Comparison of homotopy analysis method and homotopy perturbation method through an evolution equation

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Abstract

In this paper, the homotopy analysis method (HAM) proposed by Liao in 1992 and the homotopy perturbation method (HPM) proposed by He in 1998 are compared through an evolution equation used as the second example in a recent paper by Ganji *et al* (2007). It is found that the HPM is a special case of the HAM when $\hbar = -1$. However, the HPM solution is divergent for all x and t except $t = 0$. It is also found that the solution given by the variational iteration method (VIM) is divergent too. On the other hand, using the HAM, one obtains convergent series solutions which agree well with the exact solution. This example illustrates that it is very important to investigate the convergence of approximation series. Otherwise, one might get useless results.

Key words: Homotopy analysis method (HAM), analytical solution, convergence, symbolic computation
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1 Introduction

Many physics and engineering problems can be modelled by differential equations. However, it is difficult to obtain closed-form solutions for them, especially for nonlinear ones. In most cases, only approximate solutions (either analytical ones or numerical ones) can be expected. Perturbation method is one of the well-known methods for solving nonlinear problems analytically. It

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is based on the existence of small/large parameters, the so-called perturbation quantities [1]. However, many nonlinear problems do not contain such kind of perturbation quantities. In general, the perturbation method is valid only for weakly nonlinear problems. For example, considering the following heat transfer problem [2] governed by the nonlinear ordinary differential equation

$$(1+\epsilon\, u)u' + u = 0, \quad u(0) = 1, \tag{1}$$

where $\epsilon > 0$ is a physical parameter, the prime denotes differentiation with respect to the time t. Although the closed-form solution of $u(t)$ is unknown, it is easy to get the exact result $u'(0) = -1/(1+\epsilon)$, as mentioned by Abbasbandy [2]. Regarding ϵ as a perturbation quantity, one can write $u(t)$ into such a perturbation series

$$u(t) = u_0(t) + \epsilon\, u_1(t) + \epsilon^2\, u_2(t) + \epsilon^3\, u_3(t) + \cdots . \tag{2}$$

Substituting the above expression into (1) and equating the coefficients of the like powers of ϵ, one has the following linear differential equations

$$u_0' + u_0 = 0, \quad u_0(0) = 1, \tag{3}$$
$$u_1' + u_1 = -u_0\, u_0', \quad u_1(0) = 0, \tag{4}$$
$$u_2' + u_2 = -\left(u_0\, u_1' + u_1\, u_0'\right), \quad u_2(0) = 0, \tag{5}$$
$$u_3' + u_3 = -\left(u_0\, u_2' + u_1\, u_1' + u_2\, u_0'\right), \quad u_3(0) = 0, \tag{6}$$
$$\vdots$$

Solving the above equations one by one, one has

$$\begin{aligned} u_0(t) &= e^{-t}, \\ u_1(t) &= e^{-t} - e^{-2t}, \\ u_2(t) &= \frac{1}{2}e^{-t} - 2e^{-2t} + \frac{3}{2}e^{-3t}, \\ &\vdots \end{aligned} \tag{7}$$

Thus, we obtain the perturbation approximation

$$u(t) = e^{-t} + \epsilon\left(e^{-t} - e^{-2t}\right) + \epsilon^2\left(\frac{1}{2}e^{-t} - 2e^{-2t} + \frac{3}{2}e^{-3t}\right) + \cdots , \tag{8}$$

which gives at $t = 0$ the derivative

$$u'(0) = -1 + \epsilon - \epsilon^2 + \epsilon^3 - \epsilon^4 + \epsilon^5 - \epsilon^6 + \epsilon^7 - \epsilon^8 + \epsilon^9 - \epsilon^{10} + \cdots . \tag{9}$$

Obviously, the above series is divergent for $\epsilon \geq 1$, as shown in Fig. 1. This typical example illustrates that perturbation approximations are valid only for weakly nonlinear problems in general. In view of the work by Abbasbandy [2], we see that the HAM allows us to extend a series approximation beyond its initial radius of convergence.

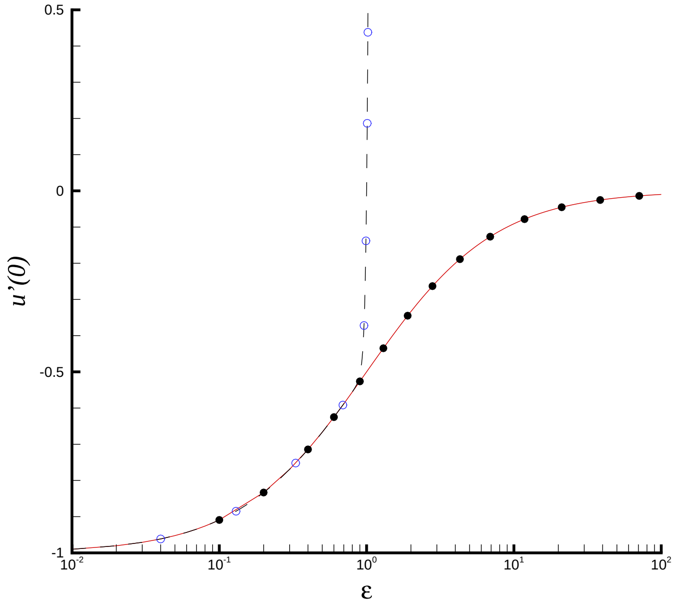

Fig. 1. Comparison of the exact and approximate solutions of (1). Solid line: exact solution $u'(0) = -(1+\epsilon)^{-1}$; Dashed-line: 31th-order perturbation approximation; Hollow symbols: 15th-order approximation given by the HPM; Filled symbols: 15th-order approximation given by the HAM when $\hbar = -1/(1+2\epsilon)$

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To overcome the restrictions of perturbation techniques, some non-perturbation techniques are proposed, such as the Lyapunov's artificial small parameter method [3], the δ-expansion method [4], the Adomian's decomposition method [5], the homotopy perturbation method [6], and the variational iteration method (VIM) [7]. Using these non-perturbation methods, one can indeed obtain approximations even if there are no small/large physical parameters. However, the convergence of solution series is not guaranteed. For example, by means of the HPM, one obtains exactly the *same* approximation of (1) as the perturbation result (9) that is divergent for $\epsilon > 1$, as shown in Fig. 1. For details, please refer to Abbasbandy [2]. This example shows the importance of the convergence of solution series for all possible physical parameters. From physical points of view, the convergence of solution series is much more important than whether or not the used analytic method itself is independent of small/large physical parameters. If one does not keep this in mind, some useless results might be obtained. For example, let us consider the following linear differential equation [8]:

$$u_t + u_x = 2u_{xxt}, \qquad x \in \mathbb{R}, t > 0, \tag{10}$$

$$u(x,0) = e^{-x}. \tag{11}$$

Its exact solution reads

$$u_{exact}(x,t) = e^{-x-t}. \tag{12}$$

By means of the homotopy perturbation method, Ganji *et al* [8] rewrote the original equation in the following form

$$(1-p)\,\frac{\partial \phi(x,t;p)}{\partial t} + p\,\left[\frac{\partial \phi\,(x,t;p)}{\partial t} + \frac{\partial \phi\,(x,t;p)}{\partial x} - 2\,\frac{\partial^3 \phi\,(x,t;p)}{\partial x^2 \partial t}\right] = 0, \tag{13}$$

subject to the initial condition

$$\phi(x,0;p) = e^{-x}, \tag{14}$$

where $p \in [0,1]$ is an embedding parameter. Then, regarding p as a small parameter, Ganji *et al* [8] expanded $\phi(x,t;p)$ in a power series

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\, p^m, \tag{15}$$

which gives the solution by setting $p = 1$. Substituting (15) into the original equation and initial condition, then equating the coefficients of the like powers of p, one can get governing equations and the initial conditions for $u_m(x,t)$. In this way, Ganji *et al* [8] obtained the mth-order approximation

$$u(x,t) \approx u_0(x,t) + \sum_{k=1}^{m} u_k(x,t), \tag{16}$$

and the 5th-order approximation reads

$$u_{HPM}(x,t) \approx \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720). \tag{17}$$

However, for *any* given $x \geq 0$, the above approximation enlarges monotonously to the positive infinity as the time t increases, as shown in Fig. 2. Unfortunately, the exact solution monotonously decreases to zero! Let

$$\delta(t) = \left|\frac{u_{exact} - u_{HPM}}{u_{exact}}\right| \tag{18}$$

denote the relative error of the HPM approximation (17). As shown in Fig. 2, the relative error $\delta(t)$ monotonously increases very quickly:

$$\delta(0) = 0, \quad \delta(0.1) = 7.8, \quad \delta(1) = 404.4, \quad \delta(10) = 1.25 \times 10^9.$$

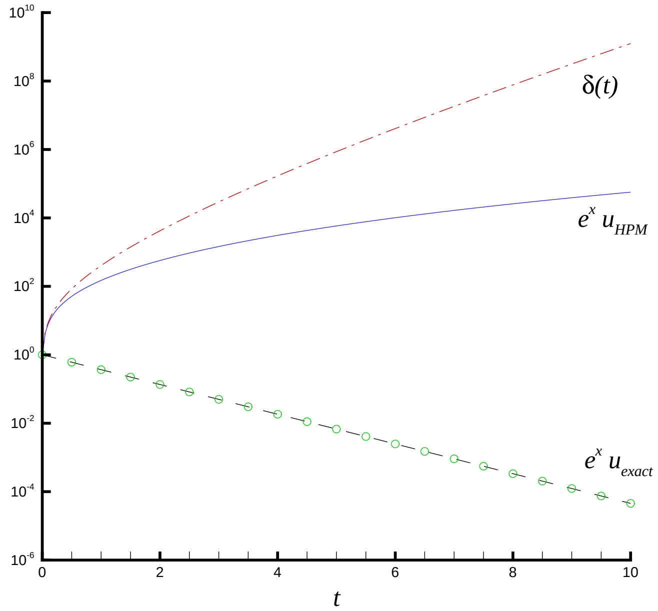

Fig. 2. Approximations of (10) given by the homotopy perturbation method. Dashed-line: exact solution (12); Solid line: the 5th-order HPM approximation (17); Dash-dotted line: the relative error $\delta(t)$ defined by (18); Hollow symbols: the 40th-order HAM approximation (42) when $\hbar = 1$.

In fact, it is easy to find that the HPM series solution (16) is divergent for *all* x and t except $t = 0$ which however corresponds to the *given* initial condition $u(x, 0) = e^{-x}$. In other words, the convergence radius of the HPM solution series (17) is zero. It should be emphasized that, using the variational iteration method (VIM) [7], Ganji *et al* [8] obtained exactly the *same* result as (17) by the 6th iteration. This example illustrates that both of the HPM and the VIM might give divergent approximations. Thus, it is very important to ensure the convergence of solution series obtained.

Note that, an analytic method for strongly nonlinear problems, namely the homotopy analysis method (HAM) [9–13], was proposed by Liao in 1992, six years earlier than the homotopy perturbation method [6] and the variational iteration method [7]. Different from perturbation techniques, the HAM is valid no matter if a nonlinear problem contains small/large physical parameters. More importantly, unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence radius of solution series. Thus, one can always get accurate approximations by means of the HAM. In the next section, we will use (10) and (11) as an example to show this point.

2 HAM solution versus HPM solution

In order to solve (10) and (11) by means of the HAM, we first construct the zeroth-order deformation equation

$$(1-p)\,\frac{\partial \phi(x,t;p)}{\partial t} = p\,\hbar\left[\frac{\partial \phi\,(x,t;p)}{\partial t} + \frac{\partial \phi\,(x,t;p)}{\partial x} - 2\,\frac{\partial^3 \phi\,(x,t;p)}{\partial x^2 \partial t}\right], \quad (19)$$

subject to the initial condition

$$\phi(x,0;p) = e^{-x}, \quad (20)$$

where $p \in [0,1]$ is an embedding parameter and $\hbar \neq 0$ is the so-called convergence-control parameter. Note that (13) is a special case of (19) when $\hbar = -1$. Obviously, when $p = 0$, we can take $\phi(x,t;0) = e^{-x}$, and when $p = 1$, the above equations are equivalent to (10) and (11) respectively, thus it holds

$$\phi(x,t;1) = u(x,t). \quad (21)$$

Expanding $\phi(x,t;p)$ in Taylor series with respect to the embedding parameter p, we have

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\; p^m, \quad (22)$$

where $u_0(x,t) = \phi(x,t;0)$ and $u_m(x,t)(m = 1,2,\ldots)$ will be determined later. Note that the above series contains the convergence-control parameter $\hbar$. Assuming that $\hbar$ is chosen so properly that the above series is convergent at $p = 1$, we have, by means of (21), the solution series

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t). \quad (23)$$

Substituting (22) into the zeroth-order deformation equations (19) and (20), and equating the coefficients of the like powers of p, we have the mth-order deformation equation

$$\frac{\partial}{\partial t}\,(u_m - \chi_m u_{m-1}) = \hbar R_m(u_{m-1}), \quad m \geq 1, \quad (24)$$

subject to the initial condition

$$u_m(x,0) = 0, \quad (25)$$

where

$$R_m(u_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \frac{\partial u_{m-1}}{\partial x} - 2\frac{\partial^3 u_{m-1}}{\partial x^2 \partial t}, \quad (26)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The solution of the mth-order deformation equation (24) for $m \geq 1$ reads

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t R_m(u_{m-1}(x,t))\, d\tau + c_1, \tag{27}$$

where the constant of integration c_1 is determined by the initial condition (25).

Using symbolic computation systems such as Maple or Mathematica, we recursively obtain

$$u_0\,(x,t) = \mathrm{e}^{-x}, \tag{28}$$

$$u_1\,(x,t) = -\hbar \mathrm{e}^{-x} t, \tag{29}$$

$$u_2\,(x,t) = \frac{\hbar \mathrm{e}^{-x} t}{2}\,(\hbar t + 2\,\hbar - 2)\,, \tag{30}$$

$$u_3\,(x,t) = -\frac{\hbar \mathrm{e}^{-x} t}{6}\left(\hbar^2 t^2 - 6\,\hbar t + 6\,\hbar^2 t + 6\,\hbar^2 - 12\,\hbar + 6\right), \tag{31}$$

$$\begin{aligned} u_4\,(x,t) = {} & \frac{\hbar \mathrm{e}^{-x} t}{24}\left(\hbar^3 t^3 - 12\,\hbar^2 t^2 + 12\,\hbar^3 t^2 + 36\,\hbar t \right. \\ & \left. -72\,\hbar^2 t + 36\,\hbar^3 t - 24 + 72\,\hbar - 72\,\hbar^2 + 24\,\hbar^3\right), \end{aligned} \tag{32}$$

$$\begin{aligned} u_5\,(x,t) = {} & -\frac{\hbar \mathrm{e}^{-x} t}{120}\left(\hbar^4 t^4 - 20\,\hbar^3 t^3 + 20\,\hbar^4 t^3 + 120\,\hbar^2 t^2 - 240\,\hbar^3 t^2 \right. \\ & +120\,\hbar^4 t^2 + 720\,\hbar^2 t - 240\,\hbar t + 240\,\hbar^4 t - 720\,\hbar^3 t - 480\,\hbar^3 \\ & \left. -480\,\hbar + 120 + 720\,\hbar^2 + 120\,\hbar^4\right), \end{aligned} \tag{33}$$

$$\begin{aligned} u_6\,(x,t) = {} & \frac{\hbar \mathrm{e}^{-x} t}{720}\left(\hbar^5 t^5 - 30\,\hbar^4 t^4 + 30\,\hbar^5 t^4 + 300\,\hbar^5 t^3 + 300\,\hbar^3 t^3 \right. \\ & -600\,\hbar^4 t^3 - 3600\,\hbar^4 t^2 + 1200\,\hbar^5 t^2 - 1200\,\hbar^2 t^2 + 3600\,\hbar^3 t^2 \\ & -7200\,\hbar^2 t + 1800\,\hbar t - 7200\,\hbar^4 t + 10800\,\hbar^3 t + 1800\,\hbar^5 t \\ & \left. +7200\,\hbar^3 + 3600\,\hbar - 720 - 7200\,\hbar^2 + 720\,\hbar^5 - 3600\,\hbar^4\right). \end{aligned} \tag{34}$$

When $\hbar = -1$, it is easily seen that the equations (30) up to (34) above are exactly the equations (3.17b) up to (3.17f) in [8], and the combination of the equations (28) and (29) is exactly the equation (3.17a) in [8] (Ganji *et al* made mistakes in the first two lines of (3.16) in [8], which makes the difference). Furthermore, when $\hbar = -1$, the 6th-order approximation

$$u(x,t) \approx \frac{e^{-x}}{720}(t^6 + 66t^5 + 1470t^4 + 13320t^3 + 46440t^2 + 45360t + 720) \tag{35}$$

is exactly the *same* as the HPM solution (17). Therefore, the HPM solution is indeed a special case of the HAM solution when $\hbar = -1$. This fact has been pointed out by many researchers, such as Abbasbandy [2], Liao *et al*

[12], Bataineh *et al* [14], Van Gorder *et al* [15], Hayat and Sajid [16][17], and Song *et al* [18].

Unfortunately, $\hbar = -1$ is *not* a proper value for the current problem, because the HPM solution (17) is far away from the exact solution $u = \exp(-x-t)$, as shown in Figs. 2 and 3. To find a proper value of $\hbar$, the curve of $u(0,1) \sim \hbar$ given by the 30th-order HAM approximation is drawn in Fig. 4, which clearly indicates that the valid region of $\hbar$ is about $0.2 \leq \hbar \leq 1.4$. So, $\hbar = -1$ is not a valid value to ensure the convergence of solution series, which explains why the HPM solution (17) is divergent, as shown in Fig. 3. Thus, as a special case of the HAM when $\hbar = -1$, the HPM can not ensure the convergence of solution series and might give useless results. This is the reason why Liao [11] introduced the convergence-control parameter $\hbar$ to improve the early version of the HAM [9]. Therefore, unlike the homotopy perturbation method and the variational iteration method, the HAM provides a convenient way to ensure the convergence of solution series.

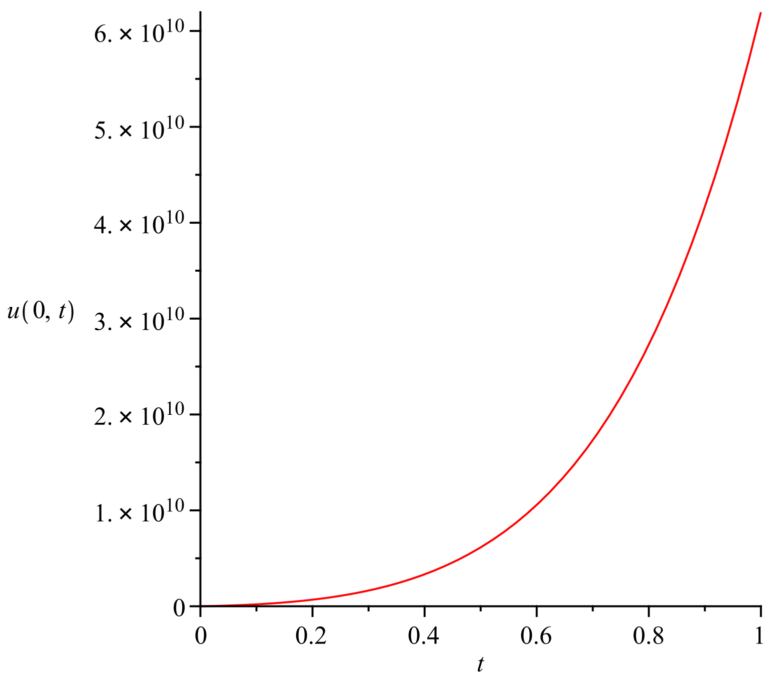

Fig. 3. The 30th-order HAM approximation of $u(0,t)$ when $\hbar = -1$, which gives exactly the HPM solution.

Note that the series solutions given by $\hbar = 1/2$ and $\hbar = 3/4$ converge to the *same* exact solution $u_{exact} = \exp(-x-t)$, as shown in Fig. 5. Especially, when $\hbar = 1$, we have the first few approximations of $u(x,t)$ as follows:

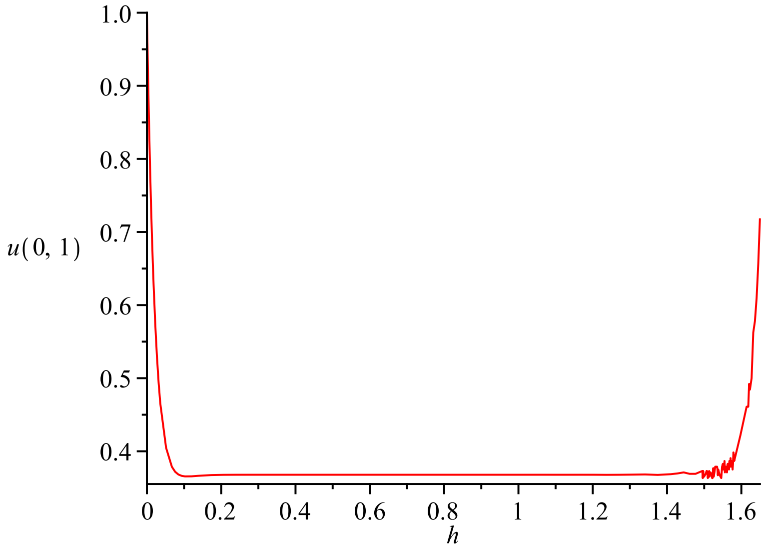

Fig. 4. The $\hbar$-curve for 30th-order HAM approximation of $u(0,1)$.

$$\sum_{i=0}^{1} u_i(x,t) = e^{-x}\,(1-t)\,, \tag{36}$$

$$\sum_{i=0}^{2} u_i(x,t) = e^{-x}\left(1 - t + \frac{t^2}{2!}\right), \tag{37}$$

$$\sum_{i=0}^{3} u_i(x,t) = e^{-x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\right), \tag{38}$$

$$\sum_{i=0}^{4} u_i(x,t) = e^{-x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!}\right), \tag{39}$$

$$\sum_{i=0}^{5} u_i(x,t) = e^{-x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}\right), \tag{40}$$

$$\sum_{i=0}^{6} u_i(x,t) = e^{-x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!}\right), \tag{41}$$

$$\vdots$$

It is easily seen that the mth-order HAM approximation (when $\hbar = 1$) reads

$$u(x,t) \approx e^{-x} \sum_{k=0}^{m} \frac{(-t)^k}{k!}, \tag{42}$$

which obviously converges to the exact solution e^{-x-t} as $m \to \infty$. This clearly illustrates that the convergence-control parameter $\hbar$ indeed provides us with a convenient way to ensure the convergence of solution series.

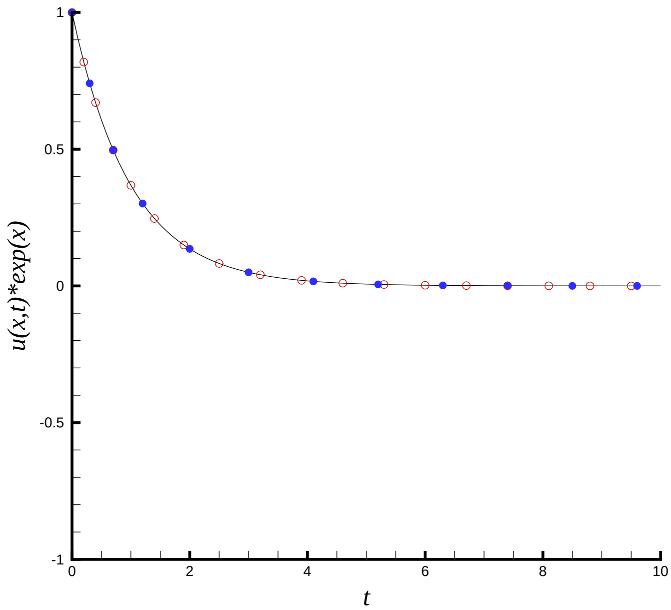

Fig. 5. Comparison of the HAM approximation with the exact solution. Solid line: the exact solution; Hollow symbols: the HAM result when $\hbar = 1/2$; Filled symbols: the HAM result when $\hbar = 3/4$.

3 Conclusion

In this paper, we compare the homotopy analysis method (HAM) proposed by Liao [9] in 1992 and the homotopy perturbation method (HPM) proposed by He [6] in 1998 through a linear partial differential equation. It is found that the HPM is indeed a special case of the HAM when $\hbar = -1$. However, the HPM result is divergent for *all* x and t except $t = 0$ (which is given as the initial condition), i.e. the convergence radius of the HPM solution is zero. Note that, using the variational iteration method (VIM), one obtains exactly the *same* result as the HPM solution (17). Therefore, the HPM (as well as the VIM) solution does not provide a useful approximation, either in the sense of convergent series, or in the sense of asymptotic series. This example illustrates that it is very important to obtain knowledge of the accuracy of any approximation.

It is true that, like other non-perturbation techniques such as Lyapunov's artificial small parameter method [3] and Adomian's decomposition method [5], the HPM can give approximations even if a problem does not contain any small/large physical parameters. However, the example above indicates that this is *not* the key point for solving nonlinear problems: using the HPM, one might get divergent results even for a *linear* problem.

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