

Branch differences and Lambert W

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Abstract—The Lambert W function possesses branches labelled by an index k . The value of W therefore depends upon the value of its argument z and the value of its branch index. Given two branches, labelled n and m , the branch difference is the difference between the two branches, when both are evaluated at the same argument z . It is shown that elementary inverse functions have trivial branch differences, but Lambert W has nontrivial differences. The inverse sine function has real-valued branch differences for real arguments, and the natural logarithm function has purely imaginary branch differences. The Lambert W function, however, has both real-valued differences and complex-valued differences. Applications and representations of the branch differences of W are given.

Keywords—complex analysis; special functions; elementary functions; Lambert W ; multivalued functions

I. INTRODUCTION

The Lambert W function is a multi-branched function (also called a multivalued function). The branches are indexed by an integer k , and written W_k . For $z \in \mathbb{C}$, the branches are defined for $|z| \rightarrow \infty$ by [1]

$$W_k(z)e^{W_k(z)} = z \quad \text{and} \quad W_k(z) \rightarrow \ln_k z ,$$

where $\ln_k z = \ln z + 2\pi ik$ is the k th branch of logarithm. The branch cuts vary between branches. For the principle branch, the cut is $(-\infty, -1/e]$, while for other branches it is $(-\infty, 0]$.

Unlike the elementary multivalued functions, such as logarithm or the inverse trigonometric functions, the branches of the Lambert W function are not trivially related. This means that the difference between branches is a new function with interesting and useful properties.

In fact, this has already been noticed in a variety of contexts, with some instances pre-dating the naming of the Lambert W function. For example, in [10], Jordan and Glasser, apparently working independently, used a substitution equivalent to $\mathfrak{W}(xe^x)$ to evaluate the definite integral

$$\int_0^\infty e^{-w/2} \sqrt{w} \, dw , \quad \text{where } w = \frac{u}{1 - e^{-u}} ,$$

a problem posed by Logan, Mallows & Shepp. Karamata [11] derived a series expansion for \mathfrak{W} as part of his solution of a problem posed by Ramanujan. Definite integrals containing the tree T function, a cognate of W , used branch differences in [3]. Further applications can be seen in [12].

In order to establish the significance of branch differences, we begin by discussing the branches of some of the elementary functions. This will explain the meaning of ‘trivial differences’ referred to above.

II. ELEMENTARY INVERSE FUNCTIONS

Branch information was added to the natural logarithm function in [7]. Although the notation \log_a is widely used to denote logarithm to the base a , this notation does not apply to natural logarithm, since the base must be e . Therefore \ln_k denotes the k th branch of logarithm. Modern convention places the branch cut for \ln along the negative real axis $(-\infty, 0]$ and hence the range of \ln_k obeys $\Im \ln_k \in ((2k - 1)\pi, (2k + 1)\pi]$. Therefore, for the logarithm function, two consecutive branches differ by the imaginary constant

$$\ln_{k+1} z - \ln_k z = 2\pi i .$$

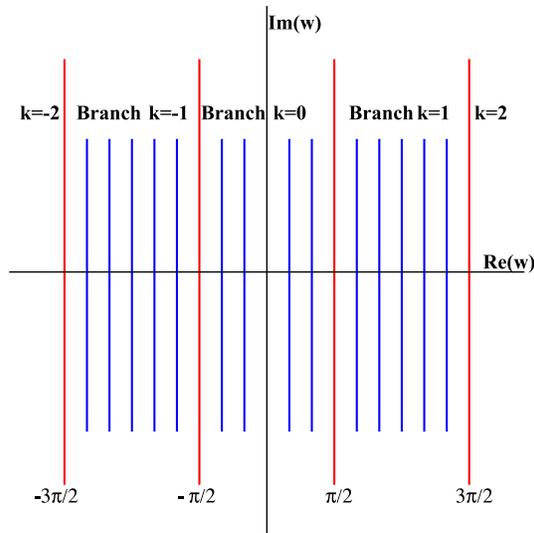


Figure 1. The domain of the sine function in the complex plane, denoted the w -plane. The variable w will become the argument of $\sin w$, or equivalently, w will be mapped to the z -plane (shown in figure 2) by the sine function. The red vertical lines separate the branch regions. Each region between the red lines maps to the entire z -plane. When returning from z back to w using the inverse sine function, the branch information must be specified. Thus this figure is also the union of the ranges of each of the branches of invsin_k .

Thus, there is no new function created by taking the difference.

In contrast with logarithm, the inverse sine function has real-valued differences. Without branch information, inverse sine is denoted \arcsin , but when branch information is added to the inverse sine, it is denoted invsin_k , for convenience in software implementation [6].

We consider the pair of equations

$$\begin{aligned} z &= \sin w , \\ w &= \text{invsin}_k z . \end{aligned}$$

To understand these equations, it is easiest to start in the w -plane, shown in figure 1, and consider the mapping under sine. The sine function does not have branches, but each region labelled as a branch maps to the entire z -plane as shown. Two points in the w -plane differing by 2π map to the same point

$$z = \sin(w) = \sin(w + 2\pi) ,$$

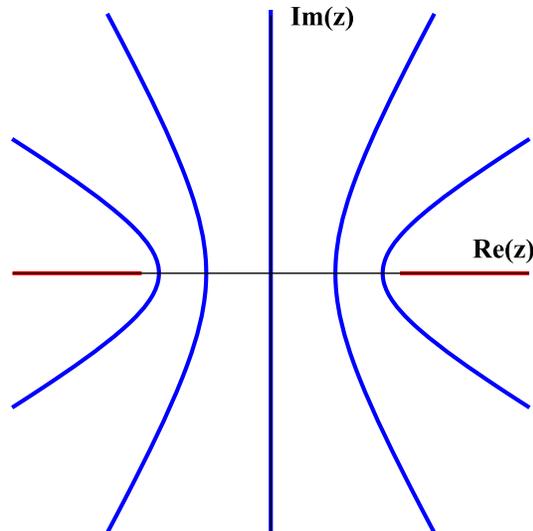


Figure 2. The range of the sine function in the complex z -plane, and the domain of the inverse sine functions. Before a point in this plane can be mapped to the w -plane, the branch of the inverse function must be specified. Notice that the real axes contain the red lines which map back to the separating lines in the w -plane, and hence are branch cuts in this plane.

and hence the branch difference is

$$\text{invsin}_{k+2} z - \text{invsin}_k z = 2\pi .$$

Note in figure 1 that points differing by 2π are in regions 2 branches apart. Similar considerations show that consecutive branches differ by

$$\text{invsin}_{k+1} z - \text{invsin}_k z = \pi - 2 \arcsin z .$$

Thus in either case, no new function is created by the branch difference, and the existing function is recovered. Further the difference is purely real.

Therefore we turn to the Lambert W function for more interesting properties.

III. ALGEBRAIC PROPERTIES

As with Lambert W , there is an algebraic equation solved by \mathfrak{W} .

Theorem 1: For $z, d \in \mathbb{C}$, d satisfies the equation

$$\frac{d}{e^{-d} - 1} \exp\left(\frac{d}{e^{-d} - 1}\right) = z \quad (1)$$

if and only if $d = \mathfrak{W}_{mn}(z)$ for $m, n \in \mathbb{Z}$.

Proof: Observe that

$$\begin{aligned} \exp(\mathfrak{W}_{mn}(z)) &= \exp(W_m(z)) / \exp(W_n(z)) \\ &= W_n(z) / W_m(z) . \end{aligned} \quad (2)$$

Since $W_m = \mathfrak{W} + W_n$, this becomes

$$\exp(\mathfrak{W}_{mn}) = \frac{W_n}{\mathfrak{W} + W_n} ,$$

which can be solved for W_n as

$$W_n(z) = \frac{\mathfrak{W}_{mn}(z)}{\exp(-\mathfrak{W}_{mn}(z)) - 1} , \quad (3)$$

and for W_m as

$$W_m(z) = \frac{\mathfrak{W}_{mn}(z)}{1 - \exp(\mathfrak{W}_{mn}(z))} . \quad (4)$$

Since $W_n \exp(W_n) = z$, then (1) follows. For the converse, assume (1) holds, then $\exists n \in \mathbb{Z}$, such that

$$\frac{d}{e^{-d} - 1} = W_n(z) .$$

Look for d in the form $d = v - W_n$. Then $e^{-d} = (z/W_n)e^{-v}$ and hence

$$v - W_n = W_n \left(\frac{ze^{-v}}{W_n} - 1 \right) = ze^{-v} - W_n .$$

Thus $ve^v = z$ and $v = W_m(z)$ for $m \in \mathbb{Z}$. ■

Lauwerier [12] studied a parametric representation of W and his work gives another equation that can be solved in terms of \mathfrak{W} .

Theorem 2: For $z \in \mathbb{R}$ and $0 < z \leq 1/e$, the equation

$$qe^{-q \coth q} \operatorname{cosech} q = z \quad (5)$$

has a real solution being given by

$$q = \pm \frac{1}{2} \mathfrak{W}_{0(-1)}(-z) .$$

Proof: In this proof, we shall abbreviate $\mathfrak{W}_{0(-1)}$ to \mathfrak{W} . The left-side of (5) is even in q , so we need consider only the case $q > 0$. The range of the left side of (5) is $(0, 1/e]$, with the maximum at $q = 0$ equal to $1/e$, which determines the restrictions on z in the theorem. Since for $x \in [-1/e, 0)$, we have $W_0(x) \in [-1, 0]$ and $W_{-1}(x) \in (-\infty, -1]$, we can note that $\mathfrak{W}(x) \in [0, \infty)$.

Using (2), we see

$$\begin{aligned} q \operatorname{cosech} q &= \frac{\mathfrak{W}(-z)}{\exp(\mathfrak{W}/2) - \exp(-\mathfrak{W}/2)} \\ &= -W_0(-z) \left(\frac{W_{-1}(-z)}{W_0(-z)} \right)^{1/2} \\ &= -W_0 \left(\frac{W_{-1}W_0}{W_0^2} \right)^{1/2} \\ &= (W_{-1}W_0)^{1/2} , \end{aligned}$$

where the signs of the quantities have been taken into account when manipulating the square roots. Similarly

$$e^{-q \coth q} = \left(\frac{z^2}{W_0(-z)W_{-1}(-z)} \right)^{1/2} .$$

Thus combining these expressions and taking the signs of the quantities into account, we see that both sides of (5) equal z . ■

Corollary 1: If we make the substitution $q = iv$ in (5), then the equation becomes

$$\frac{v}{\sin v} e^{-v \cot v} = z , \quad (6)$$

in which form it has appeared in a variety of integration problems; for example, in [13] the following integral was studied by Nuttall, Bouwkamp and Hornor & Rousseau.

$$\int_0^\pi \left\{ \frac{\sin v}{v} e^{v \cot v} \right\}^p dv = \frac{\pi p^p}{p!} , \quad p \in \mathbb{N} .$$

IV. RANGE OF BRANCH DIFFERENCE

We consider first the principal difference $\mathfrak{W}(z) = W_0(z) - W_{-1}(z)$, where we shall suppress the subscripts as being the default case. Denoting the positive real axis by \mathbb{R}^+ and the positive imaginary axis by \mathbb{I}^+ , we prove the following theorem.

Theorem 3: The range of \mathfrak{W} is the region in \mathbb{C} bounded by \mathbb{R}^+ , \mathbb{I}^+ and the curves given by the following. For $x \in [-1/e, 0)$

$$W_0(x) - W_{-2}(x) , \quad (7)$$

and for $x \in (-\infty, -1/e]$

$$W_{-1}(x) - W_{-2}(x) . \quad (8)$$

Proof: We begin by recalling important properties of the branches of W . The principal branch W_0 has branch cut $(-\infty, -1/e]$, while

the branch cut for W_{-1} is $(-\infty, 0]$. Further, both branches are closed on the top, meaning that for $x \leq -1/e$

$$W_0(x) = \lim_{y \downarrow 0} W_0(x + iy) ,$$

and for $x < 0$

$$W_{-1}(x) = \lim_{y \downarrow 0} W_{-1}(x + iy) .$$

On the bottom of the branch cuts, we have the properties that for $x \leq -1/e$

$$\lim_{y \uparrow 0} W_0(x + iy) = W_{-1}(x) , \quad (9)$$

while for $x \leq 0$, we have

$$\lim_{y \uparrow 0} W_{-1}(x + iy) = W_{-2}(x) . \quad (10)$$

Finally, W_0 is continuous at the line $-1/e < x \leq 0$.

Therefore, the branch cut for \mathfrak{W} will be $(-\infty, 0]$ with the properties that for $x \leq 0$

$$\begin{aligned} \lim_{y \downarrow 0} \mathfrak{W}(x + iy) &= \mathfrak{W}(x) \\ &= W_0(x) - W_{-1}(x) , \end{aligned} \quad (11)$$

while for $-1/e < x \leq 0$

$$\lim_{y \uparrow 0} \mathfrak{W}(x + iy) = W_0(x) - W_{-2}(x) , \quad (12)$$

and finally for $x \leq -1/e$

$$\lim_{y \uparrow 0} \mathfrak{W}(x + iy) = W_{-1}(x) - W_{-2}(x) . \quad (13)$$

The branch cut is illustrated in figure 3. The upper side of the branch cut is labelled A,B,C, with B showing the singular point $z = -1/e$, and C the singular point at the origin. The lower side of the branch cut is labelled D,E,F.

In order to map the axes in the domain of \mathfrak{W} to its range, we consider expansions around infinity and around critical points. From the known expansions for W when $|z| \rightarrow \infty$, we obtain

$$\begin{aligned} \mathfrak{W}_{mn}(z) &= 2\pi i(m - n) + \ln \frac{\ln_m z}{\ln_n z} \\ &+ O(1/\ln(z)) . \end{aligned} \quad (14)$$

Thus for the principal difference, all points at infinity map to $2\pi i$ as can be seen in figure 4. When $|z| \rightarrow 0$, W_0 is regular, while $W_{-1} \rightarrow \ln z - i\pi$, so $\mathfrak{W} \rightarrow i\pi - \ln z$. This can

be seen in the figure when $\Re(\mathfrak{W}) \rightarrow \infty$ and the imaginary component of the domain becomes asymptotic to the interval $[0, 2\pi]$. The series expansions around the branch point $z = -1/e$ depend on the side of the branch cut. On the upper side, both W_0 and W_{-1} have square-root singularities, while on the lower side W_0 remains singular, but W_{-1} is regular. In the neighbourhood of $-1/e$, the expansions use p defined by [2]

$$p = \sqrt{2(ez + 1)} ,$$

to obtain for $\Im z \geq 0$

$$\mathfrak{W}(z) = 2p + \frac{11}{36}p^3 + O(p^5) , \quad (15)$$

while for $\Im z < 0$ we have

$$\mathfrak{W}(z) = -1 + p + W_{-2}(-1/e) + O(p^2) . \quad (16)$$

From (15), we see that the special value $x = -1/e$ on the upper side of the cut obeys

$$\begin{aligned} \mathfrak{W}(-1/e) &= W_0(-1/e) - W_{-1}(-1/e) \\ &= (-1) - (-1) = 0 . \end{aligned}$$

Further, for $x < -1/e$, p , and hence (15) is purely imaginary and for $-1/e < x < 0$ we have p and (15) are real. This can also be seen because $W_{-1}(x) = \overline{W_0(x)}$, where the bar denotes complex conjugate, for $x < -1/e$, implying that for $x < -1/e$

$$\mathfrak{W}(x) = W_0(x) - \overline{W_0(x)} = 2i\Im W_0(x) ,$$

and $\Im W_0(x) > 0$ for $x < -1/e$. For $-1/e \leq x < 0$, we have $W_0(x), W_{-1}(x) \in \mathbb{R}$ and $W_0 > W_{-1}$, implying that for $-1/e < x < 0$

$$\mathfrak{W}(x) \in \mathbb{R}^+ .$$

In figure 4 this is seen in AB mapping to the imaginary axis $[0, 2\pi)$ and BC mapping to \mathbb{R}^+ .

The point E maps to

$$\begin{aligned} W_0(-1/e) - W_{-2}(-1/e) \\ &= -1 - W_{-2}(-1/e) \\ &\approx 2.0888 + 7.4615 i , \end{aligned}$$

and retains the square-root singularity because of W_0 . There are no singularities away from the branch cut and the remainder of the domain maps smoothly to the region between the boundaries. ■

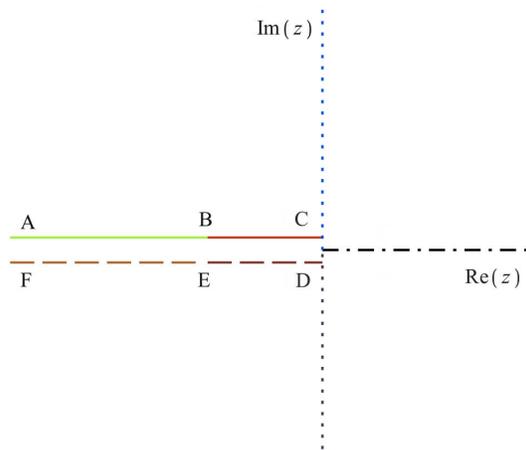


Figure 3. The domain of \mathfrak{W} (the z -plane). The branch cut is shown as two lines, one representing the upper side (ABC) and one representing the lower side (DEF). The points B and E are either side of $x = -1/e$.

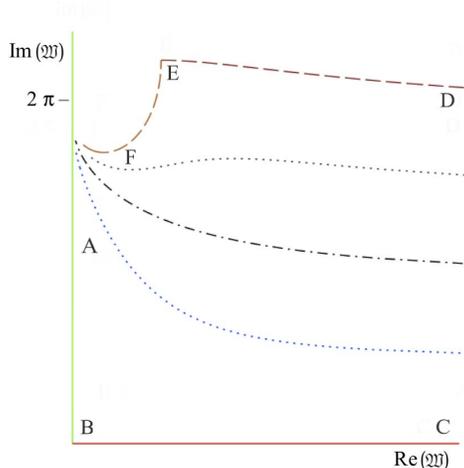


Figure 4. The range of $\mathfrak{W}_{0(-1)}(z)$. The letters correspond to the points labelled by the same letters in the domain of \mathfrak{W} , shown in figure 3. Notice that the image lines BA and EF , together with the images of the other axes in the domain converge to $2\pi i$ as a result of (14).

REFERENCES

[1] Robert M. Corless, Gaston H. Gonnet, D. E. G. Hare, David J. Jeffrey, and Donald E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[2] Robert M. Corless, David J. Jeffrey and Donald E. Knuth. A Sequence of Series for the Lambert W Function. *Proceedings of the ACM ISSAC, Maui*, 195–203, 1997.

[3] Robert M. Corless, J. Hu and D.J. Jeffrey. Some definite integrals containing the TreeT function. *Comm. Computer Algebra*, 48:2, pp 33–41, 2014.

[4] Robert M. Corless and David J. Jeffrey. The Lambert W function. In N. J. Higham, editor, *The Princeton Companion to Applied Mathematics*. Princeton University Press, 2014.

[5] German A. Kalugin and David J. Jeffrey. Convergence in C of series for the Lambert W function. *arXiv:1208.0754 [math.CA]*, 2012.

[6] D.J. Jeffrey. Multivalued elementary functions in computer-algebra systems. Proceedings AISC 2014. LNAI vol 8884, Springer 2014.

[7] D.J. Jeffrey, D.E.G.Hare, Robert M. Corless. Unwinding the branches of the Lambert W function. *Math. Scientist*. Vol 21, 1–7, 1996.

[8] German A. Kalugin, David J. Jeffrey and Robert M. Corless. Bernstein, Pick, Poisson and related integral expressions for Lambert W . *Int. Trans. Special Functions*. Vol 23(11) pp 817–829, 2012.

[9] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, Boston, 1994.

[10] William B. Jordan and M. L. Glasser. Solution of problem 68-17. *SIAM Review*, 12(1):153–154, January 1970.

[11] J. Karamata. Sur quelques problèmes posés par Ramanujan. *J. Indian Math. Soc.*, 24:343–365, 1960.

[12] H.A. Lauwerier. The asymptotic expansion of the statistical distribution of N.V. Smirnov. *Z. Wahrscheinlichkeitstheorie.*, 2:61–68, 1963.

[13] A.H. Nuttall. A Conjectured Definite Integral. Problem 85-16. *SIAM Review*, 27, p. 573. 1985.

[14] Frank W. J. Olver, Daniel W. Lozier, and Ronald F. Boisvert. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.