

An analytical approach for solving nonlinear boundary value problems in finite domains

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Abstract Based on the homotopy analysis method (HAM), a general analytical approach for obtaining approximate series solutions to nonlinear two-point boundary value problems in finite domains is proposed. To demonstrate its effectiveness, this approach is applied to solve three nonlinear problems, and the analytical solutions obtained are more accurate than the numerical solutions obtained via the shooting method and the sinc-Galerkin method.

Keywords Boundary value problem · Series solution · Homotopy analysis method · Symbolic computation · Analytical solution

1 Introduction

In the field of differential equations, a boundary value problem is a differential equation together with a set of additional restraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions. Boundary value problems arise in engineering, applied mathematics and several branches of physics, and have attracted much attention; see [1–7] for references.

However, it is usually difficult to obtain closed-form solutions for boundary value problems, especially for nonlinear boundary value problems. In most cases, only approximate solutions (either numerical solutions or analytical

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solutions) can be expected. Some numerical methods such as finite difference method [1], finite element method [2], shooting method [3], spline approximation method [4], and sinc-Galerkin method [5] have been developed for obtaining approximate solutions to boundary value problems.

Perturbation method [8] is one of the well-known analytical methods for solving nonlinear problems. However, it strongly depends on the existence of small/large parameters. Traditional non-perturbation methods such as differential transformation method [9] and Adomian's decomposition method [10] have been developed for solving boundary value problems. However, these methods cannot provide a mechanism to adjust and control the convergence region and rate of the series solutions.

In Section 2 of this paper, based on the homotopy analysis method (HAM) [11–13] which has been successfully applied to solve many types of problems [14–17], we propose a general analytical approach for solving the following type of nonlinear boundary value problems in a finite domain:

$$u^{(n)}(x) = f(x, u(x), u'(x), \dots, u^{(n-1)}(x)), \quad (1)$$

subject to the two-point boundary conditions

$$\begin{aligned} u(a) &= \alpha_0, \quad u'(a) = \alpha_1, \quad \dots, \quad u^{(r)}(a) = \alpha_r, \\ u(b) &= \beta_0, \quad u'(b) = \beta_1, \quad \dots, \quad u^{(n-r-2)}(b) = \beta_{n-r-2}, \end{aligned} \quad (2)$$

where $0 \leq r \leq n - 2$ is an integer, f is a polynomial in $x, u(x), u'(x), \dots, u^{(n-1)}(x)$, and $a, b, \alpha_0, \alpha_1, \dots, \alpha_r, \beta_0, \beta_1, \dots, \beta_{n-r-2}$ are real constants. For the cases when $n = 4$ and $n = 6$, see [18, 19].

To demonstrate its effectiveness, the approach is applied to solve three nonlinear two-point boundary value problems. More precisely, in Section 3.1, the approach is applied to solve a nonlinear second order boundary value problem [6]. It is shown that the series solution obtained via this approach is more accurate than the numerical solution obtained via the shooting method. In Section 3.2, the approach is applied to solve a nonlinear third order boundary value problem [7] which does not have a closed-form solution, and an approximate series solution is obtained which agrees very well with the numerical solution obtained by the Runge–Kutta–Fehlberg 4–5 technique. In Section 3.3, the approach is applied to solve a nonlinear fourth order boundary value problem [5]. It is shown that the series solution obtained via this approach is much more accurate than the numerical solution obtained via the sinc-Galerkin method. The success of this approach lies in the fact that the HAM provides a convenient way to adjust and control the convergence region and rate of the series solutions obtained. Finally in Section 4, some concluding remarks are given.

2 The HAM-based approach

In order to obtain a convergent series solution to the nonlinear problem (1, 2), we first construct the zeroth order deformation equation

$$(1 - p)\mathcal{L}[\phi(x; p) - u_0(x)] = p \hbar \mathcal{N}[\phi(x; p)], \tag{3}$$

where $p \in [0, 1]$ is an embedding parameter, $\hbar \neq 0$ is a convergence-control parameter, and $\phi(x; p)$ is an unknown function, respectively. According to (1), the auxiliary linear operator \mathcal{L} is given by

$$\mathcal{L}[\phi(x; p)] = \frac{\partial^n \phi(x; p)}{\partial x^n}, \tag{4}$$

and the nonlinear operator \mathcal{N} is given by

$$\mathcal{N}[\phi(x; p)] = \frac{\partial^n \phi}{\partial x^n} - f\left(x, \phi, \frac{\partial \phi}{\partial x}, \dots, \frac{\partial^{n-1} \phi}{\partial x^{n-1}}\right). \tag{5}$$

The initial guess $u_0(x)$ of the solution $u(x)$ can be determined by the rule of solution expression as follows.

In view of (1), the solution $u(x)$ can be expressed by a set of base functions

$$\{x^n | n = 0, 1, 2, \dots\} \tag{6}$$

in the form

$$u(x) = \sum_{n=0}^{+\infty} d_n x^n, \tag{7}$$

where $d_n (n = 0, 1, 2, \dots)$ are coefficients to be determined later. This provides us with the *rule of solution expression*.

According to the rule of solution expression (7), the initial guess $u_0(x)$ is given by

$$u_0(x) = x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0. \tag{8}$$

In view of the given boundary conditions (2), the coefficients $a_0, a_1, a_2, \dots, a_{n-1}$ can be determined by solving a system of n linear equations. Finally from (2), the boundary conditions to the zeroth order deformation equation (3) are given by

$$\begin{aligned} \phi(a; p) = \alpha_0, \quad \frac{\partial \phi(a; p)}{\partial x} = \alpha_1, \quad \dots, \quad \frac{\partial^r \phi(a; p)}{\partial x^r} = \alpha_r, \\ \phi(b; p) = \beta_0, \quad \frac{\partial \phi(b; p)}{\partial x} = \beta_1, \quad \dots, \quad \frac{\partial^{n-r-2} \phi(b; p)}{\partial x^{n-r-2}} = \beta_{n-r-2}. \end{aligned} \tag{9}$$

We now focus on how to obtain higher order approximations to the problem (1, 2). From (3), when $p = 0$ and $p = 1$,

$$\phi(x; 0) = u_0(x) \quad \text{and} \quad \phi(x; 1) = u(x) \tag{10}$$

both hold. Therefore, as p increases from 0 to 1, the solution $\phi(x; p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x; p)$ in Taylor series with respect to p , one has

$$\phi(x; p) = \phi(x; 0) + \sum_{m=1}^{+\infty} u_m(x) p^m, \tag{11}$$

where

$$u_m(x) = \frac{1}{m!} \left. \frac{\partial^m \phi(x; p)}{\partial p^m} \right|_{p=0}. \tag{12}$$

Now the convergence of the series (11) depends on the parameter \hbar . Assuming that \hbar is chosen so properly that the series (11) is convergent at $p = 1$, we have, by means of (10), the solution series

$$u(x) = \phi(x; 1) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) \tag{13}$$

which must be one of the solutions of the original problem (1, 2), as proved by Liao in [12].

Our next goal is to determine the higher order terms $u_m(x) (m \geq 1)$. Define the vector

$$\mathbf{u}_s(x) = \{u_0(x), u_1(x), \dots, u_s(x)\}. \tag{14}$$

Differentiating the zeroth order deformation equation (3) and its boundary conditions (9) m times with respect to p , then setting $p = 0$, finally dividing them by $m!$, we obtain the m th order deformation equation

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar R_m(\mathbf{u}_{m-1}(x)), \tag{15}$$

and its boundary conditions

$$\begin{aligned} u_m(a) &= u'_m(a) = \dots = u_m^{(r)}(a) = 0, \\ u_m(b) &= u'_m(b) = \dots = u_m^{(n-r-2)}(b) = 0, \end{aligned} \tag{16}$$

where

$$R_m(\mathbf{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(x; p)]}{\partial p^{m-1}} \right|_{p=0}, \tag{17}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Note that the m th order deformation equation (15) becomes

$$u_m^{(n)}(x) = \chi_m u_{m-1}^{(n)}(x) + \hbar R_m(\mathbf{u}_{m-1}(x)). \tag{18}$$

According to the rule of solution expression (7), the right hand side of (18) can be expressed by

$$u_m^{(n)}(x) = \sum_{k=0}^{N(m)} d_k x^k, \tag{19}$$

where the upper limit $N(m)$ depends on m .

Since the solution of $u_m^{(n)}(x) = x^k$ with the boundary conditions (16) can be expressed by

$$u_m(x) = \frac{k!}{(k+n)!} x^{k+n} + \delta_{n-1} x^{n-1} + \delta_{n-2} x^{n-2} + \dots + \delta_1 x + \delta_0, \tag{20}$$

where, in view of (16), $\delta_0, \delta_1, \dots, \delta_{n-2}, \delta_{n-1}$ can be determined by solving a system of n linear equations, then by (19), (20) and the linearity of \mathcal{L}^{-1} , we finally obtain

$$u_m(x) = \sum_{k=0}^{N(m)} d_k \left(\frac{k!}{(k+n)!} x^{k+n} + \delta_{n-1,k} x^{n-1} + \dots + \delta_{1,k} x + \delta_{0,k} \right). \tag{21}$$

In this way, we can solve $u_m(x) (m = 1, 2, 3, \dots)$ recursively.

The m th order approximation to the problem (1, 2) can be generally expressed by

$$u(x, \hbar) \approx \sum_{k=0}^m u_k(x) = \sum_{k=0}^{\sigma(m)} \gamma_{m,k}(\hbar) x^k, \tag{22}$$

where the upper limit $\sigma(m)$ depends on m , and the coefficients $\gamma_{m,k}(\hbar) (k = 0, 1, 2, \dots, \sigma(m))$ depend on m, k and \hbar . Equation (22) is a family of solutions to the problem (1, 2) expressed in terms of the parameter \hbar .

To obtain an accurate approximation (22) to the problem (1, 2), a proper value of \hbar must be found. First, the valid region of \hbar can be obtained via the \hbar -curve as follows.

Let $c_0 \in [a, b]$. Then $u(c_0, \hbar)$ is a function of \hbar , and the curve $u(c_0, \hbar)$ versus \hbar contains a horizontal line segment which corresponds to the valid region of \hbar . The reason is that all convergent series given by different values of \hbar converge to its exact value. So, if the solution is unique, then all of these series converge to the same value and therefore there exists a horizontal line segment in the curve. We call such kind of curve the \hbar -curve; see Fig. 1 for example, where the valid region of \hbar is about $-0.5 < \hbar < -0.1$.

Although the solution series (22) given by different values in the valid region of \hbar converge to the exact solution, the convergence rates of these solution series are usually different. A more accurate approximation can be obtained by assigning \hbar a proper value. By substituting the m th order approximation (22) into the original governing equation (1) and then integrating the square residual error over the whole domain $[a, b]$, one gets a function of \hbar , denoted $F(\hbar)$. Minimizing $F(\hbar)$ gives the best value of \hbar which corresponds to the best

approximation. However, it is usually difficult to minimize $F(\hbar)$. Alternatively, one can obtain a proper value of \hbar just by observation.

3 Applications

In this section, the approach proposed in Section 2 is applied to solve three nonlinear two-point boundary value problems. For the nonlinear second order problem (Section 3.1) and the nonlinear fourth order problem (Section 3.3), the approximate solutions given by the approach are much more accurate than the numerical solutions given by the shooting method and the sinc-Galerkin method. For the nonlinear third order problem (Section 3.2) whose closed-form solution is not available, the approximate solution given by the approach agrees very well with the numerical solutions given by the Runge–Kutta–Fehlberg 4–5 technique.

3.1 A second order problem

Consider the following nonlinear second order boundary value problem

$$u''(x) = 2u(x)^3 - 6u(x) - 2x^3, \quad 1 \leq x \leq 2, \tag{23}$$

$$u(1) = 2, \quad u(2) = \frac{5}{2}, \tag{24}$$

whose exact solution is

$$u_{\text{exact}}(x) = x + \frac{1}{x}. \tag{25}$$

This problem was considered in [6] by means of the shooting method.

For the zeroth order deformation equation (3), the auxiliary linear operator \mathcal{L} is given by

$$\mathcal{L}[\phi(x; p)] = \frac{\partial^2 \phi(x; p)}{\partial x^2}, \tag{26}$$

and the nonlinear operator is given by

$$\mathcal{N}[\phi(x; p)] = \frac{\partial^2 \phi(x; p)}{\partial x^2} - 2\phi(x; p)^3 + 6\phi(x; p) + 2x^3. \tag{27}$$

In view of the boundary conditions (24), the initial guess is determined as

$$u_0(x) = x^2 - \frac{5}{2}x + \frac{7}{2}, \tag{28}$$

and the boundary conditions to (3) are given by

$$\phi(1; p) = 2, \quad \phi(2; p) = \frac{5}{2}. \tag{29}$$

To obtain higher order terms $u_m(x)$, the m th order deformation equation (15) and its boundary conditions (16) are calculated:

$$u_m''(x) = \chi_m u_{m-1}''(x) + \hbar R_m(\mathbf{u}_{m-1}(x)), \tag{30}$$

$$u_m(1) = 0, \quad u_m(2) = 0, \tag{31}$$

where

$$R_m(\mathbf{u}_{m-1}(x)) = u_{m-1}''(x) + 6u_{m-1}(x) + 2(1 - \chi_m)x^3 - 2 \sum_{j=0}^{m-1} \left(u_{m-1-j}(x) \sum_{i=0}^j u_i(x) u_{j-i}(x) \right). \tag{32}$$

In this way, we can calculate $u_m(x) (m = 1, 2, \dots)$ recursively. For example, when $m = 1$, (30) becomes

$$\begin{aligned} u_1''(x) &= \hbar (u_0''(x) + 6u_0(x) + 2x^3 - 2u_0(x)^3) \\ &= -\frac{1}{4} \hbar (8x^6 - 60x^5 + 234x^4 - 553x^3 + 795x^2 - 675x + 251). \end{aligned} \tag{33}$$

Since the formula (20) now becomes

$$u_1(x) = \frac{x^{k+2}}{(k+1)(k+2)} - \frac{(2^{k+2} - 1)x}{k^2 + 3k + 2} + \frac{2^{k+2} - 2}{k^2 + 3k + 2}, \tag{34}$$

by (21), the first order term

$$\begin{aligned} u_1(x) &= -\frac{1}{560} \hbar (x - 1)(x - 2) \\ &\quad \times (20x^6 - 140x^5 + 632x^4 - 1695x^3 + 2926x^2 - 3582x + 972). \end{aligned} \tag{35}$$

$u_m(x) (m = 2, 3, \dots)$ can be calculated similarly.

The m th order approximation can be expressed by

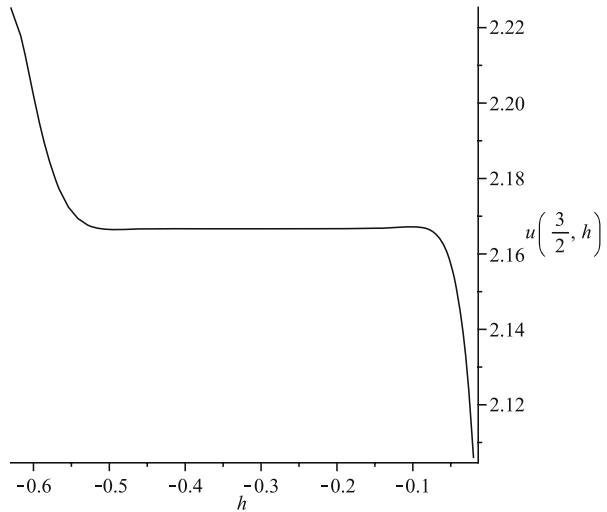
$$u(x, \hbar) \approx \sum_{k=0}^m u_k(x) = \sum_{k=0}^{6m+2} \gamma_{m,k}(\hbar) x^k, \tag{36}$$

where the coefficients $\gamma_{m,k}(\hbar) (k = 0, 1, \dots, 6m + 2)$ depend on m, k and \hbar . Equation (36) is a family of approximate solutions to the problem (23, 24) in terms of the convergence-control parameter \hbar .

To find the valid region of \hbar , the \hbar -curve given by the 15th order approximation (36) at $x = \frac{3}{2}$ is drawn in Fig. 1, which clearly indicates that the valid region of \hbar is about $-0.5 < \hbar < -0.1$.

When $\hbar = -0.167$, we obtain an approximate solution which is more accurate than the numerical solution given by the shooting method [6] as shown in Table 1, where the absolute errors of the shooting method approximation, the 5th order, 10th order, and 15th order HAM approximations are calculated and compared.

Fig. 1 h -curve for the 15th order approximation to (23, 24)



3.2 A third order problem

In this section, we consider the following nonlinear third order boundary value problem

$$u^{(3)}(x) + u(x)u''(x) - u'(x)^2 + 1 = 0, \quad 0 \leq x \leq 1, \tag{37}$$

$$u(0) = 0, u'(0) = 0, u(1) = 0. \tag{38}$$

A similar problem was considered in [7] via the finite difference method.

For the zeroth order deformation equation (3), the linear operator \mathcal{L} and the nonlinear operator \mathcal{N} are given by

$$\mathcal{L}[\phi(x; p)] = \frac{\partial^3 \phi(x; p)}{\partial x^3}, \tag{39}$$

$$\mathcal{N}[\phi(x; p)] = \frac{\partial^3 \phi(x; p)}{\partial x^3} + \phi(x; p) \frac{\partial^2 \phi(x; p)}{\partial x^2} - \left(\frac{\partial \phi(x; p)}{\partial x} \right)^2 + 1. \tag{40}$$

Table 1 Absolute errors of shooting and HAM approximations

x	Exact soln	Shooting	5thHAM	10thHAM	15thHAM
1.1	2.0090909	0.0006723	0.0005673	0.0001009	0.0000525
1.2	2.0333333	0.0014338	0.0007144	0.0000551	0.0000544
1.3	2.0692308	0.0021682	0.0004844	0.0001164	0.0000124
1.4	2.1142857	0.0028069	0.0001755	0.0003130	0.0000443
1.5	2.1666667	0.0033009	0.0001453	0.0004174	0.0000844
1.6	2.2250000	0.0035980	0.0006529	0.0003336	0.0000776
1.7	2.2882353	0.0036213	0.0017059	0.0000210	0.0000265
1.8	2.3555556	0.0032344	0.0028780	0.0004440	0.0001910
1.9	2.4263158	0.0021849	0.0030510	0.0007363	0.0001458

In view of the boundary conditions (38), the initial guess and the boundary conditions to (3) are determined as

$$u_0(x) = x^3 - x^2, \tag{41}$$

$$\phi(0; p) = 0, \frac{\partial \phi(0; p)}{\partial x} = 0, \phi(1; p) = 0. \tag{42}$$

To obtain higher order terms $u_m(x)$, the m th order deformation equation (15) and its boundary conditions (16) are calculated:

$$u_m^{(3)}(x) = \chi_m u_{m-1}^{(3)}(x) + \hbar R_m(\mathbf{u}_{m-1}(x)), \tag{43}$$

$$u_m(0) = 0, u_m'(0) = 0, u_m(1) = 0, \tag{44}$$

where $R_m(\mathbf{u}_{m-1}(x)) =$

$$u_{m-1}^{(3)}(x) + \sum_{i=0}^{m-1} u_i(x) u_{m-1-i}''(x) - \sum_{j=0}^{m-1} u_j'(x) u_{m-1-j}'(x) + 1 - \chi_m. \tag{45}$$

In this way, one can calculate $u_m(x) (m = 1, 2, \dots)$ recursively. For example, when $m = 1$, (43) becomes

$$\begin{aligned} u_1^{(3)}(x) &= \hbar \left(u_0^{(3)}(x) + u_0(x) u_0''(x) - u_0'(x)^2 + 1 \right) \\ &= -\hbar (3x^4 - 4x^3 + 2x^2 - 7). \end{aligned} \tag{46}$$

Since the formula (20) now becomes

$$u_1(x) = \frac{x^{k+3}}{(k+1)(k+2)(k+3)} - \frac{x^2}{k^3 + 6k^2 + 11k + 6}, \tag{47}$$

by (21), the first order term

$$u_1(x) = -\frac{1}{210} \hbar x^2 (x-1) (3x^4 - 4x^3 + 3x^2 + 3x - 242). \tag{48}$$

$u_m(x) (m = 2, 3, \dots)$ can be calculated similarly.

The m th order approximation can be expressed by

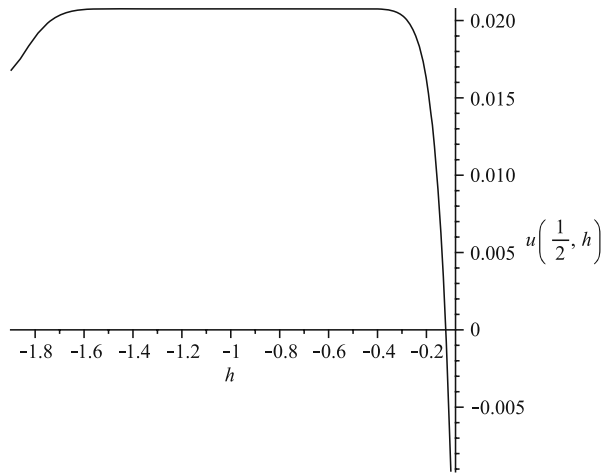
$$u(x, \hbar) \approx \sum_{k=0}^m u_k(x) = \sum_{k=0}^{4m+3} \gamma_{m,k}(\hbar) x^k. \tag{49}$$

To find the valid region of \hbar , the \hbar -curve given by the 15th order approximation (49) at $x = \frac{1}{2}$ is drawn in Fig. 2, which clearly indicates that the valid region of \hbar is about $-1.6 < \hbar < -0.4$.

When $\hbar = -1.23$, we obtain an approximate series solution which agrees very well with the numerical solution given by the Runge–Kutta–Fehlberg 4–5 technique (RKF45), as shown in Fig. 3 and Table 2, where the relative errors of the 5th order, 10th order, and 15th order HAM approximations are calculated and compared. The formula for the relative error is defined by

$$\delta(x) = \left| \frac{u_{\text{num}}(x) - u(x, \hbar)}{u_{\text{num}}(x)} \right|, \tag{50}$$

Fig. 2 h -curve for the 15th order approximation to (37, 38)



where $u_{\text{num}}(x)$ is the numerical solution given by the Runge–Kutta–Fehlberg 4–5 technique, and $u(x, h)$ is the approximate solution (49) when $h = -1.23$.

3.3 A fourth order problem

Finally we consider the following nonlinear fourth order boundary value problem

$$u^{(4)}(x) + u(x)^2 = \frac{x^{-\frac{5}{2}}}{16} (9 + 30x + 105x^2) + x^3 (1 - x)^4, \quad (51)$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \quad (52)$$

Fig. 3 Symbols: 15th order HAM approximation to (37, 38); solid line: numerical solution of (37, 38) given by RKF45

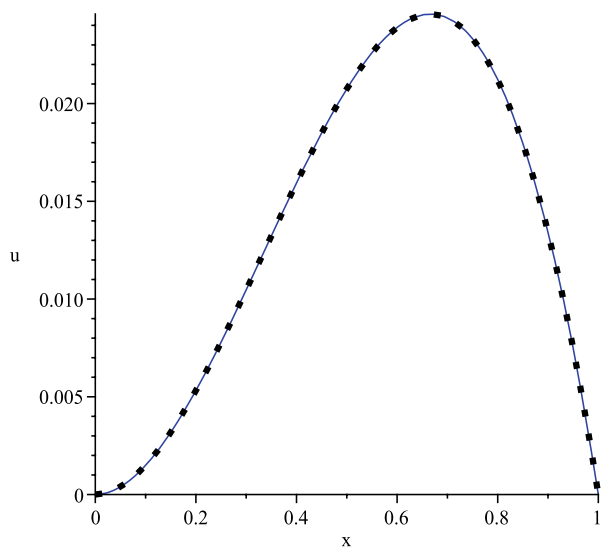


Table 2 Relative errors of HAM approximations to (37, 38)

x	RKF45 soln	5thHAM	10thHAM	15thHAM
0.1	0.001496064	2.7E-3	1.4E-5	3.2E-6
0.2	0.005317810	2.6E-3	1.3E-5	1.4E-6
0.3	0.010466190	2.6E-3	1.4E-5	7.6E-7
0.4	0.015943267	2.8E-3	1.5E-5	4.8E-7
0.5	0.020752307	3.2E-3	1.7E-5	3.5E-7
0.6	0.023897842	3.7E-3	1.8E-5	2.4E-7
0.7	0.024385830	4.6E-3	2.0E-5	1.2E-7
0.8	0.021224162	5.9E-3	2.2E-5	1.7E-8
0.9	0.013423694	7.7E-3	2.3E-5	1.1E-7

whose exact solution is

$$u_{\text{exact}}(x) = x^{\frac{3}{2}}(1 - x)^2. \tag{53}$$

This problem was considered in [5] via the sinc-Galerkin method. It is worth mentioning that although this problem does not belong to the category (1, 2), the approach proposed in Section 2 is still applicable to it.

For the zeroth order deformation equation (3), the auxiliary linear operator \mathcal{L} is given by

$$\mathcal{L}[\phi(x; p)] = \frac{\partial^4 \phi(x; p)}{\partial x^4}, \tag{54}$$

and the nonlinear operator $\mathcal{N}[\phi(x; p)]$ is given by

$$\frac{\partial^4 \phi(x; p)}{\partial x^4} + \phi(x; p)^2 - \frac{x^{-\frac{5}{2}}}{16} (9 + 30x + 105x^2) - x^3 (1 - x)^4. \tag{55}$$

By the boundary conditions (52), the initial guess is determined as

$$u_0(x) = x^4 - 2x^3 + x^2, \tag{56}$$

and the boundary conditions to (3) are given by

$$\phi(0; p) = 0, \frac{\partial \phi(0; p)}{\partial x} = 0, \phi(1; p) = 0, \frac{\partial \phi(1; p)}{\partial x} = 0. \tag{57}$$

To obtain higher order terms $u_m(x)$, the m th order deformation equation (15) and its boundary conditions (16) are calculated:

$$u_m^{(4)}(x) = \chi_m u_{m-1}^{(4)}(x) + \hbar R_m(\mathbf{u}_{m-1}(x)), \tag{58}$$

$$u_m(0) = 0, u'_m(0) = 0, u_m(1) = 0, u'_m(1) = 0, \tag{59}$$

where

$$R_m(\mathbf{u}_{m-1}(x)) = u_{m-1}^{(4)}(x) + \sum_{j=0}^{m-1} u_j(x)u_{m-1-j}(x) - \frac{x^{-\frac{5}{2}}}{16} (9 + 30x + 105x^2) (1 - \chi_m) - x^3 (1 - x)^4 (1 - \chi_m). \tag{60}$$

In this way, one can calculate $u_m(x) (m = 1, 2, \dots)$ recursively. For example, when $m = 1$, (58) becomes

$$\begin{aligned}
 u_1^{(4)}(x) &= \hbar \left[u_0^{(4)}(x) + u_0(x)^2 - \frac{x^{-\frac{5}{2}}}{16} (9 + 30x + 105x^2) - x^3 (1 - x)^4 \right] \\
 &= \hbar \left(x^8 - 5x^7 + 10x^6 - 10x^5 + 5x^4 - x^3 \right. \\
 &\quad \left. + 24 - \frac{9}{16}x^{-\frac{5}{2}} - \frac{15}{8}x^{-\frac{3}{2}} - \frac{105}{16}x^{-\frac{1}{2}} \right). \tag{61}
 \end{aligned}$$

Since the formula (20) now becomes

$$u_1(x) = \frac{k! x^{k+4}}{(k+4)!} - \frac{x^3}{(k+4)(k^2+4k+3)} + \frac{x^2}{k^3+9k^2+26k+24}, \tag{62}$$

by (21), the first order term

$$\begin{aligned}
 u_1(x) &= \hbar \left(\frac{x^{12}}{11880} - \frac{x^{11}}{1584} + \frac{x^{10}}{504} - \frac{5x^9}{1512} + \frac{x^8}{336} - \frac{x^7}{840} \right. \\
 &\quad \left. + x^4 - \frac{9503}{4752}x^3 + \frac{7919}{7920}x^2 - x^{\frac{7}{2}} + 2x^{\frac{5}{2}} - x^{\frac{3}{2}} \right). \tag{63}
 \end{aligned}$$

$u_m(x) (m = 2, 3, \dots)$ can be calculated similarly.

The m th order approximation can be expressed by

$$u(x, \hbar) \approx \sum_{k=0}^m u_k(x) = \sum_{i=2}^{8m+4} \gamma_{m,i}(\hbar) x^i + \sum_{j=1}^{8m-5} \eta_{m,j}(\hbar) x^{\frac{2j+1}{2}}, \tag{64}$$

Fig. 4 \hbar -curve for the 15th order approximation to (51, 52)

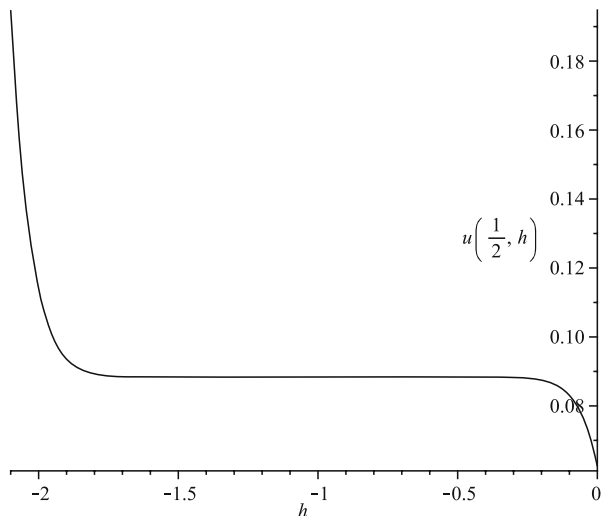


Table 3 Relative errors of sinc-Galerkin and HAM approximations

x	exactSoln	sincGalerkin	5thHAM	10thHAM	15thHAM
0.0915	0.0228445	1.3E-3	8.6E-13	2.2E-21	1.8E-31
0.1518	0.0425506	0.7E-3	1.1E-12	2.9E-21	2.4E-31
0.2410	0.0681568	0.2E-3	1.5E-12	3.7E-21	3.1E-31
0.3604	0.0885104	0.3E-3	1.8E-12	4.5E-21	3.8E-31
0.5000	0.0883883	0.2E-3	2.1E-12	5.2E-21	4.4E-31
0.6395	0.0664617	0.1E-3	2.2E-12	5.6E-21	4.7E-31
0.7590	0.0384058	0.2E-3	2.3E-12	5.7E-21	4.8E-31
0.8482	0.0180008	0.0E-3	2.3E-12	5.7E-21	4.8E-31
0.9084	0.0072645	0.0E-3	2.3E-12	2.7E-21	4.8E-31

where the coefficients $\gamma_{m,i}(\hbar)$, $\eta_{m,j}(\hbar)$ depend on m, i, \hbar and m, j, \hbar respectively. Equation (64) is a family of approximate solutions to the problem (51, 52) in terms of the convergence-control parameter \hbar .

To find the valid region of \hbar , the \hbar -curve given by the 15th order approximation (64) at $x = \frac{1}{2}$ is drawn in Fig. 4, which clearly indicates that the valid region of \hbar is about $-1.75 < \hbar < -0.25$.

When $\hbar = -1$, we obtain an approximate solution which is much more accurate than the numerical solution obtained by the sinc-Galerkin method [5] as shown in Table 3, where the relative errors of the sinc-Galerkin method approximation, the 5th order, 10th order, and 15th order HAM approximations are calculated by

$$\delta(x) = \left| \frac{u_{\text{exact}}(x) - u(x, \hbar)}{u_{\text{exact}}(x)} \right|, \tag{65}$$

where $u_{\text{exact}}(x)$ is the exact solution (53), and $u(x, \hbar)$ is the approximate solution (64).

4 Conclusions

In this paper, a general analytical approach is proposed for solving nonlinear two-point boundary value problems in a finite domain. The effectiveness of this approach is demonstrated by solving three nonlinear boundary value problems. The success of this approach lies in the fact that the HAM provides a convergence-control parameter \hbar which can be used to adjust and control the convergence region and rate of the series solutions obtained.

However, the current method cannot be extended to boundary value problems in *infinite* domains because the radius of convergence of the power series (7) is finite. It is a significant challenge to develop a general analytical approach based on the HAM for solving boundary value problems in infinite domains. The challenge lies in the fact that we have to seek a suitable set of base functions such that the radius of convergence of the corresponding series must be infinite and the corresponding initial guess must satisfy the boundary

condition (2), while we must figure out an effective way to solve the m th order deformation equation (15). We will leave this topic in our future work.

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