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Higher-order corrections to the axisymmetric interactions of nearly touching spheres

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Two unequal spheres are immersed in an axisymmetric low-Reynolds-number flow and are separated by a small nondimensional distance ϵ. The forces and stresslets exerted by the spheres on the fluid are calculated asymptotically to $O(\epsilon \ln \epsilon)$, in order to improve the convergence of numerical calculations of the same quantities. The main obstacle to the extension to $O(\epsilon \ln \epsilon)$ was the breakdown of the usual lubrication approach at higher orders in ϵ. The method adopted here obtains expressions for the functions of immediate interest, but does so by postponing the resolution of the fundamental difficulties until a higher order in ϵ.

The axisymmetric motions of two nearly touching spheres in a low-Reynolds-number flow have been studied in several papers using an approach that is often called lubrication theory.[1–4] Although different in emphasis from the theory of lubrication used in engineering, lubrication theory uses many of the same assumptions. The most important of these, from the perspective of the present work, is the assumption that one can calculate quantities of interest using only the solution for the flow in the narrow gap separating the spheres. This assumption is made possible partly by the nature of the problem, and partly because a quantity of interest is usually taken to be one whose behavior is singular in the gap width. The flow outside the gap is usually not calculated, and its only role is to justify the whole procedure by existing.

Most authors have found this a satisfactory basis for calculations, although O'Neill and Stewartson[5] expressed some caution in the introduction to their paper and so calculated the full flow field, at least to first order. They showed for their problem (which was actually an asymmetric flow rather than an axisymmetric one) that the singular terms could be found from the gap flow and that the usual procedure gave the correct result. They then deduced results to the next order, assuming the existence of a solution to a sufficiently high order outside the gap. When, however, Jeffrey and Corless[4] tried to use the method to obtain results for the stresslet functions correct to $O(\epsilon \ln \epsilon)$, the procedure broke down. The pressure at this order shows a logarithmic divergence at the "edge" of the gap and this cannot be handled by the usual arguments. Faced with this, one might try to understand the observed behavior by completing a full matched asymptotic analysis to the problem. However, since the difficulty does not arise until the third order, the calculation would clearly be one of some magnitude, and one must ask whether the results obtained from such a calculation are important enough to warrant the effort. In this regard, existing results[4,6] do show that the terms that are $O(\epsilon \ln \epsilon)$ for the resistance functions make a very useful contribution to the convergence of numerical calculations and are worth getting if the price is not too high.

The main result of this Brief Communication is to show that the extra terms can indeed be obtained at not too great a cost by an expedient that postpones the difficulties to $O(\epsilon^2 \ln \epsilon)$. The key is to notice that the difficulties did not arise in an earlier calculation[3] to $O(\epsilon \ln \epsilon)$, where the streamfunction was used to solve the flow problem and to calculate the integrals for the forces; in contrast, Jeffrey and Corless used primitive variables, because the stresslet integral cannot be expressed using the streamfunction. The integral for the force in terms of the streamfunction differs in more than a trivial way from the integral of the surface stress, and this is what we exploit.

The description of the flow problem will be highly condensed and the reader is referred to earlier work[1–4] for a full treatment. Jeffrey and Corless defined two flow problems, but the scaling is the same for each; only the boundary conditions are different. We suppose the spheres have radii a and b and that their centers are on the z axis at $z = a + h$ and $z = -b$. The nondimensional gap width is $\epsilon = h/a$ with $\epsilon \ll 1$. We introduce nondimensionalized, stretched cylindrical coordinates by the relations

$$Z = z/a\epsilon, \quad (1a)$$

and

$$R = r/a\epsilon^{1/2}. \quad (1b)$$

The scaled velocity and pressure are expanded in powers of ϵ as follows:

$$u = \epsilon^{-1/2} U = \epsilon^{-1/2}(U_1 + \epsilon U_2 + \epsilon^2 U_3), \quad (2)$$

$$w = W = W_1 + \epsilon W_2 + \epsilon^2 W_3, \quad (3)$$

$$p = \epsilon^{-2} P = \epsilon^{-2}(P_1 + \epsilon P_2 + \epsilon^2 P_3). \quad (4)$$

The governing equations for the third-order quantities are

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the obvious extensions of the equations given in Jeffrey and Corless. When they are solved, it is found that the pressure functions have asymptotic behavior for large R as follows:

$$P_1 \propto R^{-4} + O(R^{-6}), \quad P_2 \propto R^{-2} + O(R^{-4}),$$
$$P_3 \propto \log R + O(R^{-2}).$$

The forces exerted by the spheres must, by symmetry, be in the z direction, so we can calculate them from

$$F_z = -\int \mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\, dS. \tag{5}$$

Following the notation of Corless and Jeffrey,[7] we use the velocity scale v and denote $\mathbf{n}$ in cylindrical coordinates (r,θ,z) by $(\sin\phi, 0, -\cos\phi)$, and obtain

$$F_z/\pi a\mu v = -\int_0^\pi \Big[-\cos\phi\Big(-p+\mu\frac{dw}{dz}\Big) + \mu\sin\phi\Big(\frac{dw}{dr}+\frac{du}{dz}\Big)\Big] 2\sin\phi\, d\phi. \tag{6}$$

The $p\cos\phi$ is the troublesome term, because when we approximate the integral in the gap region, i.e., apply (1)–(4), we find that at $O(\epsilon)$ the integrand contains a logarithmic term introduced by P_3; previous calculations did not have this. The problem is not simply one of divergence, because many other calculations contain divergent integrands. Algebraic divergence can be handled, or more accurately can be assumed to cancel, by the standard arguments developed by earlier authors. It is the logarithmic nature that is new. The logarithmic term did not appear in Jeffrey,[3] who used a streamfunction version of (5) (Happel and Brenner,[8] Eq. 4-14.18). Studying the derivation of the streamfunction form, we see that key steps are an integration by parts, followed by an appeal to the fact that the body is closed. Taking this idea, we integrate the term containing $p\cos\phi$ by parts as follows:

$$-\int_0^\pi (p\cos\phi) 2\sin\phi\, d\phi = [-\sin^2\phi\, p]_0^\pi + \int_0^\pi \sin^2\phi \frac{dp}{dr}\, dr. \tag{7}$$

Provided the pressure is finite at $\phi = 0$ and π, the square brackets are zero, and we have changed an integral of p into an integral of dp/dr, which means the logarithmic term is replaced by an algebraic term. Notice that we have transformed the integral using the properties of the entire flow, not just the flow in the gap between the spheres. In this sense, we cannot calculate the singular terms in the force on the sphere to $O(\epsilon \ln \epsilon)$ using only the solution for the flow in the gap and the definition of force, which is a departure from previous use of lubrication theory. When we calculate the modified integral for the approaching sphere problem, we reproduce the results for $X^A_{\alpha\beta}$ already known correct to $O(\epsilon \ln \epsilon)$. From the deforming sphere problem, we get new results, which, using $\xi = 2\epsilon/(1+\lambda)$, are

$$X^G_{11} = \frac{3\lambda^2}{(1+\lambda)^3}\xi^{-1} + \frac{3\lambda(1+12\lambda-4\lambda^2)}{10(1+\lambda)^3} \times \ln \xi^{-1} + G^X_{11}(\lambda) + \frac{5+181\lambda-453\lambda^2+566\lambda^3-65\lambda^4}{140(1+\lambda)^3}\xi \ln \xi^{-1} \tag{8}$$

and

$$X^G_{12} = -\frac{12\lambda^2}{(1+\lambda)^5}\xi^{-1} - \frac{6\lambda(1+12\lambda-4\lambda^2)}{5(1+\lambda)^5} \times \ln \xi^{-1} + G^X_{12}(\lambda). - \frac{5+181\lambda-453\lambda^2+566\lambda^3-65\lambda^4}{35(1+\lambda)^5}\xi \ln \xi^{-1}. \tag{9}$$

The first check on the working is that the expression

$$X^G_{11} + \tfrac{1}{4}(1+\lambda)^2 X^G_{12} \tag{10}$$

should be free of singular terms, in accord with physical expectations.[4] This is obviously so. A second check is that

$$X^G_{11} + \tfrac{1}{4}(1+\lambda)^2 X^G_{21} - \tfrac{3}{4}(1+\lambda)(2+\xi) X^A_{11} \tag{11}$$

should also be free from singular terms,[4] as it is.

Turning now to the stresslet, the integral to evaluate is[4]

$$S_{zz} = -a\int (\mathbf{k}\cdot\mathbf{n}\mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} - \tfrac{1}{3}\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n})\, dS. \tag{12}$$

The integration by parts idea no longer works because the relevant term is now $p\cos^2\phi\sin\phi$. Thus we must look for another transformation. We notice that inside the gap, $\mathbf{n} \simeq -\mathbf{k}$, meaning that we can write

$$S_{zz} = -\Big(\frac{2a}{3}\Big)F_z - a\int\Big((\mathbf{k}\cdot\mathbf{n}+1)\mathbf{k}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} - \frac{1}{3}(\mathbf{n}+\mathbf{k})\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\Big) dS. \tag{13}$$

The integral still contains the pressure, and hence shows $\ln R$ behavior, but now the pressure is multiplied by ϵ instead of 1, and therefore the influence of the p_3 term is pushed back until $O(\epsilon^2 \ln \epsilon)$. With the new integral and the results for the force, we arrive at our other main results, which are expressions for the functions $X^M_{\alpha\beta}$. Since the previous calculations had been programmed using MAPLE, there was little extra work to repeat the calculations with the new integral expressions. We find

$$X^M_{11} = \frac{6\lambda^2}{5(1+\lambda)^3}\xi^{-1} + \frac{3\lambda(1+17\lambda-9\lambda^2)}{25(1+\lambda)^3} \times \ln \xi^{-1} + M^X_{11}(\lambda), + \frac{5+272\lambda-831\lambda^2+1322\lambda^3-415\lambda^4}{350(1+\lambda)^3}\xi \ln \xi^{-1} \tag{14}$$

and

$$X^M_{21} = \frac{48\lambda^3}{5(1+\lambda)^6}\xi^{-1} + \frac{24\lambda^2(-4\lambda^2+17\lambda-4)}{25(1+\lambda)^6} \times \ln \xi^{-1} + M^X_{21}(\lambda)$$

TABLE I. Revised convergence properties of series for $M^{X}_{\alpha\beta}$. The columns compare the convergence of infinite series expansions that do not use the $\xi \ln \xi$ terms obtained here with ones that do. The improvement in the convergence with the number of terms n is clear.

	M^{X}_{11}		M^{X}_{12}	
n	without new term	with new term	without new term	with new term
100	0.719 27	0.716 80	− 0.141 94	− 0.145 52
200	0.718 58	0.717 33	− 0.144 24	− 0.145 98
300	0.718 15	0.717 31	− 0.144 80	− 0.145 97
350	0.718 05	0.717 33	− 0.144 98	− 0.145 99

$$+\frac{4\lambda(-65\lambda^{4}+832\lambda^{3}-1041\lambda^{2}+832\lambda-65)}{175(1+\lambda)^{6}} \times \xi \ln \xi^{-1}. \tag{15}$$

Since we expect nonsingular behavior when the spheres are at rest in an ambient rate-of-strain field,[4] we expect the expression

$$X^{M}_{11}+\tfrac{1}{8}(1+\lambda)^{3}X^{M}_{21}-\tfrac{1}{5}(1+\lambda)(2+\xi)X^{G}_{11} \tag{16}$$

will be free from singular terms, and this has been verified.

In the special case $\lambda = 1$, Jeffrey and Corless attempted to obtain the coefficient of the terms in $\xi \ln \xi$ by using a plausible argument. Now that we have the coefficients, we see that they were correct for the $X^{G}_{\alpha\beta}$ functions, but incorrect for the $X^{M}_{\alpha\beta}$ functions. The corrected expressions for $\lambda = 1$ are

$$X^{M}_{11} = \tfrac{3}{20}\xi^{-1} + \tfrac{27}{200}\ln \xi^{-1} + 0.7173 + \tfrac{353}{2800}\xi \ln \xi^{-1} + 0.07\xi, \tag{17}$$

$$X^{M}_{12} = \tfrac{3}{20}\xi^{-1} + \tfrac{27}{200}\ln \xi^{-1} - 0.1460 + \tfrac{493}{2800}\xi \ln \xi^{-1} + 0.15\xi, \tag{18}$$

where $\xi = \epsilon$ when $\lambda = 1$. As further tests of the new results, the convergence tests of Jeffrey and Corless[4] and Jeffrey[9] were done again. Some results are shown in Table I, where it can be seen that the convergence is improved using the new values.

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