THE SLOW MOTION OF A CYLINDER NEXT TO A PLANE WALL

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SUMMARY

The two-dimensional flow around a cylinder that is near a plane wall is calculated assuming that the Reynolds number for the flow is small. For a cylinder translating parallel to the wall, the torque on the cylinder is zero; similarly, for a rotating cylinder the force is zero. These results, which are surprising when compared with corresponding ones for a sphere, are proved and then examined further using lubrication theory. We also consider motion perpendicular to the wall, allowing us to discuss the behaviour of a cylinder falling down an inclined plane. In addition streamline patterns are described.

1. Introduction and general method

JEFFERY (**1**) developed a general method for calculating two-dimensional Stokes flows confined to the annular region between two cylinders. Later authors independently derived similar methods (**2, 3**) or tackled related problems using different methods (**4, 5**). Jeffery used his method to calculate the torque on a cylinder rotating next to a plane wall, but missed the surprising fact that there is no lateral force on the cylinder. In this section we generalize Jeffery's method and give simple expressions for the force and torque on any cylindrical surface, in preparation for section 2, where we solve three problems explicitly, namely a cylinder translating parallel or perpendicular to a plane wall, or rotating next to it.

We define bipolar coordinates (α, β) in terms of Cartesian coordinates (x, y) by (see Fig. 1)

$$\alpha + i\beta = \log \frac{x + i(y+a)}{x + i(y-a)}, \tag{1}$$

or equivalently

$$x = h^{-1} \sin \beta \quad \text{and} \quad y = h^{-1} \sinh \alpha,$$

where $ah = \cosh \alpha - \cos \beta$, and a is a scale factor. A stream function ψ can be defined in terms of the fluid velocities by $u_\alpha = -h\, \partial\psi/\partial\beta$, and $u_\beta = h\, \partial\psi/\partial\alpha$. The general solution of $\nabla^4\psi = 0$ appropriate for Stokes flows that

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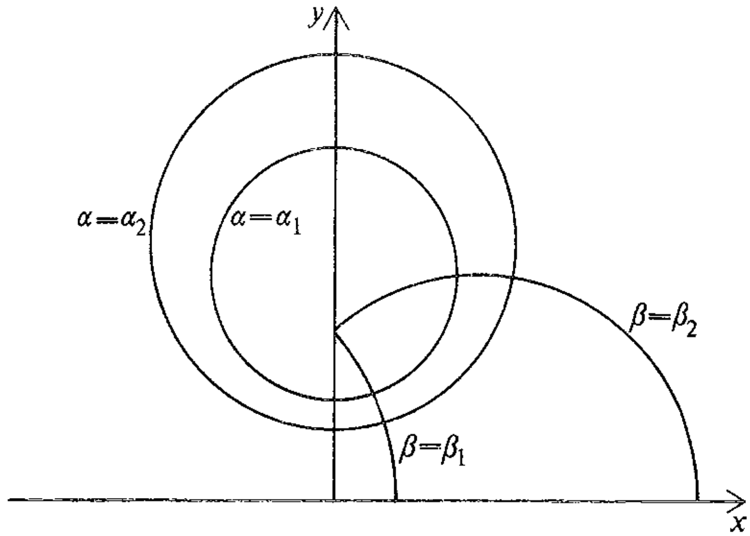

FIG. 1. Bipolar coordinates (α, β). $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$.

are bounded externally or that decay sufficiently rapidly at infinity is

$$h\psi = \mathrm{Re}\left\{\sum_{n=0}^{\infty} \chi_n(\alpha)e^{-in\beta}\right\}, \tag{2}$$

where

$$\chi_0 = A_0 \cosh \alpha + B_0\alpha \cosh \alpha + C_0 \sinh \alpha + D_0\alpha \sinh \alpha,$$
$$\chi_1 = A_1 \cosh 2\alpha + B_1 + C_1 \sinh 2\alpha + D_1\alpha,$$
$$\chi_n = A_n \cosh (n+1)\alpha + B_n \cosh (n-1)\alpha + C_n \sinh (n+1)\alpha + D_n \sinh (n-1)\alpha.$$

The constants are complex (except for $n = 0$), and not all independent, because to obtain a single-valued pressure field, we must set $\mathrm{Re}\{D_1\} = -B_0$; the pressure field is then

$$p = \frac{2\mu}{a} \mathrm{Im}\left[e^{-i\beta}\{D_0 \sinh \alpha + (B_0 + D_1) \cosh \alpha\} + \right.$$
$$\left. + \sum_{n=1}^{\infty} e^{-in\beta}\{G_n \sinh n\alpha + H_n \cosh n\alpha\}\right],$$

where

$$G_n = (n-1)(A_{n-1} + B_n) - (n+1)(A_n + B_{n+1}),$$
$$H_n = (n-1)(C_{n-1} + D_n) - (n+1)(C_n + D_{n+1}).$$

If the unit outward normal to a cylindrical surface $\alpha = \alpha_1$ is $\mathbf{n}$ and the unit anticlockwise tangential vector is $\mathbf{t}$, the torque acting on that surface is

$$T = a \operatorname{cosech} \alpha_1 \int \mathbf{t} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\, ds = 4\pi\mu a(B_0 + D_0 \coth \alpha_1).$$

The Cartesian components of the force on the surface $\alpha=\alpha_1$ are

$$F_x\mathbf{i}+F_y\mathbf{j}=\int \boldsymbol{\sigma}.\mathbf{n}\,ds=4\pi\mu D_0\mathbf{i}-4\pi\mu\,\mathrm{Im}\,\{D_1\}\mathbf{j}. \tag{3}$$

We shall be interested in the special case of a cylinder next to a plane wall. If the radius of the cylinder is r and the distance of the cylinder axis from the wall is d, then the cylinder is described by $\alpha=\alpha_1$, where

$$d=a\coth\alpha_1,\qquad r=a\,\mathrm{cosech}\,\alpha_1\quad\text{and}\quad a^2=d^2-r^2.$$

2. Particular solutions

2.1 Cylinder rotating next to a plane wall

If the cylinder rotates anticlockwise with angular velocity ω, all the coefficients in (2) are zero except for the following:

$$\begin{aligned}-C_0=2C_1=-2A_1\coth\alpha_1=2B_1\coth\alpha_1=B_0=-D_1\\ =-a\omega\cosh\alpha_1\,\mathrm{cosech}^3\,\alpha_1.\end{aligned}$$

The torque on the cylinder is thus

$$T=-4\pi\mu\omega r^2 d/(d^2-r^2)^{\frac{1}{2}},$$

and the force is zero.

The special case of a line couplet above a plane can be obtained, following (**6**), by taking the limit $r\to 0$, $d=1$ and $4\pi\mu\omega\,\mathrm{cosech}^2\,\alpha_1\to 1$. Using (1) to convert to Cartesian coordinates, we obtain

$$4\pi\mu\psi_c=\log R_1-\log R_2+2y(y+1)/R_2^2,$$

where $R_1^2=x^2+(y-1)^2$ and $R_2^2=x^2+(y+1)^2$, and the velocity components are $u_x=-\partial\psi/\partial y$, and $u_y=\partial\psi/\partial x$.

2.2 Translation parallel to wall

If the cylinder translates with speed U parallel to the x-axis, the non-zero coefficients in (2) are

$$B_0=-C_0=2C_1=-D_1=U\coth\alpha_1/\alpha_1,\qquad 2B_1=-D_0=-2A_1=U/\alpha_1.$$

The force on the cylinder is thus

$$F_x=-4\pi\mu U/\alpha_1=-4\pi\mu U/\log\left[r^{-1}\{d+(d^2-r^2)^{\frac{1}{2}}\}\right]$$

and the torque is zero, in agreement with (**4**). The fact that the torque is zero could have been predicted from the observation above of zero force on a rotating cylinder by using the reciprocal theorem.

The special case of a line force directed parallel to the plane is obtained as the limit $r\to 0$, $d=1$, and $4\pi\mu U/\alpha_1\to 1$, in which limit

$$4\pi\mu\psi_p=(1-y)\log(R_2/R_1)-2y(y+1)/R_2^2,$$

in agreement with (**7**), whose stream function is minus ours.

2.3 Translation perpendicular to wall

Finally, if the cylinder moves with speed V away from the wall, the non-zero coefficients are

$$2A_1 \coth \alpha_1 = -2B_1 \coth \alpha_1 = -2C_1 = D_1 = iV/(\alpha_1 - \tanh \alpha_1).$$

The force is thus

$$F_y = -4\pi\mu V/(\alpha_1 - \tanh \alpha_1) = -4\pi\mu V/[\log \{(d+a)/r\} - a/d],$$

where $a^2 = d^2 - r^2$ as before.

The special case of a line force directed normal to a plane is

$$4\pi\mu\psi_n = x \log (R_2/R_1) - 2xy/R_2^2.$$

3. Streamlines

The advent of computers and automatic graph-plotting routines has greatly simplified the task of plotting streamlines, although some flow patterns are still difficult to resolve without using large amounts of computer time; for these cases there are alternative methods which save computing time and which have the additional advantage of conveying greater understanding of the mechanisms at work (**8**). The recent renewal of interest in streamline patterns arises partly from the new possibilities for plotting them and partly from a basic desire to know how a flow 'looks', and until the recent series of studies of streamline patterns, the intuition of many workers in low-Reynolds-number flow would, on the subject of streamlines, have been wide of the mark. Particular attention has been paid to the occurrence of eddies and separation. It is tempting to draw an analogy between separation at low Reynolds number and that at high Reynolds number. There is, however, an important difference between the two cases. At low Reynolds number, the Stokes equations can be solved without knowing about the separation, and indeed separation is usually found from the solution. On the other hand, at high Reynolds number, it is the separation itself that determines the solution. There are thus limits to the interest that streamline patterns *per se* can hold, especially now that there exists a body of examples and heuristic principles by which our intuition has been strengthened. Future emphasis will probably be on streamline patterns that are important for other reasons (the heat-transfer calculation in (**9**) is an example).

3.1 Rotating cylinder

A set of streamlines for this flow was suggested by Ranger (**6**, Fig. 1), but his pattern is not possible as can be shown using results from (**8**). Consider, in Ranger's figure, the right-hand point where the streamline meets the plane. Ranger shows fluid moving towards this point along a streamline coming from infinity; but the fluid emerging from under the cylinder is also

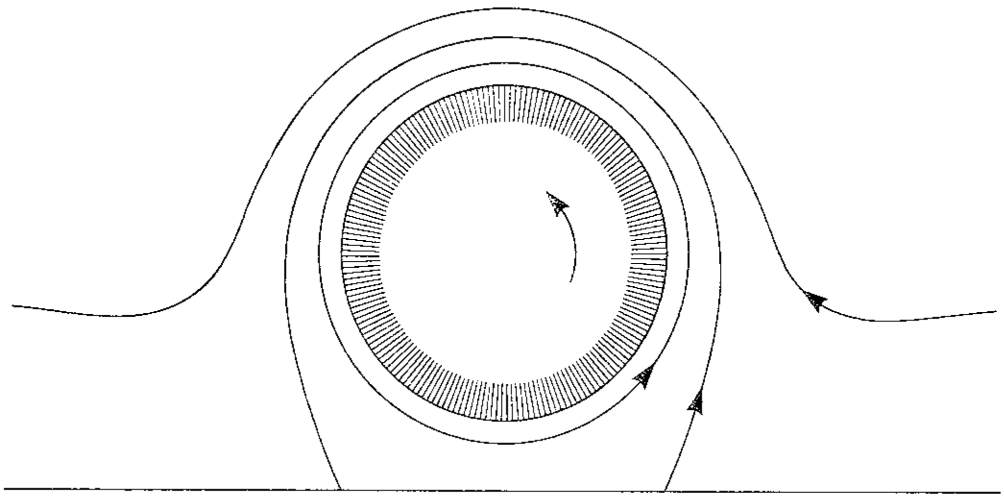

FIG. 2. The streamline pattern for low-Reynolds-number flow around a cylinder rotating next to a plane wall, plotted using a computer graphics package.

flowing towards this point, forcing the fluid coming from infinity to turn around and flow over the cylinder. The possible ways in which fluid can turn back have been studied, and the picture suggested by Ranger would only be possible if two streamlines, rather than one, met the boundary at the point in question (cf. (**8**), Fig. 2b). A check of the derivatives of ψ at this point shows that such is not the case.

The streamlines for a typical case are shown in Fig. 2. The flux of fluid over the cylinder is finite and equals $-2\omega r^2$ to the right. Streamlines meet the plane at $x = \pm a = \pm(d^2 - r^2)^{\frac{1}{2}}$. If the cylinder touches the wall, we know there is no region of closed streamlines (**5**), so to investigate this we set $d = r(1+\epsilon)$, where $\epsilon \ll 1$. We find that a streamline meets the wall at $x = (2\epsilon)^{\frac{1}{2}} r + O(\epsilon)$, while the volume flux under the cylinder is $\frac{2}{3}\epsilon\omega r^2 + O(\epsilon^2)$. The flow in the region between the cylinder and the plane can be modelled by considering a flow in a corner with plane walls, following Jeffrey and Sherwood (**8**). For the present case, we add the stream function for the flow produced by the wall $\theta = \alpha$ sliding outwards with speed V to the stream function for a line source at the origin (**10**), setting $\psi(-\alpha) = 0$. Thus

$$\psi = Vr\left(\frac{\alpha \cos\alpha \sin\theta - \theta \cos\theta \sin\alpha}{2\alpha - \sin 2\alpha} - \frac{\alpha \sin\alpha \cos\theta - \theta \sin\theta \cos\alpha}{2\alpha + \sin 2\alpha}\right) + $$

$$+ \tfrac{1}{2}Q \frac{\sin 2\theta - 2\theta \cos 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha} + \tfrac{1}{2}Q. \tag{4}$$

The streamlines for this flow are shown in Fig. 3. The distance of the separating streamline from the corner decreases with decreasing volume flux (when V is constant), which is what happens in the actual flow.

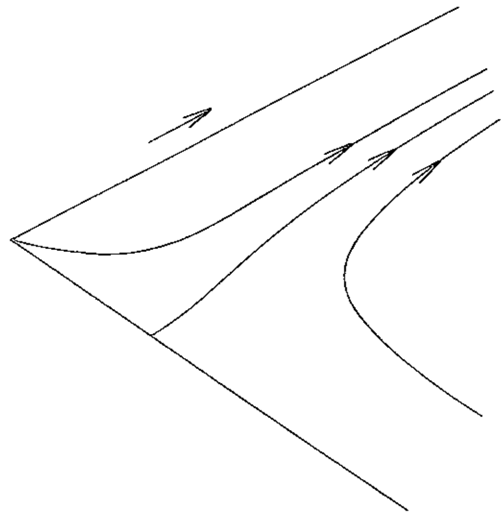

FIG. 3. Flow in a corner when one wall slides parallel to itself and there is a line source of fluid at the vertex.

3.2 The other flows

The streamlines for the other two flows do not vary qualitatively from the streamlines for the special cases of line forces, and these have been given in (**7**). We do note, though, that for the case of motion parallel to the wall, it has been shown in (**4**) that, in the frame of reference in which the cylinder is stationary, a dividing streamline meets the cylinder at $x = \pm ar/d$, $y = a^2/d$.

4. Lubrication theory

If we suppose that $d = r(1+\epsilon)$, i.e. that the smallest gap width is ϵr, we can analyse the flow in the gap between cylinder and plane using lubrication theory, and obtain further insight into the results of section 2. We define stretched coordinates (X, Y) by

$$X = x/r\epsilon^{\frac{1}{2}} \quad \text{and} \quad Y = y/r\epsilon.$$

The surface of the cylinder is given to $O(\epsilon)$ by $Y = H = 1 + \frac{1}{2}X^2$, and the normal and tangential vectors $\mathbf{n}$ and $\mathbf{t}$ are

$$\mathbf{n} = \epsilon^{\frac{1}{2}} X\mathbf{i} - \{1 + \epsilon(1-H)\}\mathbf{j},$$
$$\mathbf{t} = \{1 + \epsilon(1-H)\}\mathbf{i} + \epsilon^{\frac{1}{2}} X\mathbf{j}.$$

4.1 Rotating cylinder

We scale the velocities and pressure according to

$$u = \omega r u_0 + O(\epsilon), \qquad v = \omega r \epsilon^{\frac{1}{2}} v_0 + O(\epsilon^{\frac{3}{2}}) \quad \text{and} \quad p = \mu\omega\epsilon^{-\frac{3}{2}} P + O(\epsilon^{-\frac{1}{2}}).$$

With this scaling, the Stokes equations become, to leading order,

$$\partial P/\partial Y = 0, \qquad \partial P/\partial X = \partial^2 u_0/\partial Y^2 \quad \text{and} \quad \partial v_0/\partial Y = -\partial u_0/\partial X,$$

while the boundary conditions $\mathbf{u}.\mathbf{n}=0$ and $\mathbf{u}.\mathbf{t}=\omega r$ become

$$u_0=1 \quad \text{and} \quad v_0=X \quad \text{on} \quad Y=H.$$

The solutions for u_0 and P that make $P\to 0$ as $X\to\infty$ are

$$P=-2X/H^2 \quad \text{and} \quad u_0=\tfrac{1}{2}P'Y(Y-H)+Y/H.$$

Thus the rotation sets up a pressure field which adds a Poiseuille flow to the more obvious Couette flow. The volume flux through the gap is

$$Q=\omega r^2\epsilon\int_0^H u_0\,dY=\tfrac{2}{3}\omega r^2\epsilon,$$

and the torque and force acting on the cylinder are

$$T=-\mu\omega r^2\epsilon^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left[\frac{\partial u_0}{\partial Y}\right]_{Y=H}dX+O(1)=-4\pi\mu\omega r^2(2\epsilon)^{-\frac{1}{2}}+O(1)$$

and

$$F_x=-\mu\omega r\epsilon^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left\{\left[\frac{\partial u_0}{\partial Y}\right]_{Y=H}+XP\right\}dX=0.$$

We see that F_x is zero because the pressure field provides a force on the cylinder which exactly balances the skin friction $\partial u_0/\partial Y$. Also, $\partial u_0/\partial Y$ on the plane is zero at $X=\pm\sqrt{2}$, indicating a separation point. All these results agree with the appropriate limits taken from the general solution.

4.2 Motion tangential to wall

We scale the velocities and pressure according to

$$u=Uu_0+O(\epsilon), \qquad v=U\epsilon^{\frac{1}{2}}v_0+O(\epsilon^{\frac{3}{2}}) \quad \text{and} \quad p=\mu Ur^{-1}\epsilon^{-\frac{3}{2}}P+O(\epsilon^{-\frac{1}{2}}).$$

The boundary conditions become $u_0=1$ and $v_0=0$, but the equations are unchanged and the solutions for u_0 and P are

$$P=2X/H^2 \quad \text{and} \quad u_0=\tfrac{1}{2}P'Y(Y-H)+Y/H.$$

Here, the Poiseuille contribution to u_0 acts against the Couette one, although u_0 is still positive everywhere. The volume flux is $Ur\epsilon(H-\frac{2}{3})$ and depends on X because the upper surface is moving. The torque is zero, because the Poiseuille and Couette contributions cancel, and the force is $F_x=-4\pi\mu U(2\epsilon)^{-\frac{1}{2}}$. In the frame of reference in which the cylinder is stationary, there is a separation point on the cylinder at $X=\pm\sqrt{2}$.

4.3 Motion perpendicular to the wall

The velocities and pressure scale according to

$$u=V\epsilon^{-\frac{1}{2}}u_0+O(\epsilon^{\frac{1}{2}}), \qquad v=Vv_0+O(\epsilon) \quad \text{and} \quad p=\mu Vr^{-1}\epsilon^{-2}P+O(\epsilon^{-1}).$$

The boundary conditions are $u_0=0$ and $v_0=1$, and the solutions are

$$P=-6/H^2 \quad \text{and} \quad u_0=\tfrac{1}{2}P'Y(Y-H).$$

The force is

$$F_y=\mu V\epsilon^{-\frac{3}{2}}\int_{-\infty}^{\infty} P\,dX=-12\pi\mu V(2\epsilon)^{-\frac{3}{2}}+O(\epsilon^{-\frac{1}{2}}),$$

so that actually the next term in the approximation is also singular.

5. Cylinder or sphere moving near an inclined plane

We include in this section calculations for both the cylinder and sphere cases to show the contrasts between them. In each case the body falls under gravity near an inclined plane. We suppose that the plane is at an angle θ to the horizontal, so that the force on the body is $mg(\sin\theta\mathbf{i}-\cos\theta\mathbf{j})$, where m is mass or mass per unit length as appropriate. To calculate the rate at which the gap shrinks, we make the quasi-stationary assumption that $V=r\,d\epsilon/dt$ in the equation for F_y.

5.1 Cylinder case

The most interesting qualitative result of the analysis is that the cylinder moves without rotating. The equations of motion based on the exact expressions for the forces can be integrated numerically, but as pointed out by Moffatt (unpublished notes), it is more interesting to obtain asymptotic expressions, valid for large times and small gaps, using the lubrication results of section 4. By comparing the exact and approximate expressions for force at various gap widths, we can see that the results of section 4 are correct to within 10% when $\epsilon=0{\cdot}1$ and 1% when $\epsilon=0{\cdot}01$. If the centre of the cylinder is at $(x_c, r+r\epsilon)$, it is easy to show that for $\theta<\frac{1}{2}\pi$,

$$\epsilon=(Kt)^{-2}+O(t^{-3}), \qquad x_c=3r\tan\theta\log(Kt)+O(1),$$

where $K=mg\cos\theta/6\sqrt{2}\pi\mu r$.

5.2 Sphere case

We can obtain lubrication approximations to the forces on a sphere near a plane wall from (**11, 12, 13**). The singularities in the forces are weaker than they are in the cylinder case, F_y going as ϵ^{-1} and F_x as $\log\epsilon$, implying that the sphere must be closer to the wall for one-term approximations to be accurate. In fact, we must have $\epsilon<0{\cdot}001$ for an accuracy of 10%. The second contrast with the cylinder case is that a falling sphere rotates. Taking the leading terms from the lubrication theory for the forces, we can integrate to find the long-time asymptotic motion of the sphere. We find that the rotation Ω is given by

$$\Omega r=-\tfrac{1}{4}U+O(1/\log\epsilon).$$

Thus the sphere rotates four times slower than one's everyday ideas of rolling would predict. If the centre of the sphere is at $(x_c, r+r\varepsilon, 0)$, then

$$\varepsilon = \varepsilon_0 \exp(-kt) + O(te^{-2kt}) \quad \text{and} \quad x_c = 2r \tan\theta \log(kt) + O(1),$$

where $k = mg\cos\theta/6\pi\mu r^2$ and ϵ_0 is a notional initial value. The fact that x_c is logarithmic in time for both sphere and cylinder is a surprising coincidence.

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