Approximate solutions to a parameterized sixth order boundary value problem

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Abstract

In this paper, the homotopy analysis method (HAM) is applied to solve a parameterized sixth order boundary value problem which, for large parameter values, cannot be solved by other analytical methods for finding approximate series solutions. Convergent series solutions are obtained, no matter how large the value of the parameter is.

Key words: Homotopy analysis method, boundary value problem, analytical solution, symbolic computation

1 Introduction

Boundary value problems arise in engineering, applied mathematics and several branches of physics, and have attracted much attention. However, it is difficult to obtain closed-form solutions for boundary value problems, especially for nonlinear problems. In most cases, only approximate solutions (either numerical solutions or analytical solutions) can be expected. Some numerical methods such as finite difference method [1], finite element method [2] and shooting method [3] have been developed for obtaining approximate solutions to boundary value problems.

Perturbation method is one of the well-known methods for solving nonlinear problems analytically. However, it strongly depends on the existence of

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small/large parameters. Traditional non-perturbation methods such as Adomian's decomposition method [5], differential transformation method [6,7] and homotopy perturbation method [8] have been developed for solving boundary value problems. However, these methods have their obvious disadvantages.

Consider the following special sixth order boundary value problem involving a parameter c [8]:

$$u^{(6)}(x) = (1+c)\,u^{(4)}(x) - c\,u''(x) + c\,x, \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 0, \\ &u(1) = \frac{7}{6} + \sinh(1),\; u'(1) = \frac{1}{2} + \cosh(1),\; u''(1) = 1 + \sinh(1). \end{aligned} \tag{2}$$

The boundary value problem (1,2) is interesting because its exact solution

$$u_{exact}(x) = 1 + \frac{1}{6}x^3 + \sinh(x) \tag{3}$$

does not depend on the parameter c although itself does. This can be explained if we rewrite (1) in the following equivalent form

$$\{u^{(6)}(x) - u^{(4)}(x)\} - c\{u^{(4)}(x) - u''(x) + x\} = 0. \tag{4}$$

From (4), we see that a solution of the fourth order problem is also a solution of the sixth order problem, no matter what value of c is.

However, Noor and Mohyud-Din [8] found that the approximate solutions given by the Adomian's decomposition method and the homotopy perturbation method, both of which give the same results, are valid only for small values of c, while the approximate solution given by the differential transformation method is valid for a wide range of values of c.

At this point, one concludes that, for very large values of c, all these analytical methods are no longer valid. In other words, only divergent series solutions can be obtained by these methods. The main reason is that they cannot provide a mechanism to adjust and control the convergence region and rate of the series solutions obtained, according to the values of c.

The homotopy analysis method (HAM) [9–11] is a powerful analytical tool for solving nonlinear as well as linear problems. It has been successfully applied to solve many types of problems [12–18]. In this paper, we apply the HAM to solve the problem (1,2), and obtain convergent series solutions which agree very well with the exact solution (3), no matter what value of c is. The success lies in the fact that the HAM provides a convenient way to adjust and control the convergence region and rate of the series solutions obtained.

2 Solutions of the problem

We first construct a zeroth order deformation equation

$$(1-p)\mathcal{L}[\phi(x;p)-u_0(x)] = p\,\hbar\,\mathcal{N}[\phi(x;p)], \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a convergence-control parameter, and $\phi(x;p)$ is an unknown function, respectively. According to (1), the auxiliary linear operator is given by

$$\mathcal{L}[\phi(x;p)] = \frac{\partial^6 \phi(x;p)}{\partial x^6}, \tag{6}$$

and the nonlinear operator is given by

$$\mathcal{N}[\phi(x;p)] = \frac{\partial^6 \phi\,(x;p)}{\partial x^6} - (1+c)\,\frac{\partial^4 \phi\,(x;p)}{\partial x^4} + c\frac{\partial^2 \phi\,(x;p)}{\partial x^2} - cx. \tag{7}$$

Now suppose the initial guess of the solution is of the form

$$u_0(x) = x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \tag{8}$$

Then using the boundary conditions (2) gives a system of six linear equations in six parameters $a_0, a_1, \ldots, a_5$. Solving the resulting system gives the initial guess

$$\begin{aligned} u_0(x) = x^6 &- \frac{(19+24\,\mathrm{e}-7\,\mathrm{e}^2)\,x^5}{4\,\mathrm{e}} + \frac{(23+22\,\mathrm{e}-9\,\mathrm{e}^2)\,x^4}{2\,\mathrm{e}} \\ &- \frac{(87+82\,\mathrm{e}-39\,\mathrm{e}^2)\,x^3}{12\,\mathrm{e}} + x + 1. \end{aligned} \tag{9}$$

The boundary conditions to (5) can be set as

$$\begin{aligned} &\phi(0;p) = 1,\ \frac{\partial \phi(0;p)}{\partial x} = 1,\ \frac{\partial^2 \phi(0;p)}{\partial x^2} = 0,\ \phi(1;p) = \frac{7}{6} + \sinh(1), \\ &\frac{\partial \phi(1;p)}{\partial x} = \frac{1}{2} + \cosh(1),\ \frac{\partial^2 \phi(1;p)}{\partial x^2} = 1 + \sinh(1). \end{aligned} \tag{10}$$

We now focus on how to obtain higher order approximations to the problem (1,2). From (5), when $p = 0$ and $p = 1$,

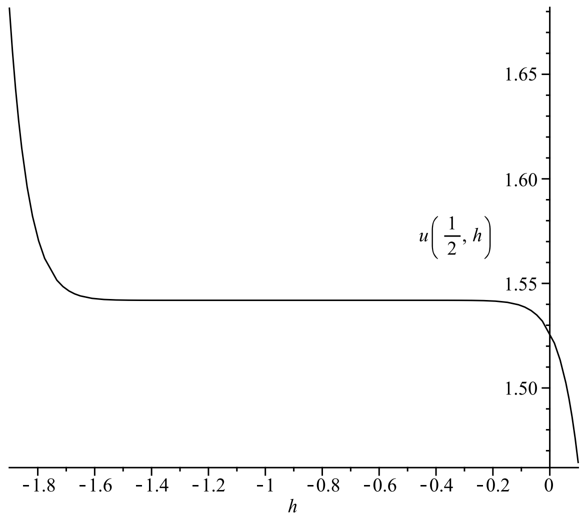

Fig. 1. $\hbar$-curve for the 15th order approximation ($c = 10$).

$$\phi(x;0) = u_0(x) \quad \text{and} \quad \phi(x;1) = u(x) \tag{11}$$

both hold. Therefore, as p increases from 0 to 1, the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, one has

$$\phi(x;p) = \phi(x;0) + \sum_{m=1}^{+\infty} u_m(x)\, p^m, \tag{12}$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;p)}{\partial p^m}\bigg|_{p=0}. \tag{13}$$

Now the convergence of the series (12) depends on the parameter $\hbar$. Assuming that $\hbar$ is chosen so properly that the series (12) is convergent at $p = 1$, we have, by means of (11), the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) \tag{14}$$

which must be one of the solutions of the original problem (1,2), as proved by Liao in [9].

The next goal is to obtain the higher order terms $u_m(x)$. Differentiating the zeroth order deformation equation (5) and its boundary conditions (10) m times with respect to p, then setting $p = 0$, finally dividing them by $m!$, we obtain the mth order deformation equation and its boundary conditions:

$$u_m^{(6)}(x) = \chi_m u_{m-1}^{(6)}(x) + \hbar R_m(\vec{u}_{m-1}(x)), \tag{15}$$

Table 1
Relative errors of HAM approximations ($c = 10$).

x	5th order	10th order	15th order
0.1	6.9E-11	3.0E-16	5.4E-22
0.2	2.6E-10	6.1E-16	8.0E-22
0.3	1.1E-9	8.7E-16	9.0E-22
0.4	1.7E-9	1.1E-15	9.2E-22
0.5	1.9E-9	1.1E-15	8.7E-22
0.6	1.5E-9	9.2E-16	7.8E-22
0.7	7.6E-10	6.3E-16	6.5E-22
0.8	1.6E-10	3.7E-16	4.9E-22
0.9	3.5E-11	1.5E-16	2.7E-22

$$u_m(0) = u'_m(0) = u''_m(0) = u_m(1) = u'_m(1) = u''_m(1) = 0, \tag{16}$$

where

$$R_m(\vec{u}_{m-1}(x)) = u^{(6)}_{m-1}(x) - (1+c)u^{(4)}_{m-1}(x) + cu''_{m-1}(x) - c(1-\chi_m)x \tag{17}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{18}$$

In this way, one can calculate $u_m(x) (m = 1, 2, \ldots)$ recursively.

For example, when $m = 1$, (15) becomes

$$\begin{aligned} u_1^{(6)}(x) &= \hbar\left(u_0^{(6)}(x) - (1+c)u_0^{(4)}(x) + cu_0''(x) - cx\right) \\ &= 30\,\hbar c x^4 - \frac{\hbar}{\mathrm{e}}\left(95\,c + 120\,c\mathrm{e} - 35\,c\mathrm{e}^2\right)x^3 \\ &+ \frac{\hbar}{\mathrm{e}}\left(138\,c - 54\,c\mathrm{e}^2 - 228\,c\mathrm{e} - 360\mathrm{e}\right)x^2 \\ &+ \frac{\hbar}{2\mathrm{e}}\left(1053\,c + 1356\,c\mathrm{e} - 381\,c\mathrm{e}^2 + 1440\mathrm{e} - 420\,\mathrm{e}^2 + 1140\right)x \\ &- \frac{12\,\hbar}{\mathrm{e}}\left(23 + 22\,c\mathrm{e} + 23\,c - 9\,c\mathrm{e}^2 - 38\,\mathrm{e} - 9\,\mathrm{e}^2\right). \end{aligned} \tag{19}$$

Integration of (19) with (16) gives the first order term

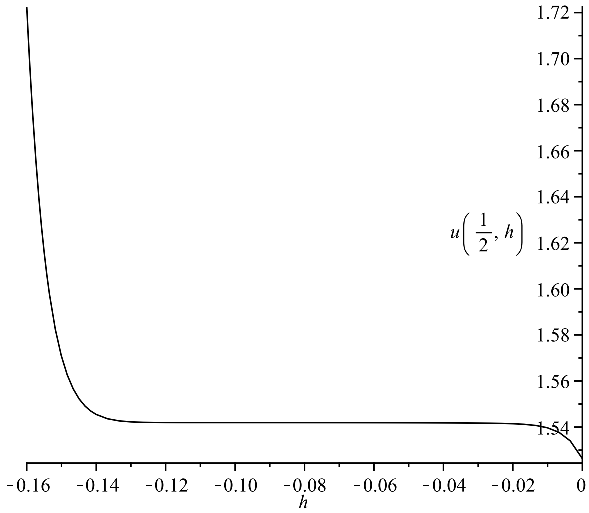

Fig. 2. $\hbar$-curve for the 15th order approximation ($c = 1000$).

$$
\begin{aligned}
u_1(x) &= \frac{\hbar c}{5040}\, x^{10} + \frac{\hbar c}{12096\,\mathrm{e}}\left(7\,\mathrm{e}^2 - 24\,\mathrm{e} - 19\right) x^9 \\
&- \frac{\hbar}{3360\,\mathrm{e}}\left(60\,\mathrm{e} + 9\,c\,\mathrm{e}^2 + 38\,c\,\mathrm{e} - 23\,c\right) x^8 \\
&+ \frac{\hbar}{3360\,\mathrm{e}}\left(351\,c - 127\,c\,\mathrm{e}^2 - 140\,\mathrm{e}^2 + 480\,\mathrm{e} + 452\,c\,\mathrm{e} + 380\right) x^7 \\
&+ \frac{\hbar}{60\,\mathrm{e}}\left(38\,\mathrm{e} - 22\,c\,\mathrm{e} - 23\,c + 9\,c\,\mathrm{e}^2 + 9\,\mathrm{e}^2 - 23\right) x^6 \\
&- \frac{\hbar}{20160\,\mathrm{e}}\left(4135\,c\,\mathrm{e}^2 - 9643\,c - 8700\,c\,\mathrm{e} + 4032\,\mathrm{e}^2 + 51984\,\mathrm{e} - 9504\right) x^5 \\
&+ \frac{\hbar}{10080\,\mathrm{e}}\left(1176\,\mathrm{e}^2 - 2513\,c - 2472 + 27972\,\mathrm{e} - 2360\,c\,\mathrm{e} + 1223\,c\,\mathrm{e}^2\right) x^4 \\
&+ \frac{\hbar}{15120\,\mathrm{e}}\left(666 - 378\,\mathrm{e}^2 - 14436\,\mathrm{e} + 723\,c\,\mathrm{e} + 674\,c - 398\,c\,\mathrm{e}^2\right) x^3. \qquad (20)
\end{aligned}
$$

$u_m(x) (m = 2, 3, \ldots)$ can be calculated similarly.

The mth order approximation can be generally expressed as

$$
u(x, \hbar) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=0}^{4m+6} \gamma_{m,k}(\hbar)\, x^k, \qquad (21)
$$

where the coefficients $\gamma_{m,k}(\hbar) (k = 0, 1, 2, \ldots, 4m+6)$ depend on m, k and $\hbar$. Equation (21) is a family of approximate solutions to the problem (1,2) in terms of the parameter $\hbar$.

The final step is to find a proper value of $\hbar$ which corresponds to an accurate approximation (21). First, the valid region of $\hbar$ can be obtained via the $\hbar$-*curve* as follows.

Table 2
Relative errors of HAM approximations ($c = 1000$).

x	5th order	10th order	15th order
0.1	9.1E-6	9.7E-6	1.9E-6
0.2	1.6E-4	2.9E-5	1.7E-6
0.3	4.4E-4	5.5E-5	3.1E-7
0.4	6.8E-4	7.6E-5	1.2E-6
0.5	7.3E-4	8.0E-5	1.7E-6
0.6	5.8E-4	6.5E-5	1.0E-6
0.7	3.2E-4	4.0E-5	2.2E-7
0.8	9.8E-5	1.8E-5	1.1E-6
0.9	4.7E-6	5.0E-6	9.8E-7

Let $\xi \in [0, 1]$. Then $u(\xi, \hbar)$ is a function of $\hbar$, and the curve $u(\xi, \hbar)$ versus $\hbar$ contains a horizontal line segment which corresponds to the valid region of $\hbar$. The reason is that all convergent series given by different values of $\hbar$ converge to its exact value. So, if the solution is unique, then all of these series converge to the same value and therefore there exists a horizontal line segment in the curve. We call such kind of curve the *$\hbar$-curve*; see Figure 1 for example, where the valid region of $\hbar$ is about $-1.6 < \hbar < -0.2$.

Although the solution series given by different values in the valid region of $\hbar$ converge to the same exact solution, the convergence rates of these solution series are usually different. A more accurate solution series can be obtained by assigning $\hbar$ a proper value which usually can be obtained by observation.

Now we are in a position to show how the parameter c in the problem (1,2) affects the approximate solution (21), and how one can always get a convergent series solution to the problem (1,2) no matter what value of c is, by choosing a proper value of $\hbar$. In the following, we will discuss four cases: (I) small values of c, (II) large values of c, (III) very large values of c, and (IV) any values of c.

(I) *Small values of c.* In this case, we take $c = 10$ as an example. To find the valid region of $\hbar$, the $\hbar-$curve given by the 15th order approximation (21) at $x = \frac{1}{2}$ is drawn in Figure 1, which clearly indicates that the valid region of $\hbar$ is about $-1.6 < \hbar < -0.2$.

When $\hbar = -0.92$, we obtain an approximate series solution which is in excellent agreement with the exact solution (3) as shown in Table 1, where the relative errors of the 5th order, 10th order and 15th order HAM approximations (21) when $c = 10$ at different points in the interval $(0, 1)$ are calculated

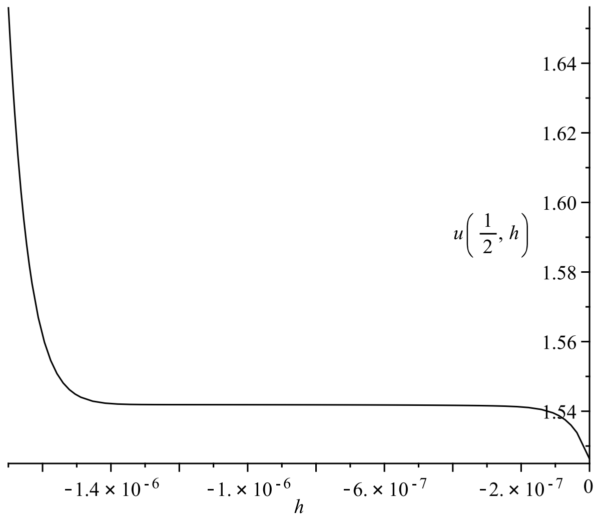

Fig. 3. $\hbar$-curve for the 15th order HAM approximation ($c = 10^8$).

by the formula

$$\delta(x) = \left| \frac{u_{exact}(x) - u(x, \hbar)}{u_{exact}(x)} \right|, \tag{22}$$

where $u_{exact}(x)$ is the exact solution (3), and $u(x, \hbar)$ is the approximate solution (21).

(II) *Large values of c*. In this case, we take $c = 1000$ as an example. As pointed out in [8], the Adomian's decomposition method is no longer valid for this case.

To find the valid region of $\hbar$, the $\hbar-$curve given by the 15th order approximation (21) when $c = 1000$ and $x = \frac{1}{2}$ is drawn in Figure 2, which clearly indicates that the valid region of $\hbar$ is about $-0.13 < \hbar < -0.02$.

When $\hbar = -0.118$, one obtains an approximate series solution which agrees very well with the exact solution (3), as shown in Table 2.

(III) *Very large values of c*. In this case, we take $c = 10^8$ as an example. Not only the Adomian's decomposition method but also the differential transformation method are no longer valid for very large values of c.

From the $\hbar$-curve in Figure 3, it is clear that the valid region of $\hbar$ is about $-1.4 \times 10^{-6} < \hbar < -2.0 \times 10^{-7}$. By choosing $\hbar = -1.3 \times 10^{-6}$, one obtains an approximate series solution which agrees very well with the exact solution (3), as shown in Table 3.

(IV) *Any values of c*. Finally in this case, we develop a relationship between the convergence-control parameter $\hbar$ and the given parameter c based on the rational interpolation technique [19].

First, we find the proper values of $\hbar$ for some given values of c as above. They

Table 3
Relative errors of HAM approximations($c = 10^8$).

x	5th order	10th order	15th order
0.1	4.9E-6	2.9E-5	1.4E-5
0.2	1.7E-4	7.1E-5	1.9E-5
0.3	5.3E-4	1.2E-4	1.9E-5
0.4	8.4E-4	1.5E-4	1.7E-5
0.5	9.1E-4	1.6E-4	1.5E-5
0.6	7.1E-4	1.3E-4	1.4E-5
0.7	3.8E-4	8.5E-5	1.4E-5
0.8	1.1E-4	4.3E-5	1.2E-5
0.9	2.5E-6	1.5E-5	7.3E-6

are

$$[c, \hbar] = \left[1, -\frac{49}{50}\right], \left[10, -\frac{23}{25}\right], \left[100, -\frac{57}{100}\right], \left[10^3, -\frac{59}{500}\right]. \tag{23}$$

Then we use the rational interpolation technique to find a rational function in c that interpolates the given points (23), which gives a relationship between c and $\hbar$:

$$\hbar(c) = -\frac{342960750 + 1115829\, c}{347425000 + 3665200\, c + 8350\, c^2}. \tag{24}$$

Substituting (24) into the mth order approximation (21) gives a solution expression

$$u(x, \hbar(c)) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=0}^{4m+6} \eta_{m,k}(c)\, x^k, \tag{25}$$

which only depends on the parameter c. It turns out that from (25) one can always get a convergent series solution which agrees very well with the exact solution (3), no matter what value of c is.

For over 1000 random values of c in the interval $[1, 10^{30}]$, we have calculated the relative errors of the 15th order approximation (25) at different points in the interval $(0, 1)$ as in the case (I), and found that all these relative errors are less than 5×10^{-5}. Figure 4 shows that the 15th order approximation (25) agrees very well with the exact solution (3) for any random value of c in the interval $[1, 10^{30}]$.

The reasons behind this miracle are as follows. The coefficients of (25) can be expressed as

$$\eta_{m,k}(c) = \frac{s_{2m,k}c^{2m} + s_{2m-1,k}c^{2m-1} + \cdots + s_{1,k}c + s_{0,k}}{t_{2m,k}c^{2m} + t_{2m-1,k}c^{2m-1} + \cdots + t_{1,k}c + t_{0,k}}, \tag{26}$$

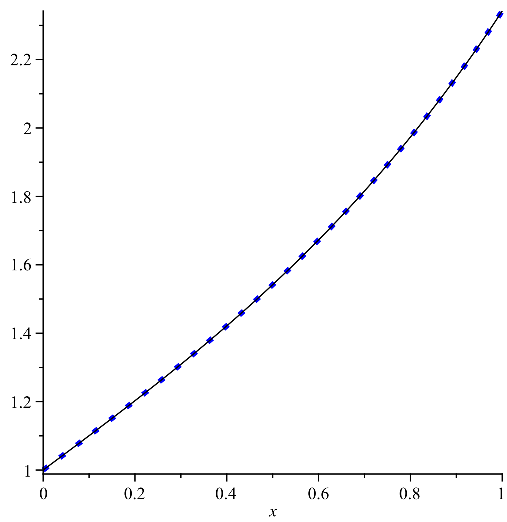

Fig. 4. Symbols: 15th order HAM approximation (21); solid line: exact solution (3).

where $s_{i,j}$ and $t_{i,j}$ are real numbers. Due to the continuity of $\hbar$ on c, equation (24) leads to good approximations for small values of c; while for large values of c, it is seen from (26) that

$$\lim_{c\to\infty} \eta_{m,k}(c) = \frac{s_{2m,k}}{t_{2m,k}} \tag{27}$$

is independent of c. Therefore, equation (25) always give good approximation, no matter what value of c is.

3 Conclusions

In this paper, the homotopy analysis method (HAM) is successfully applied to solve a parameterized sixth order boundary value problem which, for large parameter values, cannot be solved by other analytical methods for finding approximate series solutions. The success mainly lies in the fact that the HAM provides a convergence-control parameter $\hbar$ which can be used to adjust and control the convergence region and rate of the series solution obtained, according to the value of the parameter. Therefore, the HAM is a promising analytical tool for solving nonlinear as well as linear problems.

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