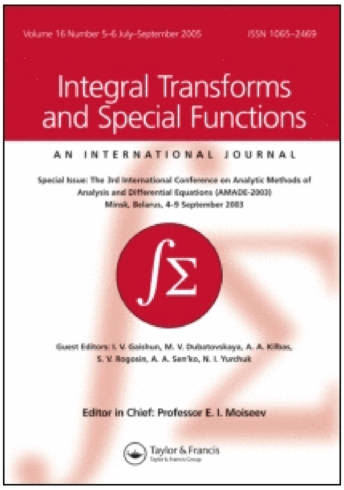

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713643686

Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville

Manuel Bronstein; Robert M. Corless [a]; James H. Davenport [b]; D. J. Jeffrey [a]
[a] Ontario Research Centre for Computer Algebra and the Department of Applied Mathematics, University of Western Ontario, London, Canada [b] Department of Mathematical Sciences, University of Bath, Bath, UK

Online Publication Date: 01 January 2008

To cite this Article Bronstein, Manuel, Corless, Robert M., Davenport, James H. and Jeffrey, D. J.(2008)'Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville',Integral Transforms and Special Functions,19:10,709 — 712

To link to this Article: DOI: 10.1080/10652460802332342

URL: http://dx.doi.org/10.1080/10652460802332342

Integral Transforms and Special Functions
Vol. 19, No. 10, October 2008, 709–712

Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville

Manuel Bronstein[†], Robert M. Corless[a], James H. Davenport[b] and D.J. Jeffrey[a]*

[a]*Ontario Research Centre for Computer Algebra and the Department of Applied Mathematics, University of Western Ontario, London, Canada, N6A 5B7;* [b]*Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK*

(*Received 14 July 2008; final version received 28 July 2008*)

It is shown that the Lambert W function cannot be expressed in terms of the elementary, Liouvillian, functions. The proof is based on a theorem due to Rosenlicht. A related function, the Wright ω function, is similarly shown to be not Liouvillian.

Keywords: implicitly elementary functions; transcendental equations; differential fields

MCS numbers: 33E30; 11J93

The Lambert W function [5,9] is a multi-valued function defined as the solution of

$$W(x)e^{W(x)} = x, \tag{1}$$

one of the simplest possible non-algebraic equations. The Wright ω function [4] also satisfies a simple transcendental equation (away from its discontinuities):

$$\omega(x) + \ln \omega(x) = x. \tag{2}$$

Both of these functions are implicitly elementary, in the sense discussed by Risch in [7]. One can ask whether there are explicit formulations of those functions in terms of known functions or whether they are genuinely new functions. A common class of 'well-known' functions are the Liouvillian functions.

Definition 1 *Let* $(k,')$ *be a differential field of characteristic* 0. *A differential extension* $(K,')$ *of* k *is called* Liouvillian *over* k *if there are* $\theta_1, \ldots, \theta_n \in K$ *such that* $K = C(x, \theta_1, \ldots, \theta_n)$ *and*

*Corresponding author. Email: djeffrey@uwo.ca
[†]This paper is dedicated to the memory of Manuel Bronstein (1963–2005).

ISSN 1065-2469 print/ISSN 1476-8291 online

DOI: 10.1080/10652460802332342
http://www.informaworld.com

for all i, at least one of the following holds:

(1) θ_i *is algebraic over* $k(\theta_1, \ldots, \theta_{i-1})$;
(2) $\theta_i' = \eta$ *for some* $\eta \in k(\theta_1, \ldots, \theta_{i-1})$;
(3) $\theta_i'/\theta_i = \eta$ *for some* $\eta \in k(\theta_1, \ldots, \theta_{i-1})$.

We say that $f(x)$ *is a* Liouvillian function *if it lies in some Liouvillian extension of* $(C(x), \mathrm{d}/\mathrm{d}x)$ *for some constant field* C.

It turns out that the possible closed-form expressions for solutions of equations of the form (1) and 2) were already studied by Liouville [6], who was certainly able to prove already that $W(x)$ is not a Liouvillian function. In any event, this result was known to Rosenlicht, who published in [8] a proposition that can be applied to prove easily that $W(x)$ and $\omega(x)$ (or many functions defined by similar transcendental equations) are not Liouvillian. Yet, questions about whether $W(x)$ is elementary or Liouvillian appear in the literature [3], possibly because Rosenlicht's paper is not as well-read as it deserves to be, so we illustrate in this note how Rosenlicht's theorem can prove that neither $W(x)$ nor $\omega(x)$ is Liouvillian.

We start by recalling Rosenlicht's result.

PROPOSITION 1 [8, Proposition, p. 21] *Let* k *be a differential field of characteristic* 0 *and let* $y_1, \ldots, y_n, z_1, \ldots, z_n$ *be elements of a Liouvillian extension of* k *having the same subfield of constants as* k. *Suppose that*

$$\frac{y_i'}{y_i} = z_i', \quad i = 1, \ldots, n,$$

and that $k(y_1, \ldots, y_n, z_1, \ldots, z_n)$ *is algebraic over each of its subfields* $k(y_1, \ldots, y_n)$ *and* $k(z_1, \ldots, z_n)$. *Then,* $y_1, \ldots, y_n, z_1, \ldots, z_n$ *are all algebraic over* k.

An immediate consequence of the case $n = 1$ of that proposition is that if $W(x)$ and $\omega(x)$ are Liouvillian functions, then they must be algebraic functions: suppose that W belongs to a Liouvillian extension K of $\mathbb{C}(x)$. Take $k = C(x)$ where C is the constant subfield of K, then K is Liouvillian over k and both fields have the same subfield of constants. Taking logarithmic derivatives on both sides of Equation (1) yields

$$\frac{W'}{W} + W' = \frac{1}{x}, \tag{3}$$

whence $y'/y = W'$ where $y = x/W \in K$. Since $k(y, W) = k(y) = k(W)$, Rosenlicht's theorem implies that W is algebraic over $k = C(x)$. The proof is similar for $\omega(x)$: differentiating both sides of Equation (2) yields $\omega' + \omega'/\omega = 1$, whence $\omega'/\omega = z'$ where $z = x - \omega$. Since $k(\omega, z) = k(\omega) = k(z)$, Rosenlicht's theorem implies that ω is algebraic over $k = C(x)$.

There are obvious analytic arguments why $W(x)$ and $\omega(x)$ cannot be algebraic functions, so they cannot be Liouvillian functions: if $W(x)$ has a pole of finite order, then $\mathrm{e}^{W(x)}$, and therefore $W(x)\mathrm{e}^{W(x)}$, has an essential singularity, so $W(x)\mathrm{e}^{W(x)}$ cannot equal x. Similarly, if $\omega(x)$ has a zero, then $\ln \omega(x)$, and therefore $\omega(x) + \ln \omega(x)$, has a logarithmic singularity, so $\omega(x) + \ln \omega(x)$ cannot equal x. Since algebraic functions with either no pole or no zero must be constants, and $W(x)$ and $\omega(x)$ cannot be constant, they cannot be algebraic.

The above argument can be cast in algebraic terms. Since Rosenlicht proved his result algebraically, we outline the algebraic proof that $W(x)$ and $\omega(x)$ cannot be algebraic functions.

Note that Equation (3) implies that $y = W(x)$ is a solution of the differential equation

$$xy'(1 + y) = y. \tag{4}$$

We first recall some notations and results from [2]: we say that a field E is an algebraic function field of one variable over a subfield $F \subset E$ if

- E is of transcendence degree 1 over F,
- for any $t \in E$ transcendental over F, $[E : F(t)]$ is finite.

By an F-place of E, we then mean the maximal ideal of a valuation ring of E containing F. For such a place p, we write $\nu_p : E^* \to \mathbb{Z}$ for its order function. It has, in particular, the following properties:

- $\nu_p(c) = 0$ for any $c \in \overline{F} \cap E^*$.
- $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and $\nu_p(a + b) \geq \min(\nu_p(a), \nu_p(b))$ for any $a, b \in E^*$.
- $\nu_p(a + b) = \min(\nu_p(a), \nu_p(b))$ for any $a, b \in E^*$ such that $\nu_p(a) \neq \nu_p(b)$.
- For any $a \in E^*$, if $\nu_p(a) \geq 0$ at all the F-places of E, then a is algebraic over F.

Let now $t \in E$ be transcendental over F and p be any F-place of E. We write $r_t(p) \in \mathbb{Z}_{>0}$ for the ramification index of p over $F(t)$. In addition, we call the place p *infinite* (w.r.t. t) if $t^{-1} \in p$, *finite* (w.r.t. t) otherwise. A finite place p contains a unique monic irreducible $P \in F[t]$, called the *center of p* (w.r.t. t).

PROPOSITION 2 *Let $(F, ')$ be a differential field containing an element x such that $x' = 1$. If F has transcendence degree 1 over its constant subfield, then the only solution $y \in F$ of Equation (4) is $y = 0$.*

Proof Let C be the constant subfield of F and suppose that F has transcendence degree 1 over C. Since $x' = 1$, x is transcendental over C, so F is algebraic over $C(x)$. Let $y \in F$ be a non-zero solution of Equation (4) and $E = \bar{C}(x, y)$, which is an algebraic function field of one variable over $\bar{C}$. Let p be any $\bar{C}$-place of E. Applying ν_p on both sides of Equation (4), we get

$$\nu_p(x) + \nu_p(y') + \nu_p(1 + y) = \nu_p(y). \tag{5}$$

Suppose that $\nu_p(y) < 0$. Then, $\nu_p(1 + y) = \min(0, \nu_p(y)) = \nu_p(y)$ and Equation (5) becomes

$$\nu_p(x) + \nu_p(y') = 0. \tag{6}$$

If p is finite w.r.t. x, then $\nu_p(x) \geq r_x(p)$. But Lemma 1.7 of [1] implies that $\nu_p(y') = \nu_p(y) - r_x(p) < -r_x(p)$, in contradiction with Equation (6). If p is infinite, then $\nu_p(x) = -r_x(p)$. But Lemma 1.8 of [1] implies that $\nu_p(y') \leq \nu_p(y) + r_x(p) < r_x(p)$, in contradiction with Equation (6). Therefore, $\nu_p(y) \geq 0$ at all the $\bar{C}$-places of E, which implies that $y \in \bar{C}$, hence that $y' = 0$, and Equation (4) becomes $0 = y$. ■

Since the only algebraic solution of Equation (4) is 0, which is not a solution of Equation (1), $W(x)$ cannot be algebraic, hence it cannot be a Liouvillian function.

The proof that $\omega(x)$ is not an algebraic function is similar, since $y = \omega(x)$ is a solution of the differential equation $y'(1 + y) = y$. The equalities (5) and (6) become, respectively, $\nu_p(y') + \nu_p(1 + y) = \nu_p(y)$ and $\nu_p(y') = 0$, and the proof of Proposition 2 remains valid.

References

[1] M. Bronstein, *Integration of elementary functions*, J. Symbolic Comput. 9(2) (1990), pp. 117–174.
[2] C. Chevalley, *Algebraic Functions of One Variable*, American Mathematical Society, New York, 1951.
[3] T. Chow, *What is a closed-form number?*, Amer. Math. Monthly 106(5) (1999), pp. 440–448.
[4] R.M. Corless and D.J. Jeffrey, *The Wright ω function*, in *Artificial Intelligence, Automated Reasoning, and Symbolic Computation*, J. Calmet, B. Benhamou, O. Caprotti, L. Henocque, and V. Sorge, eds., AISC-Calculemus 2002, LNAI 2385, Springer, 2002, pp. 76–89.
[5] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, *On the Lambert W function*, Adv. Comput. Math. 5 (1996), pp. 329–359.
[6] J. Liouville, *Mémoire sur la classification des transcendantes et sur l'impossibilité d'exprimer les racines de certaines équations en fonction finie explicite des coefficients*, J. Math. Pures Appl. 2 (1837), pp. 56–105; 3 (1838), pp. 523–547.
[7] R.H. Risch, *Implicitly Elementary Integrals*, Proc. Amer. Math. Soc. 57(1) (1976), pp. 1–7.
[8] M. Rosenlicht, *On the explicit solvability of certain transcendental equations*, Publ. Math. Inst. Haute Etudes Sci. 36 (1969), pp. 15–22.
[9] E.M. Wright, *Solution of the equation* $ze^z = a$, Bull. Amer. Math. Soc. 65 (1959), pp. 89–93.