J. Inst. Maths Applics (1978) **22**, 337–351

The Temperature Field or Electric Potential Around Two Almost Touching Spheres

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WITH APPENDIX BY

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[Received 26 October 1977 and in revised form 18 February 1978]

The temperature field or electric potential around two equal, perfectly conducting spheres which are almost touching is studied using the method of matched asymptotic expansions. The dominant "outer" approximation of the calculational scheme applies in the narrow gap between the spheres while the "inner" approximation applies in the remaining volume outside the gap. The purpose of the calculation is to investigate the properties of certain singularities whose existence has been indicated by earlier solutions of the Laplace equation around two spheres. For example, the electrostatic forces acting on the spheres or the heat flux between them can become infinite when the spheres touch. The explicit forms of the singularities are found and used to assess the accuracy of earlier solutions. In the appendix the corresponding two-dimensional problems for almost touching cylinders are analysed.

1. Introduction

THE CALCULATIONS presented in this paper are relevant to several areas of research which share the Laplace equation as their governing equation. For example, the research interests of the authors whose papers are referred to below were rain droplets (Davis, 1964), arcing phenomena between electrical contacts (Warren & Cuthrell, 1975), electrical meters (Smith & Rungis, 1975) and the properties of suspensions (Jeffrey, 1973). Although there are in the literature many solutions of the Laplace equation around two spheres (some solutions having been derived and published independently several times), points of difficulty still exist. One such point of difficulty is seen by comparing Davis (1964), Smith & Barakat (1975) and Warren & Cuthrell (1975) who all calculate the electrostatic forces acting on each of two charged spheres. Whereas the numerical results of Davis (1964) and Warren & Cuthrell (1975) suggest that the force becomes infinite when the spheres touch, Smith & Barakat (1975) calculate a finite force for touching spheres. It is shown here that all three calculations are in fact correct and the apparent disagreement is a result of the different ways the authors formulated the problem. In addition, it is shown that a singularity in the force

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can indeed exist when the spheres touch and its form is given explicitly. A similar difficulty arises in calculations of the dipole strength of each sphere (this is defined below) by Jeffrey (1973), Smith & Rungis (1975) and Love (1975): numerically it is difficult to get the various results to agree. Again all calculations are correct and the numerical difficulties come from singularities (which are calculated) in the derivative, with respect to the separation of the spheres, of the expression for dipole strength.

The difficulties just described reflect the fact that although general methods exist for the solution of two-sphere problems, the most important being bispherical coordinates (Jeffery, 1912; Davis 1964), twin multipole expansions (Ross, 1970; Jeffrey, 1973), images or reflexions (Smith & Rungis, 1975), they are all numerically inconvenient when the spheres are very close together. One purpose of the present work is to provide a way of testing the rate of convergence of these schemes. In particular it is interesting to investigate the rate of convergence of the solutions found by Jeffrey (1973) which are in the form of infinite series, because if one truncates these series one obtains expressions corresponding to solutions obtained by the still popular method of reflexions. Except when the spheres are far apart, the method of reflexions converges slowly. It is possible, however, to take a quite different view of the present work and, rather than testing the accuracy of the general solution using the asymptotic one, test the accuracy of the asymptotic solution using the general one.

From this last point of view, the present calculations are relevant to papers by Keller (1963) and Batchelor & O'Brien (1977). In these papers the thermal or electrical conductivity of a granular medium is found through a study of the fields in the small gaps separating adjacent (highly conducting) particles. In the approximation to the conductivity so obtained, the $O(1)$ term is apparently dependent upon where one places the outer boundary of the gap. This can be explained using the language of inner and outer expansions as set out in Van Dyke (1975, Section 5.9). The solution within the gap corresponds to a dominant "outer" approximation. (The terminology is unfortunate in that the gap is geometrically an inner region, and yet an "outer" region for the approximation procedure; the adjectives "inner" and "outer" have been used in both senses in the past. To keep confusion at bay, they will be used here only in the approximation sense and never in the geometrical one.) The point made by Keller (1963) and Batchelor & O'Brien (1977) is that useful results can be obtained knowing only this "outer" approximation, the "inner" approximation being very difficult to obtain, provided that the apparent dependence upon the position of the outer boundary is not regarded as a difficulty. Why it is not a difficulty can be understood from the calculation in this paper. By taking a simpler situation in which the inner approximation can be found, this paper shows that the dependence on the gap boundary is removed when the formal matching of the two approximations is carried through to completion. In addition one can comment on the accuracy of the asymptotic analysis by comparing its results with those of the general theory.

For connoisseurs of matched asymptotic expansions the details of the calculation contain an unusual feature in that, at first sight, the inner and outer approximations appear to be unique. In fact, eigensolutions can be constructed for the inner problem but not for the outer one, and it seems that the outer approximation is indeed unique to all orders. In the appendix by Van Dyke, two-dimensional problems which correspond to the problems of the main paper are studied for almost touching cylinders and the simple closed-form solutions which are found allow a discussion of some of the details of the formal asymptotic procedure which are passed over in the main paper because of algebraic complexity.

2. The Problems to be Solved

The linearity of the Laplace equation allows us to divide the main problem into four subsidiary problems; these are now described as problems in heat conduction, the conversion to the analogous electrostatic problems being obvious. Two spheres, each of radius a, are placed in a temperature gradient which tends to a constant $a^{-1}\mathbf{G}$ far from the spheres (the a^{-1} factor makes $\mathbf{G}$ take the dimensions of temperature, which simplifies later equations). Each sphere, being a perfect conductor, is at a uniform temperature. The four problems are (a) both spheres are at temperature T_a, there is no applied field ($\mathbf{G} = 0$) and the temperature tends to zero at infinity, (b) the spheres are at equal and opposite temperatures $\pm T_b$ and again $\mathbf{G} = 0$, (c) the spheres are at zero temperature and the field $\mathbf{G}_c$ is parallel to the line of centres, and (d) the spheres are at zero temperature and $\mathbf{G}_d$ is perpendicular to the line of centres. In addition, problems (b) and (c) can be combined into a problem (bc) in which T_b is determined as a function of $\mathbf{G}_c$ and the gap width by the requirement that the net heat flux into a sphere is zero.

After solving these problems, we shall calculate three quantities. Letting ϕ be the temperature field (or the electric potential), we require first the heat flux into a sphere, non-dimensionalized to remove the sphere radius:

$$Q = (2\pi a)^{-1} \int (\partial\phi/\partial n)\, dA, \tag{2.1}$$

where the integral is over any surface enclosing one sphere and $\partial\phi/\partial n$ is a normal derivative. For the electrostatic problem, Q is the charge on a sphere. We require next the dipole strength of a sphere

$$\mathbf{S} = \int \{\mathbf{x}(\partial\phi/\partial n) - \phi\mathbf{n}\}\, dA, \tag{2.2}$$

where $\mathbf{n}$ is the unit normal out of the sphere and the origin for $\mathbf{x}$ will be specified later; this definition is equivalent to Jeffrey (1973, equation (2.4)) and again the integration is over any surface enclosing a sphere. Finally we require the electrostatic force on a sphere

$$\mathbf{F} = \int \left(\frac{\partial\phi}{\partial n}\right)^2 \mathbf{n}\, dA. \tag{2.3}$$

Our main interest will lie in the forms taken by these quantities for problems (b) and (c), these being the ones containing the singularities mentioned in the introduction.

3. First-order Solutions Outside the Gap

The solutions valid outside the gap between the spheres are found using tangent-sphere coordinates. These coordinates were first suggested by Ghosh (1936) who made the mistake, however, of writing an infinite sum where now we write a Hankel transform (as in (3.3) below); this mistake is discussed further in Jeffrey & Chen (1977). The properties of the coordinates are given in Smith & Barakat (1975) and in Moon & Spencer (1961) and are only outlined here. We start by taking cylindrical coordinates (ar, az, θ) with the z axis along the line of centres and the origin midway between the spheres. Tangent-sphere coordinates (ξ, η, θ) are defined by

$$z = 2\xi/(\xi^2+\eta^2) \quad \text{and} \quad r = 2\eta/(\xi^2+\eta^2). \tag{3.1}$$

If the width of the gap is $2h$ (see Fig. 1) and $\varepsilon = h/a \ll 1$, then the equation for the upper sphere whose centre is on the positive z axis is $(z-1-\varepsilon)^2+r^2 = 1$ which, when expanded as a series in ε and transformed using (3.1), is equivalent to

$$\xi = 1+\varepsilon\tfrac{1}{2}(\eta^2-1)+O(\varepsilon^2). \tag{3.2}$$

To leading order, then, we solve $\nabla^2\phi = 0$ and apply boundary conditions on $\xi = \pm 1$, i.e. on a pair of touching spheres. Denoting these leading-order solutions for problems (a)–(d) by $T_a\phi_a^{(1)}$, $T_b\phi_b^{(1)}$, $G_c\phi_c^{(1)}$ and $G_d\phi_d^{(1)}$, we follow the methods of Smith & Barakat (1975) and use tables of Hankel transforms (Erdelyi *et al.*, 1954) to find

$$\phi_a^{(1)} = (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty e^{-s}\,\mathrm{sech}\, s\cosh s\xi J_0(s\eta)\,ds, \tag{3.3a}$$

$$\phi_b^{(1)} = (\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty e^{-s}\,\mathrm{cosech}\, s\sinh s\xi J_0(s\eta)\,ds, \tag{3.3b}$$

$$\phi_c^{(1)} = 2\xi/(\xi^2+\eta^2)-(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty 2s\,e^{-s}\,\mathrm{cosech}\, s\sinh s\xi J_0(s\eta)\,ds, \tag{3.3c}$$

$$\phi_d^{(1)} = \cos\theta 2\eta/(\xi^2+\eta^2)-\cos\theta(\xi^2+\eta^2)^{\frac{1}{2}}\int_0^\infty 2\,e^{-s}\,\mathrm{sech}\, s\cosh s\xi J_1(s\eta)\,ds. \tag{3.3d}$$

We put these solutions to immediate use and motivate the development of the solutions in the gap by calculating the leading-order approximation to Q, defined in (2.1). For problem (a):

$$\begin{aligned} Q_a^{(1)} &= T_a\int_0^\infty\left[\frac{\partial\phi_a^{(1)}}{\partial\xi}\right]_{\xi=1}\frac{2\eta\,d\eta}{1+\eta^2} \\ &= 2T_a\int_0^\infty\left\{\frac{1}{1+\eta^2}+(1+\eta^2)^{\frac{1}{2}}\int_0^\infty s\,e^{-s}\tanh sJ_0(s\eta)\,ds\right\}\frac{\eta\,d\eta}{1+\eta^2}. \end{aligned}$$

The double integral is evaluated by changing the order of integration and using tables of Hankel transforms to integrate with respect to η. Then

$$Q_a^{(1)} = T_a\left(1+\int_0^\infty e^{-2s}\tanh s\,ds\right) = T_a 2\ln 2,$$

in agreement with Smith & Barakat (1975). Similarly,

$$Q_c^{(1)} = G_c 4\int_0^\infty s\,e^{-s}\,\mathrm{cosech}\, s\,ds = \tfrac{1}{3}\pi^2 G_c,$$

and $Q_d = 0$. An attempt to calculate $Q_b^{(1)}$ from (3.3b) by this method leads to an integrand $e^{-2s}\coth s$ which would give an infinite value for $Q_b^{(1)}$. Obviously the approximation $\phi_b^{(1)}$ breaks down in the neighbourhood of the gap where its gradient becomes unbounded at the origin.

4. The Solution in the Gap

The asymptotic solution in the gap is the outer approximation of the procedure needed to solve problem (b). The leading-order solution was given by Keller (1963); in this section we shall extend the solution to the next order. In the gap we define stretched coordinates Z, R by

$$Z = z/\varepsilon \quad \text{and} \quad R = r/\varepsilon^{\frac{1}{2}}. \tag{4.1}$$

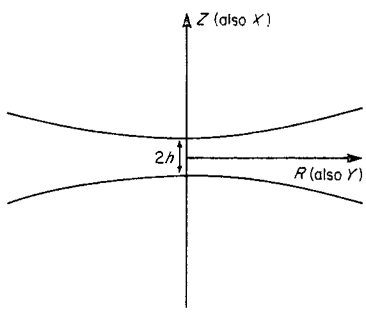

FIG. 1. The coordinates in the gap. The main paper uses cylindrical coordinates (Z, R, θ) while the Appendix uses Cartesian coordinates (X, Y).

The surface of the upper sphere in Fig. 1 is then given, within the neighbourhood of the gap, by

$$Z = H + \varepsilon\tfrac{1}{8}R^4 + O(\varepsilon^2), \tag{4.2}$$

where $H = 1 + \frac{1}{2}R^2$. We rewrite the Laplace equation in stretched coordinates and seek a solution in the form (as in (3.3b), T_b will not appear explicitly)

$$\phi_b = \Phi^{(1)} + \varepsilon\Phi^{(2)} + O(\varepsilon^2).$$

The equations for $\Phi^{(1)}$ and $\Phi^{(2)}$ are

$$\frac{\partial^2\Phi^{(1)}}{\partial Z^2} = 0 \quad \text{and} \quad \frac{\partial^2\Phi^{(2)}}{\partial Z^2} = -\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi^{(1)}}{\partial R}\right).$$

The boundary conditions come from applying $\Phi^{(1)} + \varepsilon\Phi^{(2)} = 1$ on the surface given in (4.2), and are

$$\Phi^{(1)}(H, R) = 1 \quad \text{and} \quad \Phi^{(2)}(H, R) = -\tfrac{1}{8}R^4[\partial\Phi^{(1)}/\partial Z]_{Z=H}.$$

In addition the functions are antisymmetric about $Z = 0$. The solutions to these equations are

$$\Phi^{(1)} = Z/H, \tag{4.3a}$$

$$\Phi^{(2)} = \tfrac{1}{3}Z^3(2-H)/H^3 - \tfrac{1}{6}Z(H^2-2H+3)/H^2. \qquad (4.3b)$$

Expressions related to these were given by Thwaites (1962).

To calculate the quantities defined in Section 2, we shall need $\partial\phi/\partial n$ on the sphere surface within the gap; this is most easily expressed in terms of Z:

$$\begin{aligned}\frac{\partial\phi}{\partial n} &= \varepsilon^{-1}\frac{\partial\phi}{\partial Z} + (1-Z)\frac{\partial\phi}{\partial Z} - R\frac{\partial\phi}{\partial R} + O(\varepsilon)\\ &= \varepsilon^{-1}/Z + \tfrac{1}{3}(1+1/Z) + O(\varepsilon).\end{aligned} \qquad (4.4)$$

5. The Second-order Solution Outside the Gap

Before deriving the second-order solutions outside the gap, we note that the first-order solutions to problem (b) inside and outside the gap, (4.3a) and (3.3b), match trivially. In the neighbourhood of the gap, $\eta \gg 1$ and $\xi \ll 1$, so (3.3b) becomes asymptotically

$$\phi_b^{(1)} \sim \eta\,\frac{\xi}{\eta}\int_0^{\infty} J_0(t)\,dt + O(\eta^{-1}) \sim \frac{2z}{r^2}.$$

This is the same as the asymptotic form of (4.3a) for large R:

$$\Phi^{(1)} \sim Z/\tfrac{1}{2}R^2 + O(R^{-4}).$$

It may seem, therefore, that the solutions are unique and that no new information is produced by matching. However, it is possible to construct eigensolutions which should be added to (3.3b), because strictly we are not allowed to apply our boundary condition on $\xi = 1$ everywhere including the origin ($\eta \to \infty$). Consequently, it is possible for multipoles to be placed at the origin while the problem outside the gap remains correctly posed. One example of an eigensolution is obtained by taking the solution (5.3) below and changing the first ξ in (5.3a) to $(\xi - 1)$ and the first ξ in (5.3b) to $(2\xi - 1)$. Leaving these eigensolutions out of (3.3b) is the same as anticipating the results of matching. One expects that similarly the problem inside the gap will have eigensolutions, but this does not seem to be so, making (4.3) unique, which is an unusual feature of this matching problem. Not all two-sphere potential problems are like this, for example in the potential-flow problem studied by Jeffrey & Chen (1977) and Czaykowski (1970), the solution in the gap is not unique and unknown constants must be found by matching in the conventional way.

The second-order solutions for problems (a) and (d) do not produce anything of interest, and so we concentrate here on problems (b) and (c). We extend the solutions $\phi^{(1)}$ of (3.3) by writing the solutions outside the gap as

$$\phi = \phi^{(1)} + \varepsilon\phi^{(2)} + O(\varepsilon^2).$$

The boundary condition on $\phi^{(2)}$, which is applied on $\xi = 1$, is derived from the requirement that the real boundary condition is applied to ϕ on (3.2), thus

$$\phi^{(2)}(1,\eta) = -\tfrac{1}{2}(\eta^2-1)[\partial\phi^{(1)}/\partial\xi]_{\xi=1}.$$

We substitute (3.3) into this condition and convert it to a suitable form by using the result

$$\eta^2 \int_0^\infty f(s)J_0(s\eta)\,ds = \int_0^\infty [(f/s)'-f'']J_0(s\eta)\,ds, \tag{5.1}$$

which is true provided $f'-f/s \to 0$ as $s \to 0$. The prime denotes d/ds. For problem (c) the boundary condition becomes

$$\phi_c^{(2)}(1,\eta) = -(1+\eta^2)^{\frac{1}{2}} \int_0^\infty C(s)\sinh s J_0(s\eta)\,ds,$$

where

$$C(s)\sinh s = e^{-s}-3s\,e^{-s}+2s^2\,e^{-s}+s^2\,e^{-s}\coth s - (s\,e^{-s}\coth s)'+(s^2\,e^{-s}\coth s)''.$$

The required solution for $\phi_c^{(2)}$ is then obviously

$$\phi_c^{(2)}(\xi,\eta) = -(\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty C(s)\sinh s\xi J_0(s\eta)\,ds.$$

For problem (b) things are not so straightforward. Before we can use (5.1) we must separate the s^{-1} singularity in $\coth s$; we obtain

$$\phi_b^{(2)}(1,\eta) = -\tfrac{1}{2}(1+\eta^2)+(1+\eta^2)^{\frac{1}{2}} \int_0^\infty B(s)\sinh s J_0(s\eta)\,ds, \tag{5.2}$$

where

$$B(s)\sinh s = s\,e^{-s}+\tfrac{1}{2}s\,e^{-s}\coth s - \tfrac{1}{2}(e^{-s}\coth s - e^{-s}/s)'+\tfrac{1}{2}(s\,e^{-s}\coth s - e^{-s})''.$$

We must now find a particular integral of Laplace's equation that will fit this boundary condition. We find it by substituting

$$\xi(\xi^2+\eta^2)+(\xi^2+\eta^2)^{\frac{1}{2}} \int_0^\infty W(s,\xi)J_0(s\eta)\,ds \tag{5.3a}$$

into Laplace's equation, taking an inverse Hankel transform and obtaining an equation for W:

$$\partial^2 W/\partial\xi^2 - s^2 W + 4\int_0^\infty \xi(\xi^2+\eta^2)^{-\frac{1}{2}}\eta J_0(s\eta)\,d\eta = 0.$$

Any solution of this equation that gives a convergent integral in (5.3a) will suit our purpose; the most convenient is

$$W = \frac{\xi}{s^2}(1+\xi s)\,e^{-\xi s} - \frac{1+s}{s^2}\,e^{-s}\,\frac{\sinh s\xi}{\sinh s}. \tag{5.3b}$$

When this partly arbitrary choice is combined with a general solution of Laplace's equation and (5.2) applied, the solution for $\phi_b^{(2)}$ is unique (although again eigensolutions have been suppressed):

$$\phi_b^{(2)}(\xi,\eta) = -\tfrac{1}{2}\xi(\xi^2+\eta^2)-\tfrac{1}{2}(\xi^2+\eta^2)^{\frac{1}{2}} \times \int_0^\infty [W(s,\xi)-2B(s)\sinh s\xi]J_0(s\eta)\,ds. \tag{5.4}$$

We now proceed to calculate Q, $\mathbf{S}$ and $\mathbf{F}$ and compare the expressions with other work.

6. Calculation of Q

The quantity $Q_c^{(1)}$ was calculated in Section 3. We start this section by calculating $Q_c^{(2)}$, which comes easily from $\phi_c^{(2)}$. We must first express our definition (2.1) in terms of the (ξ, η, θ) coordinates we have been using. At first sight it may seem that this requires expressions for $\partial\phi/\partial n$ and dA on $\xi = 1 + \frac{1}{2}\varepsilon(\eta^2 - 1)$, but Q is independent of the surface chosen, provided it surrounds only one sphere, and so we choose the surface $\xi = 1$. Then

$$Q = \int \left[\frac{\partial \phi}{\partial \xi}\right]_{\xi=1} \frac{2\eta\, d\eta}{1+\eta^2}.$$

The calculation of $Q_c^{(2)}$ follows exactly that of $Q_c^{(1)}$, giving

$$Q_c^{(2)} = -2\int_0^\infty C(s)\, ds = \tfrac{2}{9}\pi^2 - \tfrac{1}{3},$$

where all integrals can be evaluated using integration by parts and Gradshteyn & Ryzhik (1965, Section 3.552).

To calculate $Q_b^{(1)}$ and $Q_b^{(2)}$, we match an integral over the sphere surface in the gap with an integral over the remaining surface. We choose some point $\eta_0 \gg 1$ which lies in the region of overlap in which both solutions are valid and integrate separately the solutions (3.3b), (5.4) and (4.3) to this point. If we denote the "inner" contribution to Q_b from outside the gap $(\eta < \eta_0)$ by Q_{bi} and that from inside by Q_{bo}, then

$$Q_{bi}^{(1)} = \int_0^{\eta_0} \frac{\partial \phi_b^{(1)}}{\partial \xi} \frac{2\eta\, d\eta}{1+\eta^2}.$$

To do this integral we separate the s^{-1} singularity in $\coth s$ as we did in Section 5 and integrate it separately. To $O(\eta_0^{-1})$, the remainder integral can be taken from 0 to ∞. We obtain

$$Q_{bi}^{(1)} = T_b \ln(1+\eta_0^2) + 2T_b \int_0^\infty \left(\frac{e^{-s}}{\sinh s} - \frac{e^{-2s}}{s}\right) ds + O(\eta_0^{-1}).$$

The integral is $-2\psi(1) = 2\gamma$ (Gradshteyn & Ryzhik, 1965, Section 8.361), where γ is Euler's constant. The calculation of $Q_{bi}^{(2)}$ is similar:

$$Q_{bi}^{(2)} = -\tfrac{1}{2}T_b\eta_0^2 + \tfrac{1}{3}T_b \ln(1+\eta_0^2) + T_b(\tfrac{2}{3}\gamma + \tfrac{31}{18}).$$

All integrals are done by integrating by parts and using the definition of $\psi(1)$.

In stretched coordinates the point η_0 corresponds to Z_0, where

$$\varepsilon Z_0 = \frac{2}{1+\eta_0^2}\left(1 + \varepsilon\tfrac{1}{2}(\eta_0^2 - 3) + \frac{2\varepsilon}{1+\eta_0^2}\right) + O(\varepsilon^2).$$

The area element is $dA = \varepsilon 2\pi a\, dZ$ and so integrating (4.4) from 1 to Z_0 gives the "outer" contribution

$$Q_{bo} = T_b \ln Z_0 + \varepsilon \tfrac{1}{3} T_b (\ln Z_0 + Z_0 + 1) + O(\varepsilon^2).$$

Substituting for Z_0 gives

$$Q_{bo} = T_b[\ln 2 - \ln \varepsilon - \ln(1+\eta_0^2)] + \varepsilon T_b[\tfrac{1}{3}\ln 2 - \tfrac{1}{3}\ln \varepsilon - \tfrac{1}{3}\ln(1+\eta_0^2) + \tfrac{1}{2}\eta_0^2 - \tfrac{3}{2}] + O(\varepsilon^2) + O(\eta_0^{-1}).$$

Adding Q_{bi} and Q_{bo} cancels the singular terms and gives

$$Q_b = T_b(\ln 2 - \ln \varepsilon + 2\gamma) + \varepsilon T_b(\tfrac{1}{3}\ln 2 - \tfrac{1}{3}\ln \varepsilon - \tfrac{3}{2} + \tfrac{31}{18} + \tfrac{2}{3}\gamma) + O(\varepsilon^2).$$

We now compare this result with those derived by Keller (1963) and Batchelor & O'Brien (1977), who used just the solution (4.3a). Keller gave only the logarithmic term; by considering the $O(1)$ term of the expression above, we see that ε must be less than 10^{-6} before the logarithmic term alone is accurate to 10%. Batchelor & O'Brien (1977) included an $O(1)$ constant in their expression, although they could not actually construct a solution outside the gap to cancel the $\ln \eta_0^2$ singularity (the constant 2·48 they quote for the present two-sphere case should be 2·54). What the present result contributes is a more formal approach to justify their procedure (made possible because of the simpler situation) and an error estimate $O(\varepsilon \ln \varepsilon)$ for their expression.

We can now solve problem (bc) as well. We require $Q_b + Q_c = 0$ which gives the following relation between T_b, G_c and ε:

$$T_{bc} = G_c \tfrac{1}{3}[\pi^2 + \varepsilon(\tfrac{2}{3}\pi^2 - 1)]/[\ln 2 + 2\gamma - \ln \varepsilon + \tfrac{1}{3}\varepsilon(\tfrac{2}{3} + \ln 2 + 2\gamma - \ln \varepsilon)]. \qquad (6.1)$$

This expression shows that $T_{bc} \to 0$ as $\varepsilon \to 0$. This does not mean, however, that T_{bc} can be conveniently set to zero and forgotten for small gaps, because it is very sensitive to the value of ε. The implication is that although calculations such as those by Smith & Barakat (1975) and Smith & Rungis (1975) are formally correct in setting $T_{bc} = 0$ for touching spheres, in practice the conditions at the point of contact will be all-important in determining the value of T_{bc}. In the next section we shall see another consequence of the sensitivity of T_{bc} to ε.

7. Calculation of S

The subject of this section is the calculation of **S** for problems (b), (c) and (d) and a numerical comparison of the asymptotic formulae with the expressions given in Jeffrey (1973) which are valid for all separations of the spheres. Each method then tests the accuracy of the other. The calculation of **S** follows exactly the lines of that for Q with the difference that the gap region no longer contributes singular terms in η_0 and so the only contribution to **S** comes from outside the gap. The definition (2.2) of **S** is independent of the surface chosen to evaluate it, and so as in the last section we can choose $\xi = 1$ as the surface for integration. The origin for **x** in (2.2) will be our origin of coordinates, although this choice makes a difference only when Q is non-zero, and since $Q_{bc} = Q_d = 0$, there is no effect on S_{bc} or S_d.

Integrating (4.4) to find the contribution from the gap, we obtain

$$S_{bo} = 4\pi a^2 T_b[(1+\eta_0^2)^{-1} - 5\varepsilon(1+\eta_0^2)^{-1} + O(\varepsilon^2)],$$

which, as stated, contains no singular terms. Thus, combining this with the fact that only the z component of $\mathbf{S}$ is non-zero, we can reduce our expression for S to

$$S = 2\pi a^2 \int_0^\infty \left\{ \left[\frac{\partial \phi}{\partial \xi} \right]_{\xi=1} - \varepsilon(1-z)\phi^{(1)} \right\} \frac{4\eta\, d\eta}{(1+\eta^2)^2} + O(\varepsilon^2).$$

Performing the integrations gives (using ζ for the Riemann zeta function)

$$S_{bc} = \tfrac{4}{3}\pi a^2 T_{bc}[\tfrac{1}{2}\pi^2 + \varepsilon(\tfrac{1}{3}\pi^2 - \tfrac{1}{2})] - \tfrac{4}{3}\pi a^2 G_c[6\zeta(3) + \varepsilon(4\zeta(3) - 3\zeta(4))], \qquad (7.1)$$

where T_{bc} is given by (6.1), and $S_d^{(1)} = 3\pi a^2 \zeta(3) G_d$. Smith & Rungis (1975) and Love (1975) obtained the leading-order G_c term for S_{bc} but not the T_{bc} term.

The expressions with which we shall compare these results come from Jeffrey (1973) and are exact for all separations; in the present notation,

$$S_{bc} = \tfrac{4}{3}\pi a^2 G_c \sum_{p=0}^{\infty} A_{01p} t^p \quad \text{and} \quad S_d = \tfrac{4}{3}\pi a^2 G_d \sum_{p=0}^{\infty} A_{11p} t^p,$$

where $t = \tfrac{1}{2}(1+\varepsilon)^{-1}$ and the coefficients are given by

$$A_{mn0} = 3\delta_{1n} \quad \text{and} \quad A_{mnp} = (-1)^m \sum_{s=1}^{(p-n-3)/2} \binom{n+s}{n+m} A_{ms(p-n-s-1)}.$$

We wish to test the rate at which the infinite series converge, both to test their usefulness computationally and to test the method of reflexions. The rate of convergence of the method of reflexions is of interest because it is a method often applied to situations more complicated than those studied here, when only a small number of terms (say 10) in the infinite series can be obtained. We shall sum the series expressions for S_{bc} and S_d up to some $p = P$ and compare the numerical results with those obtained from asymptotic expressions.

Starting with what turns out to be a rapidly converging series, we calculate $S_d/\tfrac{4}{3}\pi a^2 G_d$ when the spheres touch ($t = \tfrac{1}{2}$) for $P = 12, 15, 20$ and obtain 2·718, 2·706, 2·705. The last value equals $\tfrac{9}{4}\zeta(3)$ to the accuracy shown and we conclude that for this weak interaction between the spheres a small number of terms in the series is satisfactory. Such is not the case for S_{bc}. In Fig. 2 the solid lines give $S_{bc}/\tfrac{4}{3}\pi a^2 G_c$ for $P = 70, 150$ while dotted lines show two approximations to (7.1), namely

$$6\zeta(3) + \frac{\tfrac{1}{6}\pi^4}{\ln \varepsilon} \quad \text{and} \quad 6\zeta(3) + \frac{\tfrac{1}{6}\pi^4}{(\ln \varepsilon - \ln 2 - 2\gamma)}.$$

The full expression (7.1) is indistinguishable from the second approximation because the $O(\varepsilon)$ terms largely cancel leaving a very small coefficient. It is clear that even with a very large number of terms, methods such as those using reflexions give very poor numerical results when $\varepsilon < 0{\cdot}005$.

8. Calculation of F

The final calculation is of the electrostatic force on a sphere. Almost all the leading-order non-singular contributions to the force have been calculated by Smith & Barakat (1975), and so here we shall concentrate on the singular behaviour of F_b and

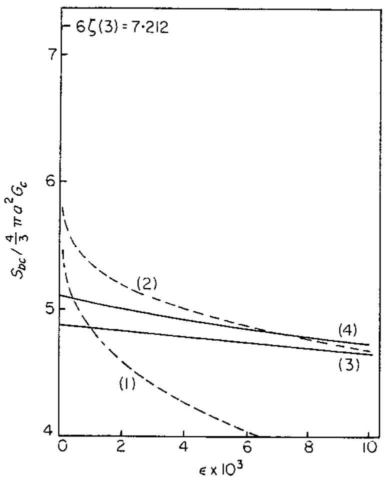

FIG. 2. Comparison of different expressions for S_{bc}. The broken lines actually start from 7·212 at $\varepsilon = 0$. The curves are: (1) $6\zeta(3)+\frac{1}{6}\pi^4/\ln\varepsilon$; (2) $6\zeta(3)+\frac{1}{6}\pi^4/(\ln\varepsilon-\ln 2-2\gamma)$; (3) the series summed to $P = 70$; (4) the series summed to $P = 150$.

F_{bc}, which has not yet been studied. Warren & Cuthrell (1975) have calculated F_b and also measured it experimentally; Davis (1964) has calculated F_{bc}; Smith & Barakat (1975) have calculated F_c. Since $T_{bc} \to 0$ as $\varepsilon \to 0$, one might expect that $F_{bc} \to F_c$ as $\varepsilon \to 0$. The numerical results of Davis (1964) and Smith & Barakat (1975) throw doubt on this and here we shall see explicitly that it is not true. In fact $F_{bc} \to \infty$ as $\varepsilon \to 0$.

Only the z components of $\mathbb{F}_b$, $\mathbb{F}_c$ and $\mathbb{F}_{bc}$ are non-zero (neglecting possible complications produced by problem (d)). Squaring (4.4) and integrating appropriately gives the contribution from the gap as

$$F_{bo} = 2\pi T_b^2\{\varepsilon^{-1}(1-1/Z_0)-\tfrac{1}{3}\ln Z_0+\tfrac{5}{3}(1-1/Z_0)\}+O(\varepsilon)$$
$$= 2\pi T_b^2\{\varepsilon^{-1}+\tfrac{1}{3}\ln\varepsilon+\tfrac{5}{3}-\tfrac{1}{3}\ln 2-\tfrac{1}{2}(1+\eta_0^2)+\tfrac{1}{3}\ln(1+\eta_0^2)\}+ O(\varepsilon)+O(\eta_0^{-2}).$$

Note that both terms of the gap solution are needed to cancel the singularities in the first term of the solution outside the gap. The contribution of (3.3b) to the force is

$$F_{bi}^{(1)} = 2\pi T_b^2 \int_0^{\eta_0} \left[\frac{\partial\phi_b^{(1)}}{\partial\xi}\right]^2_{\xi=1} \frac{\eta^2-1}{\eta^2+1}\,\eta\,d\eta. \tag{8.1}$$

We first separate those terms in $\partial\phi/\partial\xi$ which will lead to singular terms in η_0. Thus

$$\left[\frac{\partial\phi_b^{(1)}}{\partial\xi}\right]_{\xi=1} = 1+\frac{2/3}{1+\eta^2}+(1+\eta^2)^{\frac{1}{2}}\int_0^\infty s\,e^{-s}\left(\coth s-\frac{1}{S}+\tfrac{1}{3}\right)J_0(s\eta)\,ds.$$

Squaring this and integrating according to (8.1) gives

$$F_{bi}^{(1)} = 2\pi T_b^2\{\tfrac{1}{2}(1+\eta_0^2)-\tfrac{1}{3}\ln(1+\eta_0^2)+O(1)\}.$$

It will be seen below that the singular terms alone in F_b will be sufficient to reproduce the numerical results of Davis (1964) and Warren & Cuthrell (1975) and so the $O(1)$ contribution to F will not be pursued. Adding F_{bo} and F_{bi} gives

$$F_b^{(1)} = 2\pi T_b^2(\varepsilon^{-1} + \tfrac{1}{3}\ln\varepsilon + O(1)). \quad (8.2)$$

We compare (8.2) first with the results of Warren & Cuthrell (1975), who solved problem (b). Their Fig. 2 shows, in the present notation, $F_b/2\pi T_b^2$ plotted against ε. Their figure is not reproduced here because the graph of (8.2) is indistinguishable from the line they calculate. Thus (8.2) is a very good representation of the force for $\varepsilon < 0{\cdot}01$. We next compare (8.2) with the results of Davis (1964), who solved problem (bc). To do this we use

$$T_{bc} = G_c\tfrac{1}{3}\pi^2/(\ln 2 + 2\gamma - \ln\varepsilon) + O(\varepsilon). \quad (8.3)$$

In the present notation, Davis (1964) calculates $F_{bc}/8\pi T_b^2$ as a function of 2ε. Thus for $2\varepsilon = 0{\cdot}001$ and $0{\cdot}01$, he obtains 59·5 and 9·6, respectively, which agrees well with 60·5 and 10·5 calculated from (8.2) and (8.3).

I should like to thank Professor Van Dyke for writing the Appendix, and discussing with me the significance of the "trivial" matching of the inner and outer approximations.

References

BATCHELOR, G. K. & O'BRIEN, R. W. 1977 *Proc. R. Soc. Lond. A* **355,** 313.
CZAYKOWSKI, J. 1970 Ph.D. Dissertation, Oxford University.
DAVIS, M. H. 1964 *Quart. J. Mech. appl. Math.* **17,** 499.
ERDELYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1954 *Tables of Integral Transforms*. New York: McGraw-Hill.
GHOSH, S. 1936 *Bull. Calcutta Math. Soc.* **28,** 193.
GRADSHTEYN, I. S. & RYZHIK, I. W. 1965 *Table of Integrals, Series and Products*. New York: Academic Press.
JEFFERY, G. B. 1912 *Proc. R. Soc. Lond. A* **87,** 109.
JEFFREY, D. J. 1973 *Proc. R. Soc. Lond. A* **335,** 355.
JEFFREY, D. J. & CHEN, H.-S. 1977 *J. appl. Mech.* **44,** 166.
KELLER, H. B. & SACHS, D. 1964 *J. appl. Phys.* **35,** 537.
KELLER, J. B. 1963 *J. Appl. Phys.* **34,** 991.
LOVE, J. D. 1975 *J. Phys. E* **8,** 1063.
MOON, P. & SPENCER, D. 1961 *Field Theory Handbook*, p. 104. Berlin, New York, Heidelberg: Springer-Verlag.
MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics*, Part II, pp. 1210–1211. New York: McGraw-Hill.
ROSS, D. K. 1970 *Bull. Aust. Math. Soc.* **2,** 237.
SMITH, G. S. & BARAKAT, R. 1975 *Appl. Sci. Res.* **30,** 418.
SMITH, W. E. & RUNGIS, J. 1975 *J. Phys. E* **8,** 379.
THWAITES, J. J. 1962 *Proc. Instn elect. Engrs C* **109,** 600–605.
VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics*, annotated edition. Palo Alto: Parabolic Press.
WARREN, W. E. & CUTHRELL, R. E. 1975 *J. appl. Phys.* **46,** 4597.

Appendix: Two Almost Touching Circles

M. VAN DYKE

IF ONE replaces the spheres of the main paper by parallel, almost touching circular cylinders, the various approximate solutions to problems (b)–(d) can be found in simple closed form. There is no counterpart of problem (a), for if two circles are at equal temperatures there is a logarithmic singularity in the potential at infinity unless the temperature is constant everywhere.

First-order Solution Outside the Gap

We use the coordinates defined in (3.1) which here are touching circles. The change to two dimensions, however, brings with it some minor modifications to the coordinates. If we denote the Cartesian coordinates in the plane by (x, y) and place the sphere centres on the x axis (see Fig. 1), then the equivalent of (3.1) can be written

$$\zeta = \xi + i\eta = 2/(x+iy).$$

We must remember, however, that while in three dimensions the range of η is $0 \leqslant \eta < \infty$, in two dimensions it is $-\infty < \eta < \infty$. For problem (b) the first-order solution outside the gap (the inner approximation) is simply

$$\phi_b^{(1)} = \xi = 2x/(x^2+y^2), \tag{A.1}$$

where the temperature T_b has been set equal to 1. Problem (c) can be solved by the method of images, or by conformally mapping the twin circles into the unit circle in the plane of $\cot\frac{1}{4}\pi\zeta$, which gives

$$\phi_c^{(1)} = \pi \operatorname{Re}\{\cot\tfrac{1}{2}\pi\zeta\} = \pi \sin\pi\xi/(\cosh\pi\eta - \cos\pi\xi). \tag{A.2}$$

Similarly for problem (d) we find

$$\phi_d^{(1)} = \pi \operatorname{Im}\{\operatorname{cosec}\tfrac{1}{2}\pi\zeta\} = -2\pi\cos\tfrac{1}{2}\pi\xi\sinh\tfrac{1}{2}\pi\eta/(\cosh\pi\eta - \cos\pi\xi). \tag{A.3}$$

The flux into the upper circle (positive x axis) is given by

$$Q^{(1)} = \int \frac{\partial\phi}{\partial n}\,ds = \int_{-\infty}^{\infty}\left[\frac{\partial\phi^{(1)}}{\partial\xi}\right]_{\xi=1} d\eta, \tag{A.4}$$

and for problem (c) this is

$$Q_c^{(1)} = -\pi^2\int_{-\infty}^{\infty}\frac{d\eta}{1+\cosh\pi\eta} = -2\pi. \tag{A.5}$$

For problem (b), however, the integral diverges, the contribution from $-\eta_0 < \eta < \eta_0$ being $2\eta_0$ (where η_0 has the same meaning as it does in the main text).

Two-term Solution in the Gap

Using the stretched coordinates Z, R of (4.1) but denoting them X, Y in the plane, we easily find the gap solution for problem (b):

$$\Phi_b = X/H + \tfrac{1}{6}\varepsilon[(4-3H)X^3/H^3 + (2H-3)X/H^2] + O(\varepsilon^2), \qquad (A.6)$$

where $H = 1 + \frac{1}{2}Y^2$ as one would expect. The first term is the solution of Keller (1963). Although we have made no use of the boundary condition far from the circles, this matches with the solution there (i.e. this outer approximation matches the inner approximation (A.1)). That is, rewriting the one-term solution (A.1) in the gap variables X, Y, expanding for small ε and keeping two terms gives $2X/Y^2 - 2\varepsilon X^3/Y^4$, conversely, rewriting the two-term solution (A.6) in the variables used outside the gap x, y, expanding for small ε and keeping one term gives $2x/y^2 - 2x^3/y^4$; and these are the same.

This might suggest that in this singular perturbation problem matching has played no role at this stage; but that is not true. The solution (A.1) is in fact not unique, for we can add to it any multiple of any of the "eigensolutions"

$$\sin N\pi\xi \cosh N\pi\eta, \quad N = 1, 2, 3 \ldots \qquad (A.7)$$

These vanish on both circles and far away, but are increasingly singular at the point of (near) contact. We have tacitly rejected them by choosing the least singular solution; otherwise they would be removed by matching.

The flux through the upper circle is the integral of

$$\left[\frac{\partial \phi_b}{\partial n}\, ds\right] = \varepsilon^{-\frac{1}{2}}[1/H + \varepsilon\{\tfrac{1}{2} + \tfrac{1}{6}(Y^2-1)/H^2\} + O(\varepsilon^2)]\, dY, \qquad (A.8)$$

where we use Y rather than X to obtain the simplest expression.

We integrate from 0 to Y_0, where

$$\varepsilon^{\frac{1}{2}} Y_0 = 2\eta_0(1+\eta_0^2)^{-1}[1 + \varepsilon(1-\eta_0^2)/(1+\eta_0^2)] + O(\varepsilon^2), \qquad (A.9)$$

and expand the result for small ε to obtain

$$Q_b = \pi 2^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}} - 2\,\frac{1+\eta_0^2+\eta_0^4}{\eta_0(1+\eta_0^2)} + \tfrac{1}{12}\pi 2^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} + \tfrac{1}{3}\varepsilon\left[\frac{4}{\eta_0} - 8\eta_0 + \left(\frac{1+\eta_0^2}{\eta_0}\right)^3 + \frac{2\eta_0(1-\eta_0^2)}{(1+\eta_0^2)^2}\right] + O(\varepsilon^{\frac{3}{2}}). \qquad (A.10)$$

For problems (c) and (d) the solution in the gap appears to be zero to any order in ε. This is confirmed by rewriting the solutions (A.2), (A.3) in gap variables and expanding for small ε, which gives exponentially small terms.

Second Inner Approximation

We have calculated two terms of the solution in the gap which is the outer approximation and only one of the inner approximation, because the outer approximation is seen to make the primary contribution to the flux (it is for precisely this reason that it is named "outer"). In fact the role of the inner approximation is merely to remove singularities: its contribution $2\eta_0$ to the flux just cancels the singularity for large η_0 in the second term of (A.10). Of course, in other problems it can do more than this and add an $O(1)$ constant to the expression for Q (for example, Keller & Sachs (1964) predict an $O(1)$ constant $-1{\cdot}95$ in the expression for Q for a periodic array of cylinders). Thus the singularities in the last term of (A.10) must also cancel. To verify this, we continue the solution outside the gap in the form $\phi^{(1)}+\varepsilon\phi^{(2)}+\ldots$.

For problem (b) the correction to (A.1) is easily found as

$$\phi_b^{(2)} = \tfrac{1}{6}\,\mathrm{Re}\,(2\zeta+\zeta^3) = \tfrac{1}{3}\xi+\tfrac{1}{6}(\xi^3-3\xi\eta^2). \tag{A.11}$$

This shows a typical symptom of non-uniformity, being singular like the inverse cube of the distance from the origin, whereas (A.1) is singular like the inverse first power. The two-term inner approximation is found to match with the two-term outer approximation, again because we have excluded the more singular terms (A.7).

Using (A.11), we find the flux Q_{bi} through $-\eta_0 < \eta < \eta_0$ becomes

$$Q_{bi} = 2\eta_0+\varepsilon(\tfrac{5}{3}\eta_0-\tfrac{1}{3}\eta_0^3)+O(\varepsilon^2). \tag{A.12}$$

Adding the contribution (A.10) to (A.12) cancels all singular terms; and then letting η_0 tend to infinity yields the desired result:

$$Q_b = \pi 2^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}[1+\tfrac{1}{12}\varepsilon+O(\varepsilon^2)]. \tag{A.13}$$

From the exact solution of problem (b) in bipolar coordinates (Morse & Feshbach, 1953) we can easily calculate the exact result:

$$Q_b = 2\pi\,\mathrm{arcsech}\,(1+\varepsilon) = \pi 2^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}[1+\tfrac{1}{12}\varepsilon-\tfrac{17}{1440}\varepsilon^2+\ldots].$$

For problem (c) the correction to (A.2) is

$$\phi_c^{(2)} = -\tfrac{1}{6}\pi^2\,\mathrm{Re}\,\{(2\zeta+\zeta^3)/(1-\cos\pi\zeta)\}.$$

This contributes to the flux only exponentially small terms for large η_0, so that taking the limit gives

$$Q_c = -2\pi+O(\varepsilon^2). \tag{A.14}$$

For problem (d), the correction to (A.3) is

$$\phi_d^{(2)} = -\tfrac{1}{6}\pi\,\mathrm{Im}\,\{(2\zeta+\zeta^3)\cos\tfrac{1}{2}\pi\zeta/(1-\cos\pi\zeta)\}.$$

Temperature for Zero Heat Flux

If the circles are at temperatures $\pm T_{bc}$ with no net heat flux, and in a unit gradient along the line of centres far away, we can find T_{bc} by forming a linear combination of the fluxes (A.13) and (A.14). This gives

$$T_{bc} = (2\varepsilon)^{\frac{1}{2}}[1-\tfrac{1}{12}\varepsilon+O(\varepsilon^2)].$$

This work was supported by the U.S. Air Force Office of Scientific Research under Grant No. AFOSR 74–2649.