An efficient analytical approach for solving fourth order boundary value problems

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Abstract

Based on the homotopy analysis method (HAM), an efficient approach is proposed for obtaining approximate series solutions to fourth order two-point boundary value problems. We apply the approach to a linear problem which involves a parameter c and cannot be solved by other analytical methods for large values of c, and obtain convergent series solutions which agree very well with the exact solution, no matter how large the value of c is. Consequently, we give an affirmative answer to the open problem proposed by Momani and Noor in 2007 [S. Momani, M.A. Noor, Numerical comparison of methods for solving a special fourth-order boundary value problem, Appl. Math. Comput. 191(2007) 218-224]. We also apply the approach to a nonlinear problem, and obtain convergent series solutions which agree very well with the numerical solution given by the Runge-Kutta-Fehlberg 4-5 technique.

Key words: Homotopy analysis method, boundary value problem, analytical solution, numerical solution, symbolic computation
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1 Introduction

Fourth order boundary value problems have attracted much attention in recent years; see [1–5] for references. Such problems arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [6]. Some numerical methods such as finite difference method [7] and B-spline method [8], and several analytical methods such as differential transformation method [9], Adomian's decomposition method [10], homotopy

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perturbation method [11] and variational iteration method [12] have been developed for solving fourth order boundary value problems. However, these methods, especially the analytical methods, have their obvious disadvantages.

Consider the following fourth order linear boundary value problem involving a parameter c:

$$u^{(4)}(x) = (1+c)u''(x) - cu(x) + \frac{1}{2}cx^2 - 1, \tag{1}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = \frac{3}{2} + \sinh(1), \quad u'(1) = 1 + \cosh(1). \tag{2}$$

This problem was first considered by Scott and Watts via orthonormalization in 1975 [13]. The boundary value problem (1,2) is very interesting because its exact solution

$$u_{exact}(x) = 1 + \frac{1}{2}x^2 + \sinh(x) \tag{3}$$

does not depend on the parameter c although itself does. This phenomenon can be explained if we rewrite (1) in the following equivalent form

$$\{u^{(4)}(x) - u''(x) + 1\} - c\{u''(x) - u(x) + \frac{1}{2}x^2\} = 0. \tag{4}$$

From (4), we see that the solution of the second order problem is also a solution of the fourth order problem, no matter what value of c is.

However, the solutions obtained by the analytical methods mentioned above are all dependent on the parameter c. Noor and Mohyud-Din [1] found that the approximate solution to the problem (1,2) given by the variational iteration method [12] is valid only for small values of c. Golbabai and Javidi [2] discussed the same problem (1,2) via the homotopy perturbation method [11] and found that the approximate solution obtained is valid only for small values of c too.

Momani and Noor [3] compared the homopoty perturbation method, the Adomian's decomposition method [10] and the differential transformation method [9] for solving the boundary value problem (1,2). They found that the approximate solution given by the Adomian's decomposition method is the same as the solution given by the homopoty perturbation method, and thus is in good agreement with the exact solution (3) only for small values of c too, while the approximate solution given by the differential transformation method is valid for a wide range of values ($c < 10^6$).

At this point, one can conclude that, for very large values of c, $c > 10^6$, all these analytical methods are no longer valid. The main reason is that they cannot provide a mechanism to adjust and control the convergence region and rate of the series solutions obtained, according to the value of c. *Does there exist an analytical method that is valid for the problem (1,2) no matter how large the value of c is*? This is an open problem proposed in [3]. In this paper, we give an affirmative answer to this problem.

In Section 2, based on the homotopy analysis method (HAM) [14–18] which was first proposed by Liao in 1992 and has been successfully applied to solve many types of problems [19–27], we propose an efficient analytical approach for solving the following type of fourth order boundary value problems

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \tag{5}$$

subject to the two-point boundary conditions

$$u(a) = \alpha_1, \quad u'(a) = \alpha_2, \quad u(b) = \beta_1, \quad u'(b) = \beta_2, \tag{6}$$

where f is a polynomial in $x, u(x), u'(x), u''(x)$ and $u'''(x)$, while a, b, α_1, α_2, β_1 and β_2 are real constants.

In Section 3, we apply the approach to the boundary value problem (1,2), and obtain convergent series solutions which agree very well with the exact solution, no matter how large the value of c is. Therefore, we attain an affirmative answer to the open problem proposed by Momani and Noor in [3]. The success of this approach lies in the fact that the HAM provides us with a convergence-control parameter $\hbar$ which can be used to adjust and control the convergence region and rate of the series solutions obtained according to the value of c.

In Section 4, we apply the same approach to a fourth order nonlinear boundary value problem with a parameter c [4,5] which does not have a closed-form solution, and obtain convergent series solutions which agree very well with the numerical solution obtained by the Runge-Kutta-Fehlberg 4-5 technique. Finally in Section 5, some concluding remarks are given.

2 The HAM-based approach

In order to obtain a convergent series solution to the problem (5,6), we first construct the so-called *zeroth-order deformation equation*

$$(1-p)\mathcal{L}[\phi(x;p) - u_0(x)] = p\hbar\mathcal{N}[\phi(x;p)], \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is the so-called *convergence-control parameter*, and $\phi(x;p)$ is an unknown function, respectively. According to (5), the auxiliary linear operator $\mathcal{L}$ can be chosen as

$$\mathcal{L}[\phi(x;p)] = \frac{\partial^4 \phi(x;p)}{\partial x^4}, \tag{8}$$

and the nonlinear operator $\mathcal{N}$ can be chosen as

$$\mathcal{N}[\phi(x;p)] = \frac{\partial^4 \phi}{\partial x^4} - f(x, \phi, \frac{\partial \phi}{\partial x}, \frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^3 \phi}{\partial x^3}). \tag{9}$$

The initial guess $u_0(x)$ of the solution $u(x)$ can be determined by the *rule of solution expression* as follows.

In view of equation (5), the solution $u(x)$ can be expressed by a set of base functions

$$\{x^n \,|\, n = 0, 1, 2, \ldots\} \tag{10}$$

in the form

$$u(x) = \sum_{n=0}^{+\infty} d_n x^n, \tag{11}$$

where $d_n (n = 0, 1, 2, \ldots)$ are coefficients to be determined later. This provides us with the so-called *rule of solution expression*, i.e., the solution of (5,6) must be expressed in the same form as (11).

According to the rule of solution expression (11), The initial guess $u_0(x)$ can be set as

$$u_0(x) = x^4 + a\, x^3 + b\, x^2 + c\, x + d, \tag{12}$$

where the coefficients a, b, c and d can easily be determined by the given boundary conditions (6). Finally from (6), the boundary conditions to the zeroth-order deformation equation (7) can be set as

$$\phi(a;p) = \alpha_1, \quad \frac{\partial \phi(a;p)}{\partial x} = \alpha_2, \quad \phi(b;p) = \beta_1, \quad \frac{\partial \phi(b;p)}{\partial x} = \beta_2. \tag{13}$$

We now focus on how to obtain higher order approximations to the problem (5,6). From (7), when $p = 0$ and $p = 1$,

$$\phi(x;0) = u_0(x) \quad \text{and} \quad \phi(x;1) = u(x) \tag{14}$$

both hold. Therefore, as p increases from 0 to 1, the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, one has

$$\phi(x;p) = \phi(x;0) + \sum_{m=1}^{+\infty} u_m(x)\, p^m, \tag{15}$$

where

$$u_m(x) = \frac{1}{m!} \left. \frac{\partial^m \phi(x;p)}{\partial p^m} \right|_{p=0}. \tag{16}$$

Now the convergence of the series (15) depends on the parameter $\hbar$. Assuming that $\hbar$ is chosen so properly that the series (15) is convergent at $p = 1$, we have, by means of (14), the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) \tag{17}$$

which must be one of the solutions of the original problem (5,6), as proved by Liao in [16].

Our next goal is to determine the higher order terms $u_m(x)(m \geq 1)$. Define the vector

$$\vec{u}_n(x) = \{u_0(x), u_1(x), \ldots, u_n(x)\}. \tag{18}$$

Differentiating the zeroth-order deformation equation (7) and its boundary conditions (13) m times with respect to p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called *mth-order deformation equation*

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar R_m(\vec{u}_{m-1}(x)), \tag{19}$$

and its boundary conditions

$$u_m(a) = u_m'(a) = u_m(b) = u_m'(b) = 0, \tag{20}$$

where the prime denotes differentiation with respect to x, and

$$R_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\phi(x;p)]}{\partial p^{m-1}} \right|_{p=0}, \tag{21}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Note that the mth-order deformation equation (19) becomes

$$u_m^{(4)}(x) = \chi_m u_{m-1}^{(4)}(x) + \hbar R_m(\vec{u}_{m-1}(x)). \tag{22}$$

According to the rule of solution expression (11), the right hand side of (22) can be expressed as

$$u_m^{(4)}(x) = \sum_{k=0}^{N(m)} d_k \, x^k, \tag{23}$$

where the upper limit $N(m)$ depends on m.

Since the solution of $u_m^{(4)}(x) = x^k$ with the boundary conditions (20) can be easily obtained even manually as

$$u_m(x) = \frac{k!}{(k+4)!} x^{k+4} + \alpha x^3 + \beta x^2 + \gamma x + \delta, \tag{24}$$

where α, β, γ and δ depend on the boundary conditions (20), then by (23), (24) and the linearity of $\mathcal{L}^{-1}$, we finally obtain

$$u_m(x) = \sum_{k=0}^{N(m)} d_k \left(\frac{k!}{(k+4)!} x^{k+4} + \alpha x^3 + \beta x^2 + \gamma x + \delta \right). \tag{25}$$

In this way, we can solve $u_m(x)(m = 1, 2, 3, \ldots)$ recursively.

The mth-order approximation to the problem (5,6) can be generally expressed as

$$u(x, \hbar) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=0}^{\sigma(m)} \gamma_{m,k}(\hbar)\, x^k, \tag{26}$$

where the upper limit $\sigma(m)$ depends on m, and the coefficients $\gamma_{m,k}(\hbar)(k = 0, 1, 2, \ldots, \sigma(m))$ depend on m, k and $\hbar$. Equation (26) is a family of solutions to the problem (5,6) expressed in the convergence-control parameter $\hbar$.

The final step of the approach is to find a proper value of $\hbar$ which corresponds to an accurate approximation (26). First, the valid region of $\hbar$ can be obtained via the *$\hbar$-curve* as follows.

Let $c_0 \in [a, b]$. Then $u(c_0, \hbar)$ is a function of $\hbar$, and the curve $u(c_0, \hbar)$ versus $\hbar$ contains a horizontal line segment which corresponds to the valid region of $\hbar$. The reason is that all convergent series given by different values of $\hbar$ converge to its exact value. So, if the solution is unique, then all of these series converge to the same value and therefore there exists a horizontal line segment in the curve. We call such kind of curve the *$\hbar$-curve*; see Fig. 1 for example, where the valid region of $\hbar$ is about $-1.5 < \hbar < -0.2$.

Although the solution series given by different values in the valid region of $\hbar$ converge to the exact solution, the convergence rates of these solution series are usually different. A more accurate solution series can be obtained by assigning $\hbar$ a proper value.

3 Application to a linear problem

In this section, the approach proposed in Section 2 is applied to solve the fourth order linear boundary value problem (1,2).

For the zeroth-order deformation equation (7), the nonlinear operator is taken as

$$\mathcal{N}[\phi(x;p)] = \frac{\partial^4 \phi\,(x;p)}{\partial x^4} - (1+c)\,\frac{\partial^2 \phi\,(x;p)}{\partial x^2} + c\phi\,(x;p) - \frac{c}{2}\,x^2 + 1. \tag{27}$$

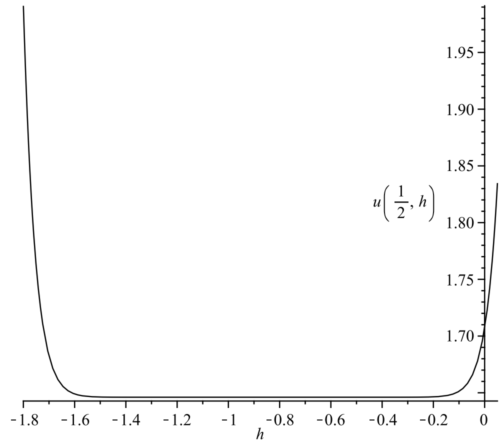

Fig. 1. $\hbar$-curve for the 20th-order HAM approximation ($c = 5$).

In view of the boundary conditions (2), the initial guess is determined as

$$u_0(x) = x^4 - \left(1 + \frac{\mathrm{e}}{2} - \frac{3}{2\mathrm{e}}\right) x^3 - \left(\frac{1}{2} - \mathrm{e} + \frac{2}{\mathrm{e}}\right) x^2 + x + 1, \tag{28}$$

and the boundary conditions to (7) can be set as

$$\phi(0;p) = 1, \frac{\partial \phi(0;p)}{\partial x} = 1, \phi(1;p) = \frac{3}{2} + \sinh(1), \frac{\partial \phi(1;p)}{\partial x} = 1 + \cosh(1). \tag{29}$$

In order to obtain the higher order terms $u_m(x)$, the mth-order deformation equation (19) and its boundary conditions (20) are calculated:

$$u_m^{(4)}(x) = \chi_m u_{m-1}^{(4)}(x) + \hbar R_m(\vec{u}_{m-1}(x)), \tag{30}$$

$$u_m(0) = u_m'(0) = u_m(1) = u_m'(1) = 0, \tag{31}$$

where

$$R_m(\vec{u}_{m-1}(x)) = u_{m-1}^{(4)}(x) - (1+c)u_{m-1}''(x) + c u_{m-1}(x) + (\chi_m - 1)\left(\frac{c}{2}x^2 - 1\right). \tag{32}$$

In this way, we can calculate $u_m(x)(m = 1, 2, \ldots)$ recursively.

For example, when $m = 1$, equation (30) becomes

$$\begin{aligned} u_1^{(4)}(x) &= \hbar\left(u_0^{(4)}(x) - (1+c)u_0''(x) + c u_0(x) - \frac{c}{2}x^2 + 1\right) \\ &= hcx^4 + \left(\frac{3}{2\mathrm{e}} - \frac{\mathrm{e}}{2} - 1\right)\hbar c x^3 + \left(\mathrm{e}c - \frac{2c}{\mathrm{e}} - 13c - 12\right)\hbar x^2 \\ &\quad + \left(3\mathrm{e}c + 7c + 3\mathrm{e} + 6 - \frac{9}{\mathrm{e}} - \frac{9c}{\mathrm{e}}\right)\hbar x \\ &\quad + \left(2c + \frac{4}{\mathrm{e}} + \frac{4c}{\mathrm{e}} - 2\mathrm{e} - 2\mathrm{e}c + 26\right)\hbar. \end{aligned} \tag{33}$$

Table 1
Relative errors of HPM solutions ($c = 5, \hbar = -1$).

x	5th order	10th order	15th order	20th order
0.1	5.9E-7	6.5E-11	7.1E-15	7.9E-19
0.2	1.9E-6	2.1E-10	2.3E-14	2.6E-18
0.3	3.2E-6	3.6E-10	4.0E-14	4.4E-18
0.4	4.1E-6	4.5E-10	4.9E-14	5.4E-18
0.5	4.1E-6	4.5E-10	4.9E-14	5.4E-18
0.6	3.3E-6	3.7E-10	4.0E-14	4.5E-18
0.7	2.2E-6	2.4E-10	2.7E-14	2.9E-18
0.8	1.1E-6	1.2E-10	1.3E-14	1.4E-18
0.9	2.7E-6	2.9E-10	3.2E-15	3.9E-19

Since the formula (24) now becomes

$$u_1(x) = \frac{k!\, x^{k+4}}{(k+4)!} - \frac{x^3}{(k+4)(k^2+4k+3)} + \frac{x^2}{k^3+9k^2+26k+24}, \tag{34}$$

by (25), the first order term

$$\begin{aligned} u_1(x) = {} & \frac{\hbar c x^8}{1680} - \left(\mathrm{e}^2 + 2\mathrm{e} - 3\right)\frac{\hbar c x^7}{1680\,\mathrm{e}} + \left(\mathrm{e}^2 c - 13\,\mathrm{e}c - 2\,c - 12\,\mathrm{e}\right)\frac{\hbar x^6}{360\,\mathrm{e}} \\ & + \left(3\,\mathrm{e}^2 + 3\,\mathrm{e}^2 c + 7\,\mathrm{e}c + 6\,\mathrm{e} - 9\,c - 9\right)\frac{\hbar x^5}{120\,\mathrm{e}} \\ & + \left(\hbar \mathrm{e}c + 2\hbar - \hbar \mathrm{e}^2 c + 13\hbar \mathrm{e} + 2\hbar c - \hbar \mathrm{e}^2\right)\frac{x^4}{12\mathrm{e}} \\ & + \left(421\,\hbar \mathrm{e}^2 c - 11004\,\hbar \mathrm{e} - 982\,\hbar \mathrm{e}c + 462\,\hbar \mathrm{e}^2\right. \\ & \left. -5040\,\mathrm{e} - 2520\,\mathrm{e}^2 + 7560 - 479\,\hbar c - 546\,\hbar\right)\frac{x^3}{5040\,\mathrm{e}} \\ & - \left(56\,\mathrm{e}^2 - 28 - 1820\,\mathrm{e} - 12\,c - 151\,\mathrm{e}c + 46\,\mathrm{e}^2 c\right)\frac{\hbar x^2}{1680\,\mathrm{e}} \end{aligned} \tag{35}$$

$u_m(x) (m = 2, 3, \ldots)$ can be calculated similarly.

The procedure has been implemented in Maple. The mth-order approximation can be generally expressed as

$$u(x, \hbar) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=0}^{4m+4} \gamma_{m,k}(\hbar)\, x^k, \tag{36}$$

where the coefficients $\gamma_{m,k}(\hbar) (k = 0, 1, 2, \ldots, 4m+4)$ depend on m, k and $\hbar$.

Table 2
Relative errors of HAM solutions ($c = 5, \hbar = -0.9$).

x	5th order	10th order	15th order	20th order
0.1	4.4E-9	2.6E-14	1.6E-19	1.1E-24
0.2	2.7E-9	2.0E-14	9.2E-20	6.0E-25
0.3	1.8E-9	1.3E-14	6.8E-20	6.0E-25
0.4	4.5E-9	1.3E-14	7.3E-20	7.0E-25
0.5	4.9E-9	1.3E-14	7.0E-20	8.5E-25
0.6	3.7E-9	1.1E-14	5.9E-20	1.1E-24
0.7	1.2E-9	8.7E-15	4.5E-20	1.6E-24
0.8	1.5E-9	1.1E-14	5.0E-20	2.6E-24
0.9	2.0E-9	1.2E-14	7.0E-20	4.0E-24

Equation (36) is a family of approximate solutions to the problem (1,2) in terms of the convergence-control parameter $\hbar$.

Our next goal is to show how the given parameter c in the problem (1,2) affects the approximate solutions (36), and how we can always get a convergent series solution to the problem (1,2) no matter how large the value of c is, by choosing a proper value of $\hbar$ in our HAM-based approach. In the following, we will discuss three cases: (I) small values of c, (II) large values of c, and (III) very large values of c. As pointed out by many researchers [20,23–27], the homotopy perturbation method (HPM) [11] is a special case of the HAM when $\hbar = -1$, and the Adomian's decomposition method (ADM) gives the same result as that given by the HPM [3]. We will show that the HPM and the ADM are valid only for small values of c. Furthermore, even in the case of small parameter for which the HPM and the ADM are valid, the solutions given by these methods are not always accurate. One can always obtain an accurate solution by choosing a proper value of $\hbar$.

(I) *Small values of c*. In this case, we take $c = 5$ as an example. To find the valid region of $\hbar$, the $\hbar-$curve given by the 20th-order HAM approximation is drawn in Fig. 1, which clearly indicates that the valid region of $\hbar$ is about $-1.5 < \hbar < -0.2$.

From Fig. 1, it is easily seen that -1 is a valid value of $\hbar$. Thus, the HPM and the ADM are valid for this case. However, the solution given by the HPM as well as the ADM is not so accurate. A more accurate solution to the problem (1,2) when $c = 5$ is obtained by choosing $\hbar = -0.9$ instead of $\hbar = -1$, as shown in Tables 1 and 2, where the relative errors of the 5th order, 10th order, 15th order and 20th order HPM and HAM approximate solutions at

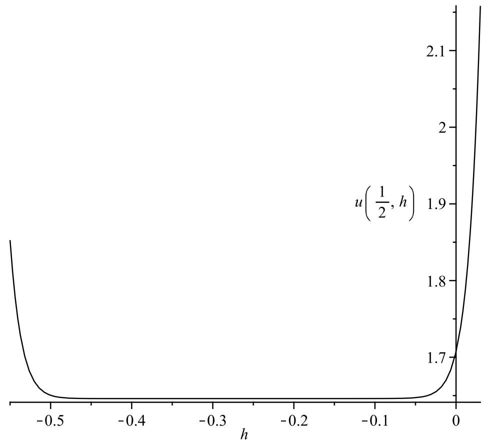

Fig. 2. $\hbar$-curve for the 20th-order HAM approximation ($c = 100$).

different points in the interval $(0, 1)$ are calculated by the formula

$$\delta(x) = \left| \frac{u_{exact}(x) - u_{approximate}(x)}{u_{exact}(x)} \right|. \tag{37}$$

It is shown that the relative errors of the HAM approximate solutions when $\hbar = -0.9$ are much smaller than the relative errors of the HPM (as well as the ADM) approximate solutions. Therefore, even in the case of small parameter for which the HPM and the ADM are valid, the approximate solutions given by these methods are not always accurate. The HAM is absolutely necessary for finding more accurate approximations.

(II) *Large values of c.* In this case, we take $c = 100$ as an example. As pointed out in [1–3], the HPM and the ADM are no longer valid for this case. In fact, it is shown that only divergent series solutions are obtained by the HPM and the ADM. On the other hand, one can obtain convergent series solutions which agree very well with the exact solution (3) by choosing a proper value of $\hbar$.

To find the valid region of $\hbar$, the $\hbar-$curve given by the 20th-order HAM approximation is drawn in Fig. 2, which clearly indicates that the valid region of $\hbar$ is about $-0.48 < \hbar < -0.05$.

Since -1 is not a valid value of $\hbar$, the HPM and the ADM are no longer valid in this case. In fact, the solutions given by the HPM (as well as the ADM) are divergent, as shown in Table 3, where the relative errors $\delta(x)$ increase exponentially to $+\infty$.

However, by using the HAM-based approach with $\hbar = -0.37$, one obtains a convergent series solution which agrees very well with the exact solution (3), as shown in Table 4.

Table 3
Relative errors of HPM solutions ($c = 100, \hbar = -1$).

x	5th order	10th order	15th order	20th order
0.1	0.8	1.3E2	2.1E4	3.3E6
0.2	2.7	4.3E2	6.8E4	1.1E7
0.3	4.6	7.3E2	1.2E5	1.8E7
0.4	5.8	9.1E2	1.4E5	2.3E7
0.5	5.8	9.1E2	1.4E5	2.3E7
0.6	4.8	7.5E2	1.2E5	1.9E7
0.7	3.1	4.9E2	7.8E4	1.2E7
0.8	1.5	2.4E2	3.8E4	5.9E6
0.9	0.3	6.0E1	9.5E4	1.6E6

Table 4
Relative errors of HAM solutions ($c = 100, \hbar = -0.37$).

x	5th order	10th order	15th order	20th order
0.1	6.1E-7	2.6E-6	1.3E-7	8.6E-9
0.2	1.0E-4	2.8E-6	6.1E-8	4.6E-9
0.3	2.3E-4	3.2E-6	3.8E-8	4.4E-9
0.4	3.0E-4	3.9E-6	3.6E-8	4.3E-9
0.5	3.0E-4	3.9E-6	3.2E-8	3.9E-9
0.6	2.5E-4	3.2E-6	2.9E-8	3.5E-9
0.7	1.5E-4	2.2E-6	2.5E-8	2.9E-9
0.8	5.8E-5	1.6E-6	3.4E-8	2.5E-9
0.9	5.2E-8	1.2E-6	5.7E-8	3.8E-9

(III) *Very large values of* c. In this case, we take $c = 10^8$ as an example. Not only the HPM and the ADM but also the differential transformation method are no longer valid for very large values of c, as pointed out in [3]. However, by means of the HAM-based approach in Section 2, one can always obtain a convergent series solution by choosing a proper value of $\hbar$.

From the $\hbar$-curve in Fig. 3, it is clear that the valid region of $\hbar$ is about $-6.5 \times 10^{-7} < \hbar < -1 \times 10^{-7}$. By choosing $\hbar = -5.9 \times 10^{-7}$, one obtains a convergent series solution which agrees very well with the exact solution (3), as shown in Table 5 and Fig. 4.

Finally, it is worth mentioning that the rate of convergence of the HAM solu-

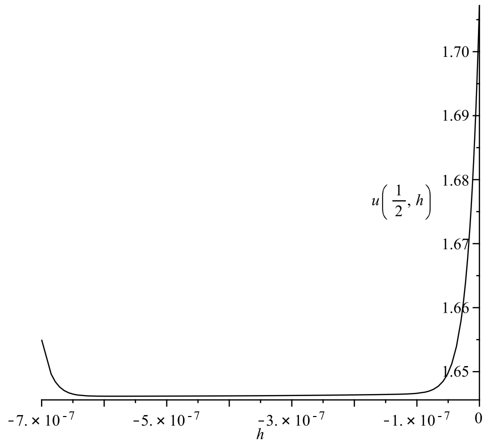

Fig. 3. $\hbar$-curve for the 20th-order HAM approximation ($c = 10^8$).

Table 5
Relative errors of HAM solutions ($c = 10^8$).

x	5th order	10th order	15th order	20th order
0.1	3.6E-5	2.6E-4	1.3E-4	8.9E-5
0.2	9.4E-4	2.8E-4	6.5E-5	4.7E-5
0.3	2.1E-3	3.0E-4	4.2E-5	3.5E-5
0.4	2.8E-3	3.6E-4	4.1E-5	3.4E-5
0.5	2.9E-3	3.6E-4	3.8E-5	3.0E-5
0.6	2.3E-3	2.9E-4	3.4E-5	2.6E-5
0.7	1.4E-3	2.0E-4	2.8E-5	2.0E-5
0.8	5.2E-4	1.5E-4	3.6E-5	2.6E-5
0.9	1.3E-5	1.1E-4	5.9E-5	4.0E-5

tions decreases as the value of the parameter c increases, as demonstrated in Tables 2,4 and 5. To accelerate the convergence of these series solutions, one can use the homotopy-Padé technique [16].

4 Application to a nonlinear problem

In this section, the approach proposed in Section 2 is applied to solve the following fourth order nonlinear boundary value problem with a parameter c:

$$u^{(4)}(x) = c\,u(x)^2 + 1, \quad 0 \leq x \leq 2, \tag{38}$$

$$u(0) = u'(0) = u(2) = u'(2) = 0. \tag{39}$$

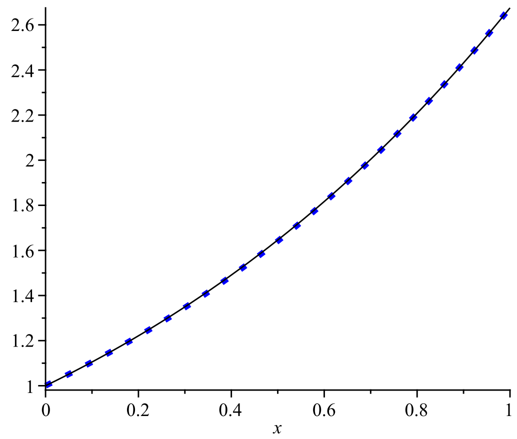

Fig. 4. Symbols: 5th-order HAM approximation ($c = 10^8$); solid line: exact solution.

Equations (38,39) describe vertical deflections of static beams subject to nonlinear forces $c\,u(x)^2 + 1$. The case when $c = 1$ was discussed in [4] and [5].

For the zeroth-order deformation equation (7), the nonlinear operator is taken as

$$\mathcal{N}[\phi(x;p)] = \frac{\partial^4 \phi\,(x;p)}{\partial x^4} - c\,\phi\,(x;p)^2 - 1. \tag{40}$$

In view of the boundary conditions (39), the initial guess is determined as

$$u_0(x) = x^4 - 4x^3 + 4x^2, \tag{41}$$

and the boundary conditions to (7) can be set as

$$\phi(0;p) = 0, \quad \frac{\partial \phi(0;p)}{\partial x} = 0, \quad \phi(2;p) = 0, \quad \frac{\partial \phi(2;p)}{\partial x} = 0. \tag{42}$$

In order to obtain the higher order terms $u_m(x)$, the mth-order deformation equation (19) and its boundary conditions (20) are calculated:

$$u_m^{(4)}(x) = \chi_m u_{m-1}^{(4)}(x) + \hbar R_m(\vec{u}_{m-1}(x)), \tag{43}$$

$$u_m(0) = u_m'(0) = u_m(2) = u_m'(2) = 0, \tag{44}$$

where

$$R_m(\vec{u}_{m-1}(x)) = u_{m-1}^{(4)}(x) - c\sum_{k=0}^{m-1} u_k(x)u_{m-1-k}(x) + \chi_m - 1. \tag{45}$$

In this way, we can calculate $u_m(x)(m = 1, 2, \ldots)$ recursively. For example, when $m = 1$, equation (43) becomes

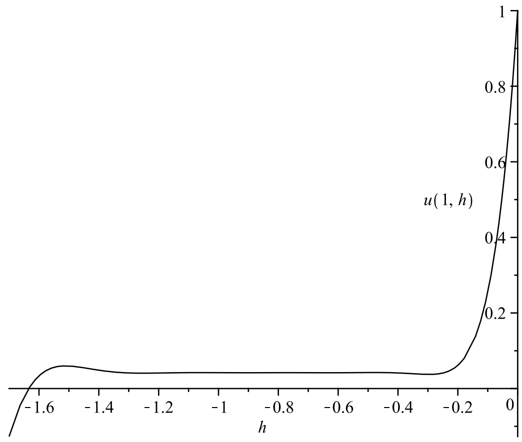

Fig. 5. $\hbar$-curve for the 15th-order HAM approximation ($c = 5$).

$$\begin{aligned} u_1^{(4)}(x) &= \hbar \left(u_0^{(4)}(x) - c\, u_0(x)^2 - 1 \right) \\ &= -h \left(cx^8 - 8\, cx^7 + 24\, cx^6 - 32\, cx^5 + 16\, cx^4 - 23 \right). \end{aligned} \tag{46}$$

Since the formula (24) now becomes

$$u_1(x) = \frac{k!\, x^{k+4}}{(k+4)!} - \frac{2^{k+1} x^3}{(k+4)(k^2+4k+3)} + \frac{2^{k+2} x^2}{k^3+9k^2+26k+24}, \tag{47}$$

by (25), the first order term

$$\begin{aligned} u_1(x) = &-\frac{\hbar c}{11880}\, x^{12} + \frac{\hbar c}{990}\, x^{11} - \frac{\hbar c}{210}\, x^{10} + \frac{2\hbar c}{189}\, x^9 - \frac{\hbar c}{105}\, x^8 \\ &+ \frac{23}{24}\, \hbar x^4 + \left(\frac{64}{945}\, c - \frac{23}{6} \right) \hbar x^3 - \left(\frac{64}{693}\, c - \frac{23}{6} \right) \hbar x^2. \end{aligned} \tag{48}$$

$u_m(x) (m = 2, 3, \ldots)$ can be calculated similarly.

The mth-order approximation can be expressed as

$$u(x, \hbar) \approx \sum_{k=0}^{m} u_k(x) = \sum_{k=2}^{8m+4} \gamma_{m,k}(\hbar)\, x^k, \tag{49}$$

where the coefficients $\gamma_{m,k}(\hbar) (k = 2, 3, \ldots, 8m+4)$ depend on m, k and $\hbar$. Equation (49) is a family of approximate solutions to the problem (38,39) in terms of the convergence-control parameter $\hbar$.

As in Section 3, our next goal is to show how the given parameter c in the problem (38,39) affects the approximate solutions (49), and how one can use the convergence-control parameter $\hbar$ to adjust and control the convergence region and rate of the solution series, according to the value of c.

Table 6
Relative errors of HPM and HAM solutions ($c = 5$).

x	10th HPM	10th HAM	15th HPM	15th HAM
0.2	2.6E-2	1.1E-2	2.0E-3	4.3E-5
0.4	2.7E-2	1.1E-2	2.1E-3	4.1E-5
0.6	2.8E-2	1.2E-2	2.2E-3	4.0E-5
0.8	2.9E-2	1.2E-2	2.3E-3	3.9E-5
1.0	2.9E-2	1.2E-2	2.3E-3	3.9E-5
1.2	2.9E-2	1.2E-2	2.3E-3	3.9E-5
1.4	2.8E-2	1.2E-2	2.2E-3	4.0E-5
1.6	2.7E-2	1.1E-2	2.1E-3	4.1E-5
1.8	2.6E-2	1.1E-2	2.0E-3	4.3E-5

When $c = 1$, we found that -1 is a valid value of $\hbar$. In fact, it is a proper value of $\hbar$. Therefore, as a special case of the HAM, the HPM does give accurate approximation to the problem (38,39) when $c = 1$. However, as the absolute value of c increases, the HPM again will no longer give accurate solutions; it even gives divergent series solutions.

We first take $c = 5$ as an example. From the $\hbar$-curve in Fig. 5, one sees that -1 is a valid value of $\hbar$, so the HPM is still valid for the problem (38,39).

However, the approximate solution given by the HPM is not so accurate. A more accurate solution is obtained by choosing $\hbar = -0.57$ instead of $\hbar = -1$, as shown in Table 6, where the relative errors of the 10th order HPM and HAM approximate solutions and the relative errors of the 15th order HPM and HAM approximate solutions are calculated and compared, which clearly indicates that the HAM solution when $\hbar = -0.57$ is better than the HPM solution. The formula for the relative error is given by

$$\delta(x) = \left| \frac{u_{numerical}(x) - u_{approximate}(x)}{u_{numerical}(x)} \right|. \qquad (50)$$

Since the closed-form solution to the problem (38,39) is not available, the numerical solution $u_{numerical}(x)$ is calculated via the Runge-Kutta-Fehlberg 4-5 technique, and is considered as the exact solution in the relative error computation.

However, if we take $c = -12$, then the HPM gives divergent series solutions, as shown in Table 7. On the other hand, when $\hbar = -0.634$, the HAM gives convergent series solutions which agree very well with the numerical solution

Table 7
Relative errors of HPM and HAM solutions ($c = -12$).

x	10th HPM	10th HAM	15th HPM	15th HAM
0.2	2.9E2	1.3E-1	4.5E3	5.8E-5
0.4	3.0E2	1.4E-1	4.7E3	5.8E-5
0.6	3.2E2	1.5E-1	4.9E3	5.4E-5
0.8	3.2E2	1.5E-1	5.1E3	5.0E-5
1.0	3.3E2	1.5E-1	5.1E3	4.9E-5
1.2	3.2E2	1.5E-1	5.1E3	5.0E-5
1.4	3.2E2	1.5E-1	4.9E3	5.4E-5
1.6	3.0E2	1.4E-1	4.7E3	5.8E-5
1.8	2.9E2	1.3E-1	4.5E3	5.8E-5

given by the Runge-Kutta-Fehlberg 4-5 technique, as shown in Table 7 and Fig. 6.

5 Conclusion

In this paper, a HAM-based approach has been proposed for obtaining approximate analytical solutions to fourth order boundary value problems. The efficiency of the approach has been demonstrated by solving some linear and nonlinear boundary value problems. Consequently, an affirmative answer to the open problem proposed by Momani and Noor in 2007 [3] has been given.

It has been found that, for parameterized boundary value problems, the homotopy perturbation method (HPM), a special case of the homotopy analysis method (HAM), is valid only for a small portion of the valid values of the parameters. Furthermore, even in the case the HPM is valid, the approximate solution given by the HPM is not always accurate. On the other hand, by means of the HAM, one can always obtain an accurate approximate solution by choosing a proper value of $\hbar$, the convergence-control parameter in HAM. The fundamental reason is that the HPM, as well as other analytical tools, cannot provide a mechanism to adjust and control the convergence region and rate of the series solutions obtained, according to the value of the parameter, but the HAM can, via the convergence-control parameter $\hbar$.

Finally, it is worth mentioning that, although the approach proposed in this paper is focused on fourth order problems, it can also be applied to other order boundary value problems by minor modification.

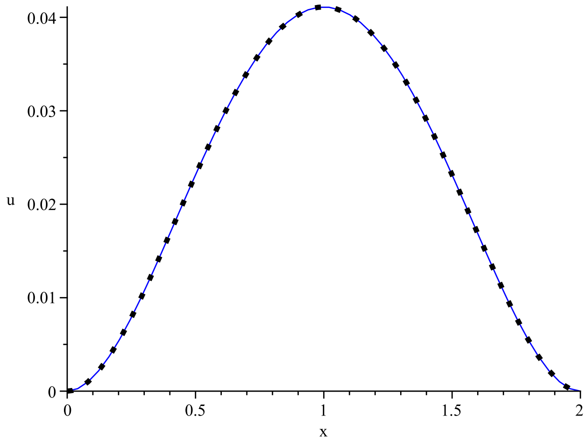

Fig. 6. Symbols: 15th-order HAM approximation ($c = -12$); solid line: numerical solution.

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