $F_1(z) \to a/W^2(a)$ as $z \to \infty$. Using (1.13) we find $F_{0,1} \sim c/z$ as $z \to 0$, where $c = (1 + W(a))^2/W(a)$.

We now have a result even stronger than Theorem 2.4 in the following corollary.

Corollary 2.5 *With the constant c defined in Remark 5 the differences $F_{0,1} - c/z$ are Stieltjes functions for fixed $a \in (0, e]$.*

Proof Follows from Remark *5* and the property (ix) given in Theorem 1.3. ∎

2.3. *Is W a Stieltjes function?*

The principal branch of the Lambert W function itself is not a Stieltjes function in the sense of Definition 1.2. This can be shown in different ways. For example, one can apply Theorem 1.4 to $W(z)$ to see that the second condition (1.20) fails. Indeed, when $z = is$ we have by (1.11) and (1.12)

$$|sW(is)| = s\sqrt{u^2 + v^2} = v^2 \sec^2(v) e^{v \tan v} \to \infty \quad \text{as} \quad v \to \pi/2.$$

The same conclusion can be reached using Theorem 1.5 because (1.4) contradicts (1.21). Finally, W is not a Stieltjes function because it is not an anti-Herglotz function (cf. Remark *1*).

Note, however, that W function can be regarded as a Stieltjes function in the sense of a definition given in [24] and [10] or used in [25] and different from (1.17) by the factor z in the right hand-side. The W function can also be considered as a generalized Stieltjes transform by the definition in [18] (which is different from that of the generalized Stieltjes transform defined in [26, p. 30] and studied, for example, in [20] and [22]). Finally, in [19], the terms Stieltjes function and Stieltjes representation are not treated as equivalent (compare definitions [19, p. 11] and [19, p. 55]). By these definitions $W(z)$ has a Stieltjes representation (which is the result of multiplication of the representation (2.3) by z) though it is not a Stieltjes function.

3. Completely monotonic functions

We denote by $\mathcal{CM}$ the set of all completely monotonic functions, which are defined as follows [3].

Definition 3.1 *A function* $f : (0, \infty) \to \mathbb{R}$ *is called a completely monotonic function if* f *has derivatives of all orders and satisfies* $(-1)^n f^{(n)}(x) \geq 0$ *for* $x > 0$, $n = 0, 1, 2, ...$

The set of Stieltjes functions is contained in the set of completely monotonic functions, and thus all of the functions listed in Theorem 2.2 are completely monotone. The set $\mathcal{CM}$ is a convex cone containing the positive constant functions; a product of completely monotonic functions is again completely monotone [9, p. 61]. By Bernstein's theorem [9, Theorem 9.3], a function $f \in \mathcal{CM}$ if and only if it is of the form

$$f(x) = \int_0^\infty e^{-x\xi} d\nu(\xi) \quad (x > 0), \tag{3.1}$$

where ν is an uniquely determined positive measure on $[0, \infty)$. Completely monotonic functions are in turn connected with the set of Bernstein functions denoted by $\mathcal{B}$.

Definition 3.2 *[8, Definition 5.1] A function* $f : (0, \infty) \to [0, \infty)$ *is called a Bernstein function if it is* C^∞ *and* f' *is completely monotonic.*

Since $W' \in \mathcal{S} \subset \mathcal{CM}$, W is a Bernstein function. The same fact has been established in [15] in a different way, based on the properties of the polynomials appearing in the higher derivatives of W.

A Bernstein function $f(x)$ admits the Lévy-Khintchine representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-x\xi}\right) d\nu(\xi) \ , \tag{3.2}$$

where $a, b \geq 0$ and ν is a positive measure on $(0, \infty)$ satisfying $\int_0^\infty \xi(1+\xi)^{-1} d\nu(\xi) < \infty$. It is called the Lévy measure. The equation (3.2) is obtained by integrating (3.1) written for f' [8].

An important relation between the classes $\mathcal{S}$ and $\mathcal{B}$ is given by the assertion [8, Theorem 5.4]

$$g \in \mathcal{S} \setminus \{0\} \Rightarrow 1/g \in \mathcal{B}. \tag{3.3}$$

Combining this with the function composition result [8, Corollary 5.3] that $f \in \mathcal{CM}$ and $g \in \mathcal{B}$ implies $f \circ g \in \mathcal{B}$ we obtain the following lemma.

Lemma 3.3 *If* $f \in \mathcal{CM}$ *and* $g \in \mathcal{S} \setminus \{0\}$ *then* $f(1/g) \in \mathcal{CM}$.

This lemma extends the list of completely monotonic functions containing W.

Theorem 3.4 *The following functions are completely monotonic*

(a) $x^{\lambda}W(x)$ $(x>0,\ \lambda\le -1)$.
(b) $x^{\lambda}W^{\alpha}(x^{\beta})\left[1+W(x^{\beta})\right]^{\gamma}$ $(x>0,\ \alpha,\gamma\ge 0,\ -1\le\beta\le 0,\ \lambda\le 0)$.
(c) $x^{\lambda}W^{\alpha}(x^{-\beta})\left[1+W(x^{-\beta})\right]^{\gamma}$ $(x>0,\ \alpha,\gamma\le 0,\ -1\le\beta\le 0,\ \lambda\le 0)$.
(d) $1-x^{-\alpha\beta\gamma}W^{\alpha\gamma}(x^{\beta})[1+W(x^{\beta})]^{\gamma-1}$ $(x>0,\ 0\le\alpha\le 1,\ -1\le\beta\le 0,\ 0\le\gamma\le 1)$.

Proof

(a) Since $W(x)/x\in\mathcal{S}\subset\mathcal{CM}$ and $x^{\alpha}\in\mathcal{CM}$ for $\alpha\le 0$, the function $x^{\lambda}W(x)$ $(\lambda\le -1)$ is a product of two completely monotonic functions and the statement (a) follows.
(b) Take function $f_{\alpha}(x)=x^{-\alpha}\in\mathcal{CM}$ $(x>0,\ \alpha\ge 0)$ and functions $g(x)=1/W(x^{\beta})$ and $h(x)=1/(1+W(x^{\beta}))$ where $-1\le\beta\le 0$. Since $1/g\in\mathcal{S}$ and $1/h\in\mathcal{S}$ by Theorem 2.2 (f) with $c=0$ and $c=1$ respectively, by Lemma 3.3 we have $f_{\alpha}(g(x))=g^{-\alpha}(x)\in\mathcal{CM}$ and $f_{\gamma}(h(x))=h^{-\gamma}(x)\in\mathcal{CM}$ $(\gamma\ge 0)$. Substituting functions $g(x)$ and $h(x)$ in the power functions and taking a product of obtained completely monotonic functions with $x^{\lambda}\in\mathcal{CM}$ $(x>0,\ \lambda\le 0)$, the statement (b) follows.
(c) Consider function $f_{\lambda}(x)=x^{\lambda}\in\mathcal{CM}$ $(x>0,\ \lambda\le 0)$ and functions $g(x)=W(x^{-\beta})$ and $h(x)=1+W(x^{-\beta})$ where $-1\le\beta\le 0$. Since $1/g\in\mathcal{S}$ and $1/h\in\mathcal{S}$ by Theorem 2.2 (g) with $c=0$ and $c=1$ respectively, by Lemma 3.3 we have $f_{\alpha}(g(x))=g^{\alpha}(x)\in\mathcal{CM}$ and $f_{\gamma}(h(x))=h^{\gamma}(x)\in\mathcal{CM}$ for $\alpha\le 0$ and $\gamma\le 0$. Substituting functions $g(x)$ and $h(x)$ and taking a product of obtained functions with $f_{\lambda}(x)$, the statement (c) follows.
(d) By Theorem 2.2 (h) and the assertion (3.3), for $x>0$, $0\le\alpha\le 1$, $-1\le\beta\le 0$, $0\le\gamma\le 1$ we have $f(x)=g^{\alpha\gamma}(x)[1+W(x^{\beta})]^{\gamma-1}\in\mathcal{B}$, where $g(x)=x^{-\beta}W(x^{\beta})$. In addition, the function $f(x)$ is bounded, particularly, $0<f(x)<1$ because $0<[1+W(x^{\beta})]^{\gamma-1}<1$ and $0<g(x)<1$ (the latter follows from the fact that $g(x)$ goes to 0 and 1 as x tends to 0 and ∞ respectively and $g'(x)>0$, which can be established using (1.7), (1.8) and (1.13)). Then by [8, Remark 5.5] the assertion (d) follows.

■

We note that we have considered only sufficient conditions for a function to be completely monotonic. To find the necessary and sufficient conditions is a much more complicated problem, so that in some cases it requires (at least as the first step) using the methods of experimental mathematics [21].

4. Complete Bernstein functions

A very important subclass in $\mathcal{B}$ is the class of complete Bernstein functions denoted by $\mathcal{CB}$.

Definition 4.1 *[19, Definition 6.1] A Bernstein function f is called a complete Bernstein function if the Lévy measure in* (3.2) *is such that $d\nu(t)/dt$ is a completely monotonic function.*

We point out four connections between classes $\mathcal{CB}$ and $\mathcal{S}$ used in this paper (for additional relations between these classes see [19, Chapter 7]). By Proposition 7.7 in [19],

$$f\in\mathcal{S}\Rightarrow f(0)-f(x)\in\mathcal{CB}\ , \tag{4.1}$$

where the limit of $f(x)$ at $x = 0$ (from the right) is assumed to be finite. Also if f is bounded and $f \in \mathcal{CB}$, there exists a bounded $g \in \mathcal{S}$ with $\lim_{x\to\infty} g(x) = 0$ such that

$$f(x) = f(0) + g(0) - g(x) \ . \tag{4.2}$$

In addition, [19, Theorem 7.3] and [19, Theorem 6.2(i),(ii)] establish

$$f \in \mathcal{CB} \Leftrightarrow 1/f \in \mathcal{S} \setminus \{0\} \ , \tag{4.3}$$

$$f \in \mathcal{CB} \Leftrightarrow f(x)/x \in \mathcal{S} \ . \tag{4.4}$$

We note at once that the statement (4.3) together with that $1/W \in \mathcal{S}$ (by Theorem 2.2(a) with $c = 0$) immediately results in a conclusion that W is a complete Bernstein function.

Now we go back to the properties of the set $\mathcal{S}$ listed in Section 1.2 to prove the last three properties therein. Let $f \in \mathcal{S} \setminus \{0\}$.

(x) Apply sequentially (vii), (4.1), (4.3), (i), to obtain $f^\alpha \in \mathcal{S}$ $(0 \le \alpha \le 1) \Rightarrow f^\alpha(0) - f^\alpha(x) \in \mathcal{CB} \Rightarrow g(x) = [f^\alpha(0) - f^\alpha(x)]^{-1} \in \mathcal{S} \Rightarrow 1/g(1/x) = f^\alpha(0) - f^\alpha(1/x) \in \mathcal{S}$;

(xi) Apply sequentially (4.1), (4.3), (ii), to obtain $f(0) - f(x) \in \mathcal{CB} \Rightarrow g(x) = [f(0) - f(x)]^{-1} \in \mathcal{S} \Rightarrow 1/(xg(x)) = (f(0) - f(x))/x \in \mathcal{S} \Rightarrow (1 - f(x)/f(0))/x \in \mathcal{S}$;

(xii) By (vii), $f^\alpha \in \mathcal{S}$ $(0 \le \alpha \le 1)$. Suppose that $\lim_{x\to 0} f(x) = b \le \infty$ and $\lim_{x\to\infty} f(x) = c$ where $0 < c < \infty$. Then $b^{-\alpha} \le f^{-\alpha} \le c^{-\alpha}$, i.e. $f^{-\alpha}$ is bounded. In addition, $f^{-\alpha} \in \mathcal{CB}$ by (4.3). Therefore the statement (4.2) can be applied, i.e. there exists a bounded function $g \in \mathcal{S}, \lim_{x\to\infty} g(x) = 0$ such that we can write $g(x) = g(0) + b^{-\alpha} - f^{-\alpha}(x)$. Taking the last equation in the limit $x \to \infty$ we obtain $g(0) + b^{-\alpha} = c^{-\alpha}$, hence $g = c^{-\alpha} - f^{-\alpha}$ and the assertion follows.

5. Concluding remarks

We have verified the statement made in in Section 1 that many functions containing the principal branch of the Lambert W function belong to various function classes which are characterized by their own integral forms. As a consequence, the W function is rich in integral representations. In this paper we considered in detail the classes of Stieltjes functions as well as the classes of completely monotonic functions, Bernstein functions and complete Bernstein functions. Through specific examples of functions based on W, we demonstrated a number of different ways to establish whether a particular function belongs to one of these classes. This paper has not exhausted the classes of functions with integral representations, and relations between functions of W to further function classes, together with explicit Stieltjes and other integral representations of W, will be given in a subsequent paper.

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