Integration of the signum, piecewise and related functions

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Abstract

When a computer algebra system has an assumption facility, it is possible to distinguish between integration problems with respect to a real variable, and those with respect to a complex variable. Here, a class of integration problems is defined in which the integrand consists of compositions of continuous functions and signum functions, and integration is with respect to a real variable. Algorithms are given for evaluating such integrals.

1 Introduction

In recent years, 'assume' or 'declare' facilities have been implemented in most of the available computer algebra systems (CAS). As well, such facilities have been gaining wider acceptance within the user community. The presence of these facilities has altered the way CAS behave, and many established areas of symbolic computation need to be reconsidered. The topic of this paper is an example of the impact on one traditional field of computer algebra, namely, symbolic integration.

Because the early versions of many present-day CAS could not record the domain of a variable, they assumed that the variable was complex. In particular the problem $\int f(x)\,\mathrm{d}x$ was usually interpreted as requiring the evaluation of a complex integral, valid for $x \in \mathbb{C}$. This immediately ruled out the possibility of formulating problems such as $\int |x|\,\mathrm{d}x$, because the absolute value function is not differentiable in the complex plane. With domain information available, it is possible to specify an integration problem

$$\int f(x)\,\mathrm{d}x \qquad \text{for} \quad x \in \mathbb{R}\ ,$$

and one consequence of this is the possibility that the above problem may have a different answer from the problem

$$\int f(x)\,\mathrm{d}x \qquad \text{for} \quad x \in \mathbb{C}\ .$$

For example, consider the integral

$$\int 3x^2\sqrt{1+\frac{1}{x^2}}\,\mathrm{d}x\ .$$

Maple V and Mathematica evaluate this integral as

$$\int 3x^2\sqrt{1+\frac{1}{x^2}}\,\mathrm{d}x = (x^3+x)\sqrt{1+\frac{1}{x^2}}\ , \tag{1}$$

because they assume $x \in \mathbb{C}$. In contrast, Derive and Macsyma assume $x \in \mathbb{R}$ and return

$$\int 3x^2\sqrt{1+\frac{1}{x^2}}\,\mathrm{d}x = [(x^2+1)^{3/2}-1]\operatorname{sgn} x\ . \tag{2}$$

Maple can also obtain this answer, as will be shown later. Both answers are correct, and the difference lies in the assumptions. Notice in particular that the integrand in equation (**1**) is continuous on $\mathbb{R}$, but the right side of (**2**) is discontinuous at $x = 0$.

The example just given can be treated as a member of the class of integral problems studied here. In this paper, we consider some classes of integration problems obtained through the composition of continuous functions and signum functions, or equivalents and discuss the implementations in both Derive and in Maple V.

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The interest in this class of problems arises because functions that have piecewise definitions are widely used in engineering, physics, and other areas. Such functions are often constructed explicitly by users of CAS to represent discontinuous processes. They can also appear as the result of algebraic simplifications performed by a CAS on an integrand, even if that integrand contained no signum functions explicitly when first presented. An important feature of the computations discussed here is the fact that they ensure that the expressions obtained are valid on domains of maximum extent.

We remark that functions equivalent to signum are supported by all the major CAS. However, the support takes various forms and the definitions used by the different systems are not completely equivalent. Examples include the SIGN function in Derive, the `signum` and `piecewise` functions in Maple V and the `UnitStep` in Mathematica.

2 Definitions of functions

The signum function is defined differently in each of the major CAS. This is not really surprising given that different areas of mathematics also use different definitions of a signum function. However, these disagreements do not affect the integration question, and a discussion of variations would only distract attention from the main problem. Therefore one particular definition, and a specific unambiguous notation, is used here, so that the issue of variations in definition does not intrude on this discussion of integration. A signum function $S_{nn} : \mathbb{R} \to \mathbb{R}$ that is 1 for all non-negative real numbers, briefly an n-n signum, is defined by

$$S_{nn}(x) = \begin{cases} 1\ , & \text{for } x \geq 0, \\ -1\ , & \text{for } x < 0. \end{cases} \tag{3}$$

Notice that $S_{nn}(x)$ is antisymmetric only on $\mathbb{R}\backslash\{0\}$. Some comments on the implementation of this definition will be made below.

The functions *absolute value* and *Heaviside step* are defined in terms of S_{nn} by

$$|x| = xS_{nn}(x) \quad \text{and} \quad H(x) = \tfrac{1}{2} + \tfrac{1}{2}S_{nn}(x)\ . \tag{4}$$

The characteristic function χ of a closed interval $[a,b] \subset \mathbb{R}$ is defined also in terms of S_{nn}:

$$\chi(x,[a,b]) = \tfrac{1}{2}S_{nn}(x-a) + \tfrac{1}{2}S_{nn}(b-x)\ . \tag{5}$$

Notice that this definition implies that a point function, nonzero only at a point a, can be defined as $\chi(x,[a,a])$. It is also useful to define the characteristic function of an open interval $]a,b[$.

$$\chi(x,]a,b[) = \chi(x,[a,b]) - \chi(x,[a,a]) - \chi(x,[b,b])\ . \tag{6}$$

For semi-infinite intervals, the χ function reverts to one equivalent to S_{nn}.

An alternative to signum functions has been introduced by Maple. Maple V release 4 defines the function `piecewise` by

$$\mathtt{piecewise}(c_1, f_1, \ldots, c_n, f_n, f) = \begin{cases} f_1\ , & c_1 \text{ true}, \\ \ldots & \\ f_n\ , & c_n \text{ true}, \\ f, & \text{otherwise}, \end{cases} \tag{7}$$

where the c_i are Boolean expressions of the Maple type *relation* and the f_i are algebraic expressions. The relations c_i are evaluated in order from left to right, until one is found to be true. In terms of this function, S_{nn} is

$$S_{nn}(x) = \mathtt{piecewise}(x < 0, -1, 1)\ . \tag{8}$$

Since `piecewise` is more general than signum, the converse is more lengthy. Let condition c_i be true on a union of disjoint intervals $I_i = \bigcup_j I_{ij}$, where each $I_{ij} \subset \mathbb{R}$, and let $J_i = I_i \backslash \bigcup_{k<i} I_k$, then a `piecewise` function can be expressed as a sum of χ functions.

$$\mathtt{piecewise}(c_1, f_1, \ldots, f) = f + \sum_i (f_i - f)\chi(x, J_i)\ . \tag{9}$$

Maple V can make a similar conversion of a piecewise function to a sum of Heaviside functions

$$f(x) = f_0(x) + \sum_{i=1}^{p} f_i(x) H(x - a_i). \tag{10}$$

3 Definition of integration

The example in the introduction showed that different definitions of integration are possible. Therefore it is necessary to define the integration problem and verify the existence of a solution.

Definition 1. Let $f : [a,b] \to \mathbb{R}$ be a function that is continuous except at the n points $\mathbb{D} = \{x_1^{\mathrm{b}}, x_2^{\mathrm{b}}, \ldots, x_n^{\mathrm{b}}\}$ where $x_i^{\mathrm{b}} \in [a,b]$. The function f is then piecewise continuous on $[a,b]$, and the points x_i^{b} are the break points of f. □

Definition 2. If the left and right limits of a piecewise continuous function f separately exist at a break point x^{b}, then x^{b} is called a bounded break point of f, otherwise it is called an unbounded breakpoint. □

Remark 1. A bounded break point is also called a jump discontinuity. There is a possibility that the terms bounded and unbounded will be taken to refer to number of breakpoints rather than the behaviour of the function at a particular breakpoint; in the former case it is the set $\mathbb{D}$ that is bounded or not, rather than the point itself. □

If x_i^{b} is a bounded breakpoint of f, then we denote the left and right limits of f at x_i^{b} by $f(x_i^{\mathrm{b}}-)$ and $f(x_i^{\mathrm{b}}+)$. In the case of bounded breakpoints a continuous integral always exists.

Theorem 1. Let $f : [a,b] \to \mathbb{R}$ be a piecewise-continuous function with a set $\mathbb{D}$ of bounded breakpoints. There exists a function g, called an integral of f on $[a,b]$, written

$$\int f(x)\,\mathrm{d}x = g(x)\ , \tag{11}$$

with the properties that g is continuous on $[a,b]$ and g is differentiable on $[a,b]\backslash\mathbb{D}$, where its derivative is $g' = f$. □

Remark 2. If g_1 and g_2 are integrals of f on $[a,b]$, then $g_1 = g_2 + K$ for some constant K. For on any open subinterval defined by successive breakpoints, namely $]x_i^{\mathrm{b}}, x_{i+1}^{\mathrm{b}}[$, the functions differ by a constant, and one can write $g_1 - g_2 = K_i$. Since g_1 and g_2 are separately continuous at each breakpoint,

$$\begin{aligned} K_i &= g_1(x^{\mathrm{b}}_{i+1}-) - g_2(x^{\mathrm{b}}_{i+1}-) \\ &= g_1(x^{\mathrm{b}}_{i+1}+) - g_2(x^{\mathrm{b}}_{i+1}+) = K_{i+1} \ . \end{aligned} \tag{12}$$

□

Remark 3. Clearly there exist functions g with the property that $g' = f$ on $[a,b]\backslash\mathbb{D}$, but without the property that g is continuous on $[a,b]$. Such functions are called *anti-derivatives* or *primitives* of f, but are not called integrals, the last term being reserved for functions continuous on $[a,b]$. □

Theorem 2. Let $f : [a,b] \to \mathbb{R}$ be a piecewise continuous function with a set $\mathbb{D}$ of unbounded breakpoints. For each closed subinterval $I_i = [x^{\mathrm{b}}_{i-1}, x^{\mathrm{b}}_i]$, let the quantity

$$\int_{I_i} f(x)dx$$

exist in the sense of improper integrals. There exists a function g, called an integral of f on $[a,b]$, which is continuous on $[a,b]$, and for which $g' = f$ on $[a,b]\backslash\mathbb{D}$.
Proof: Let g_i be an integral of f on $]x^{\mathrm{b}}_{i-1}, x^{\mathrm{b}}_i[$. By definition, $\hat{g}_i = g_i - g_i(x^{\mathrm{b}}_{i-1}+)$ exists. The function

$$g(x) = \sum_i \left(\hat{g}_i(x)\chi(x, [x^{\mathrm{b}}_{i-1}, x^{\mathrm{b}}_i[) + \sum_{j<i} \hat{g}_j(x^{\mathrm{b}}_j-) \right)$$

is the required function. The definition of χ is an obvious extension of the one given above. □

A complete problem specification is therefore as follows. Given a piecewise-continuous function f, *together* with an interval $[a,b]$ on which it is integrable, find g, an integral of f on $[a,b]$. In practice, users do not specify the interval $[a,b]$ when posing indefinite integration problems to CAS, and furthermore no CAS offers the syntax to accept such a specification. Therefore we now define the domain of maximum extent [1] in order to overcome this difficulty.

Definition 3: Given a function $f : \mathbb{R} \to \mathbb{R}$ that is integrable on domains $C_i \subset \mathbb{R}$, a function g is an integral on the domain of maximum extent if g is continuous everywhere on $\bigcup C_i$ and differentiable almost everywhere on $\bigcup C_i$, and moreover $g' = f$ almost everywhere on $\bigcup C_i$. □

Therefore the modified problem definition requires that in the absence of a specified $[a,b]$, the integral should be valid on the domain of maximum extent. If the function f is integrable only on the open subintervals $I_i =]x_{i-1}, x_i[$, then all one has to find are the functions g_i such that $g_i' = f$ on I_i.

4 Integration rules for signum

For reasons of efficiency, and because different CAS offer different syntax to users, it is necessary to develop several related algorithms for integration. We begin with integrands expressed using signum.

Lemma 3. Let $\{s_i | i = 1..p\}$, $p \in N$, be a set of constants, each taking one of the values ± 1. Let $f(x, \{s_i | i = 1..p\})$ be a function that is continuous for fixed $\{s_i\}$ on $[a,b]$. Let $\{S_{nn}(\alpha_i x + \beta_i) | i = 1..p\}$ be a set of signum functions with $\forall i, \alpha_i \neq 0$, then

$$\int f(x, \{S_{nn}(\alpha_i x + \beta_i) | i = 1..p\}) \, \mathrm{d}x = \int \hat{f}(x, \{S_{nn}(x - x^{\mathrm{b}}_i) | i = 1..p\}) \, \mathrm{d}x \ , \tag{13}$$

where $\hat{f}(x, \{s_i | i = 1..p\}) = f(x, \{S_{nn}(\alpha_i) s_i | i = 1..p\})$ and $x^{\mathrm{b}}_i = -\beta_i/\alpha_i$.
Proof. For $\alpha_i > 0$, the function $S_{nn}(\alpha_i x + \beta_i)$ and the function $S_{nn}(\alpha_i) S_{nn}(x - x^{\mathrm{b}}_i)$ are identical. For $\alpha_i < 0$, the two functions differ only at $x = x^{\mathrm{b}}_i$, and functions that are bounded and equal almost everywhere have the same integral. The redefinition of f is trivial. Clearly, the points x^{b}_i are breakpoints of $\hat{f}$. □

We consider first the integration of a function having one breakpoint and then generalize the result to many breakpoints.

Theorem 4: Let $x \in [a,b]$ and $s = \pm 1$, and let $f(x,s)$ be a function that is continuous with respect to x on $[a,b]$. Let $g(x,s)$ be an integral of f with respect to x on $[a,b]$. Then

$$\begin{aligned} \int f(x, S_{nn}(x - x^{\mathrm{b}})) \, \mathrm{d}x &= G(x) \\ &= g(x, S_{nn}(x - x^{\mathrm{b}})) - J S_{nn}(x - x^{\mathrm{b}}) \ , \end{aligned} \tag{14}$$

where

$$J = \tfrac{1}{2} g(x^{\mathrm{b}}, 1) - \tfrac{1}{2} g(x^{\mathrm{b}}, -1) \ .$$

Proof: We write $\partial_x = \partial/\partial_x$. For the case $x^{\mathrm{b}} \notin [a,b]$, $S_{nn}(x - x^{\mathrm{b}})$ is constant on $[a,b]$, and

$$\partial_x g(x, S_{nn}(x - x^{\mathrm{b}})) = f(x, S_{nn}(x - x^{\mathrm{b}})) \ .$$

If $x^{\mathrm{b}} \in [a,b]$, then for $x > x^{\mathrm{b}}$, $S_{nn}(x - x^{\mathrm{b}}) = 1$. Therefore for $x \in]x^{\mathrm{b}}, b]$,

$$\begin{aligned} &\partial_x[g(x, S_{nn}(x - x^{\mathrm{b}})) - J S_{nn}(x - x^{\mathrm{b}})] \\ &= \partial_x[g(x,1) - J] = f(x,1) \\ &= f(x, S_{nn}(x - x^{\mathrm{b}})) \ . \end{aligned}$$

On $[a, x^{\mathrm{b}}[$, subject to the change $S_{nn}(x - x^{\mathrm{b}}) = -1$, the same proof applies. At the point $x = x^{\mathrm{b}}$, the right-hand side of (**14**) must be continuous. The right limit of $G(x)$ at x^{b} is

$$\begin{aligned} &\lim_{x \to x^{\mathrm{b}}+} [g(x, S_{nn}(x - x^{\mathrm{b}}) - J S_{nn}(x - x^{\mathrm{b}})] \\ &= \lim_{x \to x^{\mathrm{b}}+} [g(x,1) - J] \\ &= g(x^{\mathrm{b}}+, 1) - J \ . \end{aligned}$$

Since $g(x,s)$ is the integral of $f(x,s)$, $g(x^{\mathrm{b}}+, 1) = g(x^{\mathrm{b}}, 1)$, and the limit is

$$\lim_{x \to x^{\mathrm{b}}+} G(x) = \tfrac{1}{2} g(x^{\mathrm{b}}, 1) + \tfrac{1}{2} g(x^{\mathrm{b}}, -1) \ .$$

A similar calculation shows the left limit has the same value. Finally, substituting $x = x^{\mathrm{b}}$ into the right-hand side of (14) gives

$$\begin{aligned} G(x^{\mathrm{b}}) &= g(x^{\mathrm{b}}, S_{nn}(0)) - JS_{nn}(0) = g(x^{\mathrm{b}},1) - J \\ &= \tfrac{1}{2}g(x^{\mathrm{b}},1) + \tfrac{1}{2}g(x^{\mathrm{b}},-1) \ . \end{aligned}$$

Therefore, G is continuous as required. Notice that for many other signum functions, in particular for the one defining $\mathrm{sgn}(0) = 0$, the theorem would not be true. □

We now generalize the theorem to integrands containing a finite number of signum functions.

Theorem 5: Let $f(x, \{s_i | i = 1..p\})$ be a function that is continuous with respect to x on $[a, b]$, and let $\{x_i^{\mathrm{b}} | i = 1..p\}$ be a set of distinct points with $x_1^{\mathrm{b}} > x_2^{\mathrm{b}} > \ldots > x_p^{\mathrm{b}}$. Let $g(x, \{s_i | i = 1..p\})$ be an integral of f on $[a, b]$, then

$$\begin{aligned} \int f(x, S_{nn}(x - x_i^{\mathrm{b}}) | i = 1..p\})dx & \\ = G(x) = g(x, \{S_{nn}(x - x_i^{\mathrm{b}}) | i = 1..p\}) & \\ - \sum_{i=1}^{p} J_i S_{nn}(x - x_i^{\mathrm{b}}) \ , & \qquad (15) \end{aligned}$$

where

$$\begin{aligned} J_j &= \tfrac{1}{2}g(x_j^{\mathrm{b}}, S_j^<, 1, S_j^>) - \tfrac{1}{2}g(x_j^{\mathrm{b}}, S_j^<, -1, S_j^>) \ , \\ S_j^< &= \{\mathrm{sgn}(x_j^{\mathrm{b}} - x_i^{\mathrm{b}}) = -1 | i < j\} \ , \\ S_j^> &= \{\mathrm{sgn}(x_j^{\mathrm{b}} - x_i^{\mathrm{b}}) = 1 | i > j\} \ . \end{aligned}$$

Proof. Let the interval $[a, b]$ be partitioned into subintervals $[a_i, a_{i+1}]$, where $a_0 = a$, $x_i^{\mathrm{b}} < a_i < x_{i+1}^{\mathrm{b}}$ and $a_{p+1} = b$. By the previous theorem, (15) is an integral on each $[a_i, a_{i+1}]$, and by construction and hypothesis it is continuous on each $[x_i^{\mathrm{b}}, x_{i+1}^{\mathrm{b}}]$, therefore it is continuous and an integral on $[a, b]$. □

In the case of integration on an unspecified domain we require.

Theorem 6. If $f(x, \{s_i | i = 1..p\})$ is integrable with respect to x, for $\{s_i\}$ fixed and equal to ± 1, on domains $R_i \subset R$, and $g(x, \{s_i\})$ is the integral of f on the domain of maximum extent, i.e. $\bigcup R_i$, then G(x) defined in (15) is also an integral on the domain of maximum extent, provided $J_i = 0$ if $x_i^{\mathrm{b}} \notin \bigcup R_i$. □

The next theorem applies to the case in which an integrable singularity coincides with the breakpoint of a signum function.

Theorem 7. Let $x \in [a, b]$ and $s = \pm 1$, and let $f(x, s)$ be a function such that f is continuous with respect to x on $[a, b]$ except at $x_b \in [a, b]$. Let $f(x, s)$ be integrable at x^{b}. Then

$$\int f(x, S_{nn}(x - x^{\mathrm{b}}))dx = G(x) = g(x, S_{nn}(x - x^{\mathrm{b}})) \ , \quad (16)$$

where $g(x, s)$ is an integral of $f(x, s)$ on $[a, b]$ subject to $g(x^{\mathrm{b}}, s) = 0$.

Proof. Let $g_a(x, s) = \int_a^x f(y, s)dy$. By theorem 3, g_a is continuous on $[a, b]$. Define $g(x, s) = g_a(x, s) - g_a(x^{\mathrm{b}}, s)$. For $x < x^{\mathrm{b}}$,

$$\partial_x g(x, S_{nn}(x - x^{\mathrm{b}})) = \partial_x g(x, -1) = f(x, S_{nn}(x - x^{\mathrm{b}})) \ ,$$

and likewise for $x > x^{\mathrm{b}}$. At x^{b},

$$\lim_{x \to x^{\mathrm{b}}-} G(x) = \lim_{x \to x^{\mathrm{b}}+} = G(x^{\mathrm{b}}) = 0$$

□

5 Integration of signum

The above theorems can be summarized in the following algorithm. Given an integral with respect to a real variable containing signum or Heaviside functions, the algorithm used by Derive roughly proceeds as follows.

1. Use the definitions (4) to convert Heaviside to signum functions.
2. Check each signum has a linear argument, and replace each $\mathrm{sgn}(\alpha_i x + \beta_i)$ with $S_{nn}(\alpha_i) S_{nn}(x - x_i^{\mathrm{b}})$. If any signums contain other arguments, the algorithm fails.
3. Order the breakpoints so that the integrand is in the form $\hat{f}(x, \{S_{nn}(x - x_k^{\mathrm{b}}) | k = 1..p\})$ where p is an integer, the $\{x_k^{\mathrm{b}}\}$ are the ordered breakpoints of the integrand, with $x_1^{\mathrm{b}} < x_2^{\mathrm{b}} < \ldots < x_p^{\mathrm{b}}$. Further the function $\hat{f}(x, \{s_k | k = 1..p\})$, with the $\{s_k\}$ being symbolic constants, contains no signum function.
4. Pass the function $\hat{f}(x, \{s_k\})$ to the system integrator. Assume it returns a function $g(x, \{s_k\})$, else FAIL.
5. For k from 1 to p, compute

   $$J_k = \tfrac{1}{2}G(x_k^{\mathrm{b}}, \{S^<, 1, S^>\}) - \tfrac{1}{2}G(x_k^{\mathrm{b}}, \{S^<, -1, S^>\})$$

   where $S^<$ is a set of $k-1$ entries equal to 1 and $S^>$ is a set of $p - k$ entries -1.
6. Return the expression

   $$G(x, \{S_{nn}(x - x_k^{\mathrm{b}}) | k = 1..p\}) - \sum_{i=1}^{p} J_k S_{nn}(x - x_k^{\mathrm{b}}) \ .$$

□

Example 1. Consider the example given in the introduction. This integral is evaluated as follows.

$$\int 3x^2 \sqrt{1 + 1/x^2}\, \mathrm{d}x = \int 3x\, \mathrm{sgn}(x) \sqrt{x^2 + 1}\, \mathrm{d}x \ .$$

In the integrand, we have left the traditional signum notation. Now, since

$$\int 3xs\sqrt{x^2 + 1}\, \mathrm{d}x = s(x^2 + 1)^{3/2} \ ,$$

we find $J = 2$ and hence obtain

$$\int 3x^2\sqrt{1+1/x^2}\,\mathrm{d}x = S_{nn}(x)\left[(1+x^2)^{3/2}-1\right] \ .$$

Continuity at the origin has already been noted. □

Example 2. In this example we illustrate the need for definition (3). For the integral of $(x+2)^{1+\operatorname{sgn} x}$, $f(x,s) = (x+2)^{1+s}$ and $g = (x+2)^{2+s}/(2+s)$ and $J = 2/3$. Thus

$$\int (x+2)^{1+\operatorname{sgn} x} dx = \frac{(x+2)^{2+S_{nn}(x)}}{2+S_{nn}(x)} - \frac{S_{nn}(x)}{3} \qquad (17)$$

At $x = 0$, this evaluates to the correct $5/3$ using definition (1). In contrast, the other common definition in which $\operatorname{sgn}(0) = 0$ would yield the value 2 and hence create a removable discontinuity at $x = 0$. □

Example 3. The algorithm relies on the underlying integration system to return a continuous expression for $g(x,s)$, in the notation of the theorems. For example, the result

$$\int \frac{3\operatorname{sgn}(x-\pi)}{5-4\cos x}dx = \left(x - \pi + 2\arctan\frac{\sin x}{2-\cos x}\right) S_{nn}(x-\pi)$$

cannot be obtained if the system computes the integration: $\int 3s/(5-4\cos x)dx = 2s\arctan(3\tan(x/2))$. □

Example 4. Note that there is no difficulty if the integrand is singular at the break point of a signum. For example,

$$\int \frac{\operatorname{sgn} x\ \mathrm{d}x}{x^{1/3}} = \tfrac{3}{2}x^{2/3}S_{nn}(x) \ ,$$

where the fractional powers are interpreted as real-valued. □

Example 5. In this example we use Heaviside functions set in the context of a simple differential equation, even though the present algorithm applies only to integration. From engineering beam theory, the bending moment $M(x)$ in a beam that extends from $x = 0$ to $x = l$ and supports point loads P_a and P_b at $x = a$ and $x = b$ is given by the equation

$$\frac{dM}{dx} = \begin{cases} K, & \text{for } 0 \le x \le a \ ; \\ K + P_a, & \text{for } a \le x \le b \ ; \\ K + P_a + P_b, & \text{for } b \le x \le l \ . \end{cases}$$

Here K is a constant to be determined from the boundary conditions, which are $M(0) = M(l) = 0$ for the case of free ends. Engineers commonly solve equations like this using a specialised system of notation called Macaulay brackets [2], which essentially develop a subset of the results above. Instead, the equation is written

$$\frac{dM}{dx} = K + P_a H(x-a) + P_b H(x-b) \ , \qquad (18)$$

and integrated. The integral of the Heaviside function is

$$\int H(x-a)\ dx = \int (\tfrac{1}{2} + \tfrac{1}{2}S_{nn}(x-a))\ dx = (x-a)H(x-a)$$

and then integrating gives

$$M = Kx + M_0 + P_a(x-a)H(x-a) + P_b(x-b)H(x-b) \ .$$

Since $M(0) = 0$, we have $M_0 = 0$, and $M(l) = 0$ gives

$$K = -[P_a(l-a) + P_b(l-b)]/l$$

The result for M can be integrated further to obtain the displacement of the beam. □

6 Integration of Piecewise Functions in Maple

In this section we discuss the algorithm used in Maple for integration of piecewise functions. We also show how this algorithm can be extended to cover absolute value and signum functions in the obvious way. In this case all functions are converted to Heaviside step functions which are then in turn simplified using the normal form algorithm of v. Mohrenschildt [3]. The integration is then performed on these step functions with care taken to construct the functions in order to remain continuous over as large a region as possible.

Let f be a piecewise function of the form

$$f(x) = \texttt{piecewise}(c_1, f_1(x), \ldots, c_p, f_p(x), f_{p+1}(x)) \qquad (19)$$

where the c_i are boolean combinations of linear ordering relations in one variable. Thus $c_i = x \ge a \wedge x < b$ or $c_i = \neg x \le a$ and the functions f_i are bounded in $[x^{\mathrm{b}}_{i-1}, x^{\mathrm{b}}_i]$.

To compute the integral of this piecewise functions we first convert to its Heaviside representation

$$f(x) = h_0(x) + \sum_{i=1}^{p} h_i(x)H(x - x^{\mathrm{b}}_i) \qquad (20)$$

using the rules T defined by :

$$\begin{array}{rcl} T(x > x^{\mathrm{b}}_i) & \rightarrow & H(x - x^{\mathrm{b}}_i) \ \text{(also same for } T(x \ge x^{\mathrm{b}}_i)) \\ T(x < x^{\mathrm{b}}_i) & \rightarrow & H(-x + x^{\mathrm{b}}_i) \ \text{(also same for } T(x \le x^{\mathrm{b}}_i)) \\ T(\neg c_i) & \rightarrow & 1 - T(c_1) \\ T(c_i \wedge c_j) & \rightarrow & T(c_i)T(c_j) \\ T(c_i \vee c_j) & \rightarrow & H(T(c_i) + T(c_j)) \end{array}$$

and

$$T(\texttt{piecewise}(c_1, f_1(x), c_2, \cdots, f_{p+1}(x))) \rightarrow f_{p+1}(x)$$
$$+T(c_1)(f_1(x) - f_{p+1}(x)) + T(c_2)(f_2(x) - f_1(x) - f_{p+1}(x)) + \cdots$$

Using

$$\begin{array}{rcl} H(x-a)H(x-b) & \rightarrow & H(x - \max(a,b)) \\ H(-x+a) & \rightarrow & 1 - H(x-a) \\ H(fH(x-a)+g) & \rightarrow & H(x-a)H(f+g) \\ & & +(1 - H(x-a))H(g) \end{array}$$

where in the last equation $f = \pm 1$ and g is a linear combination of H, we end up in form (20) after a finite number of reductions (c.f. v. Mohrenschildt [3]). Note, that by using the identity $H(x-a) = 1 - H(-x+a)$ we can change the value of the function in a finite number of points. However this does not alter the value of the integral. For example $\texttt{piecewise}(x > a \wedge x \le b, f(x))$, assuming $a < b$, converts to

$$f(x)H(x-a)H(-x+b)$$

which in turn reduces to

$$f(x)H(x-a) - f(x)H(x-b).$$

Lemma 9. Let $f:\mathbb{R}\to\mathbb{R}$ be specified by

$$f(x) = f_0(x) + \sum_{i=1}^{p} f_i(x)H(x-x_i^{\mathrm{b}}). \qquad (21)$$

Then an integral of this function is given by

$$g(x) = \int_{-\infty}^{\infty} f_0(x)dx + \sum_{i=1}^{p}(\hat{g}_i(x) - \hat{g}_i(x_i^{\mathrm{b}}))H(x-x_i^{\mathrm{b}}) \quad (22)$$

where

$$\hat{g}_i(x) = \int_{x_k^{\mathrm{b}}}^{x} f_i(x)dx.$$

Proof. Clearly $g' = f$, since for each i, $\hat{g}_i(x)' = f_i(x)$. Also $g(x)$ is continuous since $g(x)$ is a sum of continuous functions $(\hat{g}_i(x) - \hat{g}_i(x_i^{\mathrm{b}}))H(x-x_i^{\mathrm{b}})$. □

Once an integrand has been converted to Heaviside functions and integrated the result can be converted back to a piecewise function. Indeed an expression

$$f(x) = f_0(x) + \sum_{i=1}^{n} f_i(x)H(x-x_i^{\mathrm{b}})$$

is converted back to a piecewise representation by

$$\begin{cases} f_0(x) & x \le -x_1^{\mathrm{b}} \\ f_0(x) + f_1(x) & x \le x_2^{\mathrm{b}} \\ \dots & \dots \\ f_0(x) + \dots + f_p(x) & \end{cases}.$$

Similar conversions back are also possible in the case of signum functions and absolute value functions.

Example 7. Suppose

$$f(x) = \begin{cases} \cos(x) & x < 0 \\ \sin(x) & x < \pi \\ x^2 & x < 6 \\ \ln(x) & \text{otherwise} \end{cases}.$$

Then we convert to Heaviside, integrate and convert back:

$$\begin{aligned} f(x) &= (\sin(x) - \cos(x))\,H(x) + \left(x^2 - \sin(x)\right)H(x-\pi) \\ &\quad + \left(\ln(x) - x^2\right)H(x-6) + \cos(x) \end{aligned}$$

$$\begin{aligned} \int f(x)dx &= (1 - \sin(x) - \cos(x))\,H(x) + \left(\frac{x^3}{3} - \frac{\pi^3}{3}\right. \\ &\quad \left. + \cos(x) + 1\right)H(x-\pi) + \left(x\ln(x) - x\right. \\ &\quad \left. -6\,\ln(6) - 1/3\,x^3 + 78\right)H(x-6) + \sin(x) \end{aligned}$$

which converts to piecewise functions via

$$\int f(x)dx = \begin{cases} \sin(x) & x \le 0 \\ 1 - \cos(x) & x \le \pi \\ 2 + 1/3\,x^3 - 1/3\,\pi^3 & x \le 6 \\ 80 + x\ln(x) - x - 6\,\ln(6) - \frac{\pi^3}{3} & 6 < x \end{cases}$$

□

The example in the introduction can be handled in Maple V release 4 as follows.

```
> assume(x,real):
> f:= simplify( 3*x^2*sqrt(1+1/x^2) ) ;
          3*x*signum(x)*(x^2+1)^(1/2)
> f:= convert(f,piecewise);
```

$$\begin{cases} -3\,x\,\sqrt{x^2+1} & x \le 0 \\ 3\,x\,\sqrt{x^2+1} & 0 < x \end{cases}$$

```
> int(f,x)
```

$$\begin{cases} -\left(x^2+1\right)^{3/2} & x \le 0 \\ \left(x^2+1\right)^{3/2} - 2 & 0 < x \end{cases}$$

The last result differs from equation (2) by the constant -1.

Example 8. The assume system works together with the `piecewise` function.

```
> assume(a<b);
> f  := piecewise(x<a,-x,x<b,2*x,x>=b,1):
> int(f,x);
```

$$\begin{cases} -1/2\,x^2 & x \le a \\ x^2 - 3/2\,a^2 & x \le b \\ -3/2\,a^2 + b^2 + x - b & b < x \end{cases}$$

□

Example 9. The conditions of a `piecewise` function do not have to be linear. Maple can linearize using simplify (which in fact is the normal form conversion discussed previously). The simplify below shows the linear form.

```
> f:= piecewise( x^2<1,x^4,x^2):
> f1:= piecewise( x<2, f, x + 2);
```

$$f1 := \begin{cases} \begin{cases} x^4 & x^2 < 1 \\ x^2 & \text{otherwise} \end{cases} & x < 2 \\ x+2 & \text{otherwise} \end{cases}$$

```
> f2:= simplify(f1);
```

$$\begin{cases} x^2 & x \le -1 \\ x^4 & x \le 1 \\ x^2 & x \le 2 \\ x+2 & 2 < x \end{cases}$$

```
> int(f1,x);
```

$$\begin{cases} 1/3\,x^3 & x \le -1 \\ 1/5\,x^5 - 2/15 & x \le 1 \\ -\frac{4}{15} + 1/3\,x^3 & x \le 2 \\ -\frac{18}{5} + 1/2\,x^2 + 2\,x & 2 < x \end{cases}$$

We remark that the simplification is done automatically in the first step of the integration algorithm during the conversion to Heaviside functions. □

Example 10. The function $\mathrm{abs}(2 - \mathrm{abs}(x))$ can be integrated in two ways in Maple. The first uses the standard integrator

```
int(abs(2-abs(x)),x);
```

$$1/2\,\frac{|2-|x||\,x\,(-4+|x|)}{-2+|x|}$$

leaving a result with discontinuities at -2 and 2. By converting first to piecewise functions we get a continuous integral of the function.

```
> f := convert(abs(2-abs(x)),piecewise);
```

$$f := \begin{cases} -2-x & x \leq -2 \\ 2+x & x \leq 0 \\ 2-x & x \leq 2 \\ x-2 & 2 < x \end{cases}$$

```
> g := int(f,x);
```

$$g := \begin{cases} -2\,x-1/2\,x^2 & x \leq -2 \\ 2\,x+4+1/2\,x^2 & x \leq 0 \\ 2\,x+4-1/2\,x^2 & x \leq 2 \\ -2\,x+8+1/2\,x^2 & 2 < x \end{cases}$$

```
> convert( g, abs );
```

$$-\frac{x}{2}\,|x|+(\frac{x}{2}+1)\,|x+2|+(\frac{x}{2}-1)\,|x-2|-2\,x+4$$

Some further applications can be found in [4]. □

7 Conclusions

The existing implementations in Derive and Maple are not completely reflected in this presentation. For example, the definition $S_{nn}(x)$ is not used by Derive. In Derive, Example 1 is evaluated at $x = 0$ by taking a limit. The signum function in Maple can be modified to make it act like $S_{nn}(x)$ by setting an environment variable.

Although similar facilities may be present in other systems, we do not have access to them.

The correct integration of piecewise-continuous functions is not solely a CAS issue. The average user of a CAS has received little instruction from elementary mathematics books on working with functions as simple as $|x|$ — indeed no table of integrals contains an entry for this function — and without that background users might be slow to accept them. In addition, the integration of piecewise functions requires users to understand the difference between integration with respect to a complex variable and with respect to a real variable. There has already been a significant impact by CAS on the practice and teaching of mathematics, and piecewise-continuous functions could be another area in which CAS will lead the way.

References

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[3] Mohrenschildt, M. v., Symbolic Solutions of Discontinuous Differential Equations, *Ph.D Thesis, ETH Zürich* (1994)

[4] Mohrenschildt, M. v., A Normal Form for Function Rings of Piecewise Functions, *Tech Report CS-96-14, Univ. of Waterloo* (1996)