Series transformations to improve and extend convergence

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Abstract. We consider a new invariant transformation of some previously known series for the Lambert W function. The transformations contain a parameter p which can be varied, while retaining the basic series structure. The parameter can be used to expand the domain of convergence of the series. The speed of convergence, that is the accuracy for a given number of terms, can increase or decrease with p. Theoretical and experimental investigations that rely heavily on the computer-algebra system MAPLE are described.

1 Introduction

The Lambert W function is the inverse of the mapping $z \mapsto ze^z$. The inverse is a multivalued function denoted W_k, and the branches of this multivalued function are fixed by defining W_k through the equations [1]

$$\forall z \in \mathbb{C}, W_k(z)\exp(W_k(z)) = z \ , \tag{1}$$

$$W_k(z) \to \ln_k z \text{ for } \Re z \to \infty \ . \tag{2}$$

Here, $\ln_k z$ is the kth branch of logarithm, namely $\ln_k z = \ln z + 2\pi i k$, where $\ln z$ is the principal branch of natural logarithm [2]. Lambert W and its branches are important in the study of delay-differential equations. The simplest delay equation, using the notation $\dot{y} = \frac{dy}{dt}$ for the derivative with respect to time, is

$$\dot{y}(t) = ay(t-1) \ ,$$

subject to the condition on $-1 \le t \le 0$ that $y(t) = f(t)$, a known function. The solution can be expressed as the sum [3]

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \ \exp\left(W_k(a)t\right) \ ,$$

where the c_k can be determined from the initial conditions. One sees immediately that the solution will grow exponentially if any of the $W_k(a)$ has a positive real part, which leads to important stability theorems in the theory of delay equations. Other applications are given in [1].

In this paper, we use the computer-algebra system MAPLE to investigate the properties of series expansions for W. We focus on a number of asymptotic expansions for large z; these are also valid for non-principal branches around $z = 0$. One practical application of the series is to provide initial estimates for the numerical evaluation of W; these estimates can then be refined using iterative schemes to provide the arbitrary precision computations used in computer algebra systems. The series also have intrinsic interest. For example, the definition above of the branches W_k is based on partitioning the plane using the asymptotic series. Another interest is the fact that the asymptotic series are also convergent, and the nature of the convergence is one particular interest of this paper. In this paper, we shall mostly be concerned with the principal branch $k = 0$, which is the only branch that is finite at the origin. We shall abbreviate W_0 to W for the rest of the article.

The first asymptotic series is that found by de Bruijn [4] and Comtet [5] as

$$W(z) = \ln z - \ln\ln z + u \ , \tag{3}$$

where u has the series development

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{n}(-1)^{n-m}\left[{n \atop n-m+1}\right]\frac{\sigma^{n-m}\tau^m}{m!} \ , \tag{4}$$

where $\sigma = 1/\ln z$ and $\tau = \ln\ln z/\ln z$, and where $\left[{n \atop n-m+1}\right]$ denotes Stirling Cycle Numbers, also called the unsigned Stirling numbers of the first kind [6, 7]. This series was rearranged in [8] by introducing the new variable $\zeta = 1/(1+\sigma)$ to get

$$u = \sum_{m=1}^{\infty}\frac{\tau^m}{m!}\sum_{k=0}^{m-1}\left\{{k+m-1 \atop k}\right\}_{\geq 2}(-1)^{k+m-1}\zeta^{k+m} \ , \tag{5}$$

where the 2-associated Stirling Subset Numbers [6, 7] appear. Two further expansions introduce the variables $L_\tau = \ln(1-\tau)$ and $\eta = \sigma/(1-\tau)$.

$$u = -L_\tau + \sum_{n=1}^{\infty}(-\eta)^n\sum_{m=1}^{n}(-1)^m\left[{n \atop n-m+1}\right]\frac{L_\tau^m}{m!}, \tag{6}$$

$$u = -L_\tau + \sum_{m=1}^{\infty}\frac{1}{m!}L_\tau^m\eta^m\sum_{k=0}^{m-1}\left\{{k+m-1 \atop k}\right\}_{\geq 2}\frac{(-1)^{k+m-1}}{(1+\eta)^{k+m}} \ . \tag{7}$$

All of these expansions are limited in their domain of applicability by the fact that σ and τ are each singular at $z = 1$, restricting their utility to $z > 1$. In addition to the domain of validity of the variables, there is the question of the domain of convergence of the series. For example, we show below that for $z \in \mathbb{R}$, series (4) is convergent only for $z > e$.

In this paper, we consider transformations of the above series. We shall concentrate on the properties of the series for $z \in \mathbb{R}$. Our aims are to improve the convergence properties with respect to domain of convergence and with respect to rate of convergence. We shall do this using theoretical and experimental methods.

2 Computer algebra tools

We shall be using a number of tools from MAPLE in the work below. The coefficients appearing in the expansions (4) and (5) can be computed from their generating functions as follows. The 2-associated Stirling subset numbers are defined by the generating function

$$(e^z - 1 - z)^m = m! \sum_{n \geq 0} \frac{z^n}{n!} \left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{\geq 2} .$$

Given numerical values for n and m, we expand the left-hand side symbolically up to the term of nth order and then extract the appropriate numerical coefficient. The next lines show an implementation of this procedure with examples in Maple.

```
> StirlingSubset2:=proc(n::integer, m::integer)
                      option remember;
                      local f,z;
                      f:=series( (exp(z)-1-z)^m , z , n+1);
                      if n<2*m then
                           0
                        else
                           coeff(f,z,n)*n!/m!;
                        end if;
                      end proc;

> StirlingSubset2(6,3),StirlingSubset2(9,4),StirlingSubset2(12,5);
                   15,  1260,  190575
```

It can be noted that a similar method to this is used in the standard MAPLE library for Stirling Cycle numbers, which are used in (4). In practice, it is more efficient to store all of the coefficients from any series expansion, but this level of detail is not shown here. Similar techniques can be used for the Eulerian numbers used below in (23).

Another important tool from MAPLE for this paper is computation to arbitrary precision. It is a standard topic in numerical analysis that summing series requires a close watch on the effects of working precision, otherwise one runs the risk of generating ‘numerical monsters’ which are completely artificial effects of the computation and do not reflect any actual mathematical properties [9]. In all of the calculations below, the MAPLE environment variable `Digits` was set and monitored to ensure that the results were reliable.

3 An invariant transformation

We reconsider the derivation of (4), replacing (3) with the *ansatz*

$$W = \ln z - \ln(p + \ln z) + u \; . \tag{8}$$

Substituting into the defining equation $We^W = z$, we obtain

$$\left(\ln z - \ln(p + \ln z) + u\right) \frac{ze^u}{p + \ln z} = z$$

From this, it is clear that if we define

$$\sigma = \frac{1}{p + \ln z} \text{ and } \tau = \frac{p + \ln(p + \ln z)}{p + \ln z} \ , \tag{9}$$

then we recover the equation originally given by de Bruijn for u.

$$1 - \tau + \sigma u - e^{-u} = 0 \ . \tag{10}$$

The remarkable property is that (10) is invariant with respect to p, with only the definitions of σ and τ being changed. From (10), the expansion (4) is derived [5].

We now consider the properties of the transformations for $z \in \mathbb{R}$. We shall start with $p \in \mathbb{R}$ and later consider briefly one complex value of p. Both σ and τ are singular at $z_s = e^{-p}$, with the special case $p = 0$ recovering the previous observations regarding the singularities at $z = 1$. We note σ is monotonically decreasing on $z > z_s$. For τ, we have $\tau(z_0) = 0$ at $z_0 = \exp(z_s - p)$, with τ positive for larger z and negative for smaller. Also we note that τ has a maximum at $z = \exp(ez_s - p)$. In Figure 1, we plot σ and τ, defined by (9), for different values of p. We see that for all $z > z_s$, σ decreases with increasing p, but τ increases. In view of the form of the double sums above it is not obvious whether convergence is increased or decreased as a result of these opposed changes. This is what we wish to investigate here.

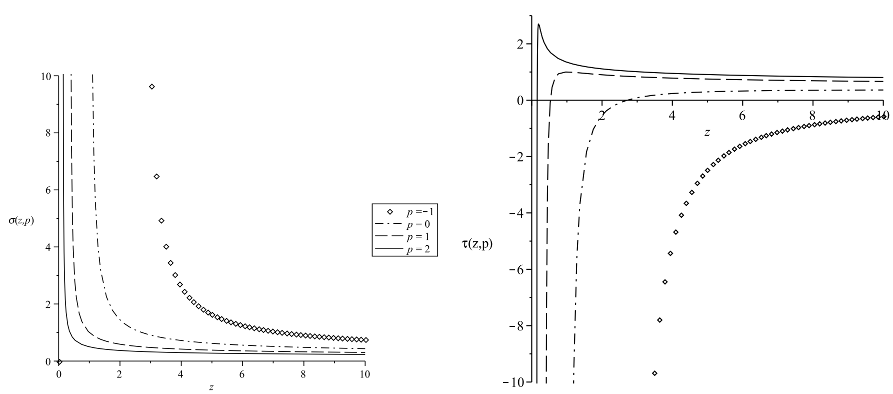

Fig. 1. Dependence σ and τ on z for different values of parameter p.

4 Domain of Convergence

We wish to investigate first the domains of $z \in \mathbb{R}$ for which the various series above converge, and how the domains vary with p. We begin with a theoretical result for $p = 0$.

Theorem 1. *The series (4) converges for $p = 0$ for all $z \geq e$.*

Proof. For $p = 0$, we have $\tau = -\sigma \ln \sigma$. We write (10) in the form

$$1 - \tau + \sigma u - e^{-u} = g(u) + f(u; \sigma, \tau) = 0 \ , \qquad (11)$$
$$g(u) = 1 - e^{-u} \quad \text{and} \quad f(u; \sigma, \tau) = \sigma u - \tau \ .$$

We now consider this equation in the complex plane of u. For any analytic function $F(\zeta)$ with a single isolated zero at $\zeta = u$ inside a contour C, we can use Cauchy's integral formula to write

$$u = \frac{1}{2\pi i} \int_C \frac{F'(\zeta)}{F(\zeta)} \, \zeta \, \mathrm{d}\zeta \ . \qquad (12)$$

Thus for our case, we have

$$u = \frac{1}{2\pi i} \int_C \frac{e^{-\zeta} + \sigma}{1 - e^{-\zeta} + \sigma\zeta - \tau} \zeta \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_C \frac{e^{-\zeta} + \sigma}{g(\zeta) + f(\zeta; \sigma, \tau)} \, \zeta \, \mathrm{d}\zeta \ , \qquad (13)$$

provided we can find the contour C.

For $z \approx e$ while $z > e$, we have $\sigma \approx 1$ and $\sigma < 1$. We define $\delta > 0$ by $\sigma = (1 - \delta)$. A contour which satisfies the requirements is the rectangular contour

$$\zeta = \begin{cases} \delta + it \ , & -2\delta^{1/2} \leq t \leq 2\delta^{1/2} \ , \\ t + 2\delta^{1/2} i \ , & -2 \leq t \leq \delta \ , \\ -2 + it \ , & -2\delta^{1/2} \leq t \leq 2\delta^{1/2} \ , \\ t - 2\delta^{1/2} i \ , & -2 \leq t \leq \delta \ . \end{cases} \qquad (14)$$

It is straightforward for Maple show that on this contour $|g| > |f|$. Rouché's theorem states that g and $f + g$ have the same number of zeros within C. Since $g(u) = 0$ for $u = 0$, the function $f + g$ has a single isolated zero as desired.

In addition to satisfying the conditions of the integration, the contour allows us to evaluate the integral by expanding the denominator of the integrand as an absolutely and uniformly convergent power series in f/g.

$$\frac{1}{1 - e^{-\zeta} + \sigma\zeta - \tau} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (1 - e^{-\zeta})^{-k-m-1} \zeta^k \sigma^k \tau^m (-1)^{k+1} C_m^{m+k} \ . \qquad (15)$$

Substituting this expansion into (13) and integrating term by term, we obtain u as the sum of an absolutely convergent double power series in σ and τ, provided $z > e$.

The domain of convergence cannot be extended to $z < e$, because the series for du/dz diverges at $z = e$. This can be seen by noting that $\tau = 0$ at $z = e$ (for $p = 0$). All terms reduce to zero except $m = 1$ which gives the sum

$$\frac{1}{e}\sum_{k=0}^{\infty}(-1)^k ,$$

which is divergent. □

In general, the precise domain of convergence is not of high importance, although its characterization remains an interesting mathematical challenge. The important point is to establish whether the domain of convergence increases or decreases, so that numerical procedures can be designed accordingly. Therefore, rather than devote space here to accumulating formal proofs for all the different cases, we can use numerical means as a rapid method to ascertain trends in the domains of convergence for all series. The method is simply to compute the partial sum of a series to a high number of terms, using extended floating-point precision as necessary, and then to plot the ratio of the partial sum to the exact value (the exact value is obtained by means other than series summation). The edge of the domain of convergence is then signaled by rapid oscillations and by marked deviations from the desired ratio of 1. Thus for the series just discussed, namely (4), we have plotted in Figure 2 the partial sum to 40 terms for different values of p. For $p = 0$, we see a nice illustration of theorem 1, with the partial sum becoming unstable in the vicinity of $z = e$. For positive p, we see the domain of convergence increased and for negative p it is decreased.

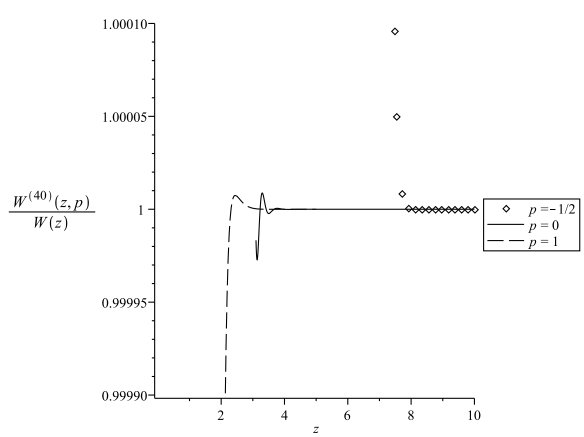

Fig. 2. For series (4), the ratio $W^{(40)}(z,p)/W(z)$ as functions of z for $p = -1/2, 0, 1$.

Similar effects can be seen for (5), we plot in Figure 3 the partial sums for 40 terms as p varies. The domain of convergence for each p is clearly seen, and confirms that the point of divergence moves to larger z for decreasing p and to the left for increasing p.

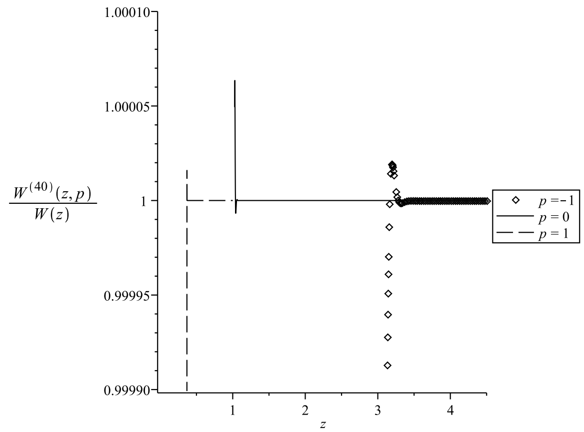

Fig. 3. For series (5), the ratio $W^{(40)}(z,p)/W(z)$ as functions of z for $p = -1, 0, 1$. Compared with Figure 2, this shows convergence down to smaller z.

A similar investigation of series (6) shows an interesting non-monotonic change in the domain of convergence. In Figure 4 the partial sums are plotted and the boundary of the domain of convergence moves to the right for $p \neq 0$.

We can summarize these findings by noting that series (5) has the widest domain of convergence, and the best behaviour with p, while the domains of convergence for series (4) and (6) become worse in that order.

5 Rate of convergence

By rate of convergence, we are referring to the accuracy obtained by partial sums of a series. Given two series, each summed to N terms, the series giving on average a closer approximation to the converged value is said to converge more quickly. The qualification 'on average' is needed because it will be seen in the plots below that the error regarded as a function of z can show fine structure which confuses the search for a general trend. Further, the comparison of rate of convergence between different series can vary with z and p. For some ranges of z, one series will be best, while for other ranges of z a different series will be best. Although one series may converge on a wider domain than another, there is no guarantee that the same series will converge more quickly on the part of the domain they have in common. The practical application of these series is to obtain rapid estimates for W using a small number of terms, and for this the quickest convergence is best, but this will be dependent on the domain of z.

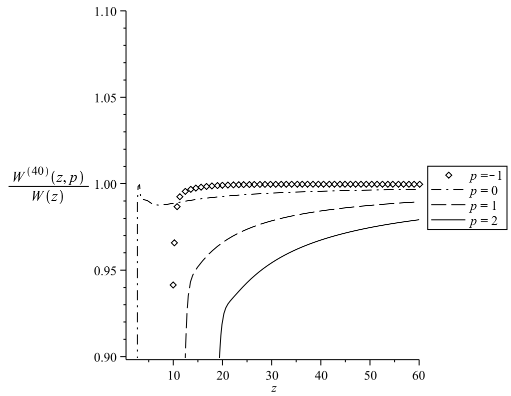

Fig. 4. For series (6), the ratio $W^{(40)}(z,p)/W(z)$ against z for $p = -1, 0, 1, 2$. Compared with figures 2 and 3, the changes in convergence are no longer monotonic in p.

The previous section showed that positive values of the parameter p extend the domain of convergence of the series, but its effect on rate of convergence is different. Figures 5, 6 and 7 show the dependence on z of the accuracy of computations of the series (4),(5) and (7) respectively with $N = 10$ for $p = -1, -1/2, 0$ and 1. One can see that the behaviour of the accuracy is non-monotone with respect to both z and p although some particular conclusions can be made. For example, one can observe that for the series (4) at least for $z < 30$ within the common domain of convergence the accuracy for $p = -1/2, 0$ and 1 is higher than for $p = -1$. The series (5) and (7) have the same domain of convergence and a very similar behaviour of the accuracy. Specifically, for these series an increase of positive values of p reduces a rate of convergence within the common domain of convergence i.e. for $z > 1.5$. However, at the same time for $z > 11$ computations with $p = -1$ are more accurate than those with positive p and for $5 < z < 18$ the highest accuracy occurs when $p = -1/2$.

The next two figures 8 and 9 display the dependence of convergence properties of the series (4) and (5) respectively on parameter p for different numbers of terms $N = 10, 20$ and 40. Again, the curves in these figures confirm that the accuracy strongly depends on parameter p and is non-monotone and show that on the whole an increase of the number of terms improves the accuracy. It is also interesting that there exists a value of p for which the accuracy at the given point is maximum; this value depends very slightly on N and approximately is $p \approx -0.75$ in Figure 8 and $p \approx -0.5$ in Figure 9.

The explained behaviour of the accuracy depending on parameter p shows that introducing parameter p in the series can result in significant changes in accuracy. The pointed out non-monotone effects of parameter p on a rate of convergence can be due to the aforementioned non-monotone behaviour of τ.

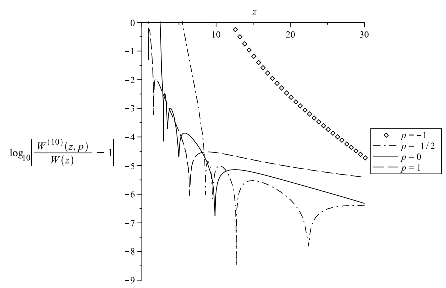

Fig. 5. For series (4) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

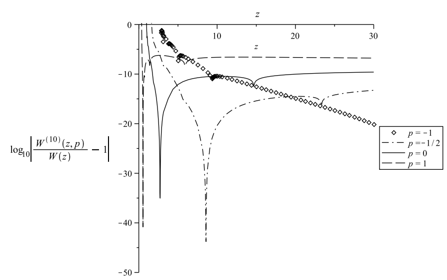

Fig. 6. For series (5) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

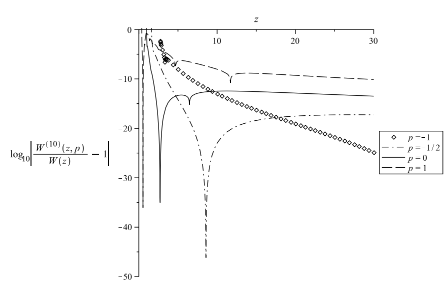

Fig. 7. For series (7) with $N = 10$, changes in accuracy in z for $p = -1, -1/2, 0$ and 1.

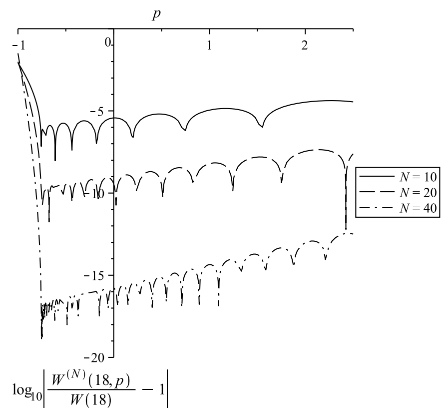

Fig. 8. For series (4), the accuracy as a function of p at fixed point $z = 18$ for $N = 10, 20$ and 40.

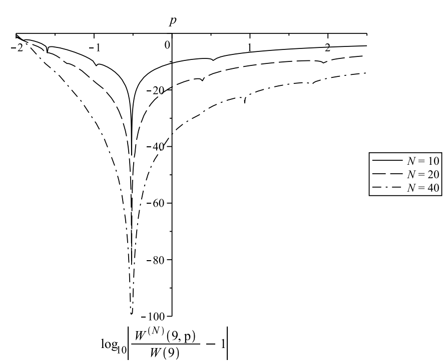

Fig. 9. For series (5), the accuracy as a function of p at fixed point $z = 9$ for $N = 10, 20$ and 40.

6 Branch −1 and complex p

The above discussion has considered only real values for the parameter p. We briefly shift our consideration to complex p and to branch −1. For z in the domain $-1/e < z < 0$, we have that $W_{-1}(z)$ takes real values in the range $[-1, -\infty)$. The general asymptotic expansion (2) takes the form

$$W_{-1}(z) = \ln(z) - 2\pi i - \ln(\ln(z) - 2\pi i) + u \ . \tag{16}$$

This will clearly be very inefficient for $z \in [-1/e, 0)$ because each term in the series will be complex, and yet the series must sum to a real number. If, however, we utilize the parameter p, we can improve convergence enormously.

We again adopt the *ansatz* used above to write

$$W_k(z) = [\ln_k z + p] - [p + \ln(p + \ln_k z)] + \frac{p + \ln(p + \ln_k z)}{p + \ln_k z} + v \ , \tag{17}$$

where v stands for the remaining series which will not be pursued here. By setting $p = i\pi$, we can rewrite $[\ln_{-1} z + i\pi]$ as $\ln(-z)$. A numerical comparison of partial sums can be used to show the improvement. We compare

$$W_{-1}^{(1)} = \ln(z) - 2\pi i - \ln(\ln(z) - 2\pi i) + \frac{\ln(\ln(z) - 2\pi i)}{\ln(z) - 2\pi i} \ , \tag{18}$$

$$\hat{W}_{-1} = \ln(-z) - \ln(-\ln(-z)) + \frac{\ln(-\ln(-z))}{\ln(-z)} \ . \tag{19}$$

The results are shown in table 1. We note that the transformed series is exactly correct at $z = -1/e$ and asymptotically correct as $z \to 0$, and therefore the error is a maximum somewhere in the domain. In contrast the untransformed series has an error that increases as $z \to -1/e$.

The accuracy is also shown graphically in figure 10. Notice that although the approximation $\hat{W}_{-1}$ given in (19) is exactly equal to W_{-1} at $z = -e^{-1}$, the local behaviour is different. We know that W_{-1} has a square-root singularity, while $\hat{W}_{-1}$ is regular there. This is why the maximum error occurs at $z = -e^{-1}$.

z	$W_{-1}(z)$	$\hat{W}_{-1}(z)$	$W_{-1}^{(1)}(z)$
−0.01	−6.4728	−6.4640	$-6.3210 - 0.04815i$
−0.1	−3.5772	−3.4988	$-3.4124 - 0.3223i$
−0.2	−2.5426	−2.3810	$-2.5182 - 0.5153i$
−0.3	−1.7813	−1.5438	$-2.0087 - 0.6621i$
$-1/e$	−1	−1	$-1.7597 - 0.7450i$

Table 1. Numerical comparison of series transformation with $p = i\pi$.

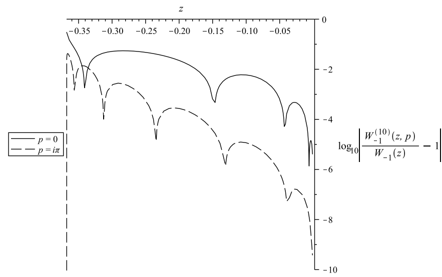

Fig. 10. Errors in approximations (18) and (19) for W_{-1}.

7 Taylor series

We have seen that the transformation allows us to obtain series valid for a wider range of z. We now observe that the Taylor series for $W(z)$ around $z = 0$ is well known [1]

$$W(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n. \tag{20}$$

This converges for $|z| < e^{-1}$. We can bridge the gap between the series above and the Taylor series by a series around $z = 1$. We have [7, 10] with $\omega = W(1)$

$$W(x) = \omega + \sum_{n=1}^{\infty} a_n (\ln x)^n \tag{21}$$

or by setting $t = \ln x$

$$W(e^t) = \omega + \sum_{n=1}^{\infty} a_n t^n, \tag{22}$$

where

$$a_n = \frac{1}{n!(1+\omega)^{2n-1}} \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k \omega^{k+1} \ . \tag{23}$$

This formula represents the expansion coefficients in terms of the second-order Eulerian numbers [6, 7]. We now show that these coefficients can also be represented through the unsigned associated Stirling numbers of the first kind $d(m, k)$ given by [5]

$$[\ln(1+v) - v]^k = k! \sum_{m=2k}^{\infty} (-1)^{m+k} \, d(m, k) \frac{v^m}{m!} \tag{24}$$

and the 2-associated Stirling subset numbers used in the series (5).

Both representations can be obtained on the basis of a relation [2]

$$W(e^t) + \ln W(e^t) = t \tag{25}$$

and the Lagrange Inversion Theorem [11]. To apply this theorem it is convenient to introduce a function that is zero at $t = 0$. We consider the function

$$v = v(t) = W(e^t)/\omega - 1 \tag{26}$$

and write (25) as

$$t = \omega\, v + \ln(1 + v). \tag{27}$$

Then by the Lagrange Inversion Theorem we obtain

$$v = \sum_{n=1}^{\infty} \frac{t^n}{n} [v^{n-1}] \left(\omega + \frac{\ln(1+v)}{v}\right)^{-n} \tag{28}$$

where the operator $[v^p]$ represents the coefficient of v^p in a series expansion in v. Comparing (26),(22) and (28) leads to a formula for the coefficients a_n, which after applying the binomial theorem becomes

$$a_n = \frac{\omega}{n(1+\omega)^n} [v^{n-1}] \sum_{k=0}^{\infty} (-1)^k \binom{n-1+k}{n-1} \frac{[\ln(1+v) - v]^k}{v^k (1+\omega)^k} \tag{29}$$

or by (24)

$$a_n = \frac{\omega}{n!} \sum_{k=0}^{n-1} \frac{(-1)^{n+k-1}\, d(n+k-1, k)}{(1+\omega)^{n+k}}. \tag{30}$$

If instead of function (26), we take

$$h = h(t) = W(e^t) - \omega - t \tag{31}$$

and apply the Lagrange Inversion Theorem to invert a relation

$$t = (e^{-h} - 1)\, \omega - h \tag{32}$$

coming from (25), then we find in a similar way

$$a_n = \frac{1}{n!} \sum_{k=0}^{n-1} \left\{ {n+k-1 \atop k} \right\}_{\geq 2} \frac{(-1)^{k+1} \omega^k}{(1+\omega)^{n+k}}. \tag{33}$$

Finally, one more representation for the coefficients a_n can be found in the following way. Let us consider a function

$$\psi = \psi(t) = W(e^t) - t \tag{34}$$

which is a simplified version of functions (26) and (31): now one does not need to provide the zero function value at $t = 0$ and here $\psi(0) = \omega$. Then it follows from (25) that

$$t = e^{-\psi} - \psi. \tag{35}$$

This equation can also be obtained from the fundamental relation (10) by transformation $u = \psi + \ln t, \sigma = 1/t$ and $\tau = \ln t/t$.

Differentiating (35) in t and excluding the term $e^{-\psi}$ from the result again using (35) result in an initial value problem for ordinary differential equation

$$\frac{d\psi}{dt} = -\frac{1}{1+t+\psi}. \tag{36}$$

Searching a solution in the form of series

$$\psi(t) = \omega + \sum_{n=1}^{\infty} c_n t^n \tag{37}$$

by substituting it into the differential equation and equating coefficients of the same power in t one can finally find

$$c_1 = -\frac{1}{1+\omega}, \quad c_n = -\frac{1}{n(1+\omega)}\left((n-1)c_{n-1} + \sum_{k=1}^{n-1} k c_k c_{n-k}\right), \text{ for } n \geq 2\ . \tag{38}$$

At length combining (37),(34) and (22) gives

$$a_1 = 1 + c_1, \quad a_n = c_n \text{ for } n \geq 2. \tag{39}$$

In practice, computing the coefficients using (38) and (39) is found to be more effective than using other representations. However, we have found some remarkable combinatorial identities. For example, equating the right-hand sides of (23) and (33) we obtain

$$\frac{1}{(1+\omega)^{n-1}} \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^{k+1}\omega^{k+1} = \sum_{k=0}^{n-1} \left\{ {n+k-1 \atop k} \right\}_{\geq 2} \frac{(-1)^k \omega^k}{(1+\omega)^k}.$$

8 Concluding remarks

We found an invariant transformation defined by the parameter p and applied it to the series for the Lambert W function to obtain a family of series. We studied an effect of parameter p on convergence properties of the transformed series. It is shown that an increase of p results in an extension of the domain of convergence of the series and thus the series obtained under the transformation with positive values of p have a wider domain of convergence than the original series does. However, at the same time a rate of convergence can be found to be reduced when the parameter p increases. Therefore in such a case within the common domain of convergence of the series with different positive values of p the series with the minimum value of p would be the most effective. The found relationships can be used, e.g. in evaluating of the Lambert W function in computer algebra systems.

We also considered the well-known expansion of $W(x)$ in powers of $\ln x$ and found three more forms for a represenation of the expansion coefficients in terms of the associated Stirling numbers of the first kind (30), the 2-associated Stirling subset numbers (33) and iterative formulas (39)-(38). As a consequence some combinatorial identities are obtained.

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