Modern Physics Letters A
Vol. 21, No. 4 (2006) 1–4

DIMINISHED UPPER BOUNDS ON THE UNIFICATION MASS SCALES FOR HEAVY HIGGS BOSON MASSES

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Received 23 December 2005

We consider dominant three-, four- and five-loop contributions to λ, the quartic scalar coupling-constant's β-function in the Standard Model. We find that these terms accelerate the evolution of λ to nonperturbative values, thereby lowering the unification bound for which scalar-couplings are still perburbative. We also find that these higher order contributions imply a substantial lowering of λ itself before the anticipated onset of nonperturbative physics in the Higgs sector.

The dominant running coupling constants of the standard model evolve with μ, the renormalization scale, according to two-loop renormalization group equations

$$\mu\frac{d\lambda}{d\mu} = \frac{1}{16\pi^2}\left\{4\lambda^2 + 12\lambda h^2 - 36h^4 - 9\lambda g_2^2 - \frac{9}{5}\lambda g_1^2 + \frac{81}{100}g_1^4 + \frac{27}{10}g_1^2 g_2^2 + \frac{27}{4}g_2^4\right\} + \frac{1}{(16\pi^2)^2}\left\{-\frac{26}{3}\lambda^3 - 24\lambda^2 h^2 - 3\lambda h^4 + 180h^6 + 80\lambda g_3^2 h^2 - 192h^4 g_3^2 + \cdots\right\}, \tag{1}$$

$$\mu\frac{dh}{d\mu} = \frac{1}{16\pi^2}\left\{\frac{9}{2}h^3 - 8g_3^2 h - \frac{9}{4}g_2^2 h - \frac{17}{20}g_1^2 h\right\} + \frac{1}{(16\pi^2)^2}\left\{-12h^5 - 2\lambda h^3 + \frac{1}{6}\lambda^2 h + 36g_3^2 h^3 - 108g_3^4 h + \cdots\right\}, \tag{2}$$

$$\mu\frac{dg_3}{d\mu} = \frac{1}{16\pi^2}\{-7g_3^3\} + \frac{1}{(16\pi^2)^2}\left\{-26g_3^5 - 2h^2 g_3^3 + \frac{11}{10}g_1^2 g_3^3 + \frac{9}{2}g_2^2 g_3^3 + \cdots\right\}, \tag{3}$$

$$\mu \frac{dg_2}{d\mu} = \frac{1}{16\pi^2}\left\{\frac{-19}{6} g_2^3\right\} + \frac{1}{(16\pi^2)^2}\left\{\frac{35}{6} g_2^5 + 12 g_3^2 g_2^3 + \frac{9}{10} g_1^2 g_2^3 - \frac{3}{2} h^2 g_2^3 + \cdots\right\}, \tag{4}$$

$$\mu \frac{dg_1}{d\mu} = \frac{1}{16\pi^2}\left\{\frac{41}{10} g_1^3\right\} + \frac{1}{(16\pi^2)^2}\left\{\frac{199}{50} g_1^5 + \frac{27}{10} g_2^2 g_1^3 + \frac{44}{5} g_3^2 g_1^3 - \frac{17}{10} g_1^3 h^2 + \cdots\right\}. \tag{5}$$

In the above equations the initial conditions for the gauge coupling constants g_3, g_2 and g_1 are obtained from low-energy phenomenology [$\alpha_s(M_z) = 119$, $\alpha(M_z) = 1/128$, $\sin^2\theta_w = 0.225$]. The top quark mass leads to a numerical initial value for the Yukawa coupling constant $h(\mu)$. These numerical initial conditions are

$$\begin{aligned} g_1(M_z) &= 0.4595\,, \\ g_2(M_z) &= 0.6605\,, \\ g_3(M_z) &= 1.2228\,, \\ h(M_t) &= 1.0020\,. \end{aligned} \tag{6}$$

Only λ has an unspecified initial condition. The initial value for λ may be expressed in terms of the Higgs boson mass

$$\lambda(M_{\rm H}) = 3M_{\rm H}^2/v^2\,, \tag{7}$$

where $v = 246$ GeV is the electroweak vacuum expectation value. Thus a large Higgs boson mass necessarily implies a large value of $\lambda(M_{\rm H})$ that will evolve to increasingly large values of $\lambda(\mu)$ as μ increases.

The idea that the scalar interaction (as well as all other standard model interactions) remain perturbative up to unification[2] necessarily implies an upper bound on the unification mass scale M for a given choice of $M_{\rm H}$ by the requirement $\lambda(M) = \lambda_{\rm max}$, where $\lambda_{\rm max}$ is the largest value of λ for which scalar field theory should remain perturbative. This criterion has been used by Riesselmann and collaborators[3] to correlate upper bounds on M with $M_{\rm H}$.

Here we re-assess these bounds by considering the purely scalar field (i.e. purely λ) three-, four- and five-loop contributions to the β-function (1). These have been known for some time; the scalar field theory projection of the standard model is just a globally $O(4)$-symmetric real scalar field theory whose β-function is given to five-loop order by[4]

$$\mu \frac{d}{d\mu} Y = 4Y^2 - \frac{26}{3} Y^3 + 55.661 Y^4 - 532.99 Y^5 + 6317.7 Y^6 \ldots\,, \tag{8}$$

where $Y \equiv \lambda/16\pi^2$.

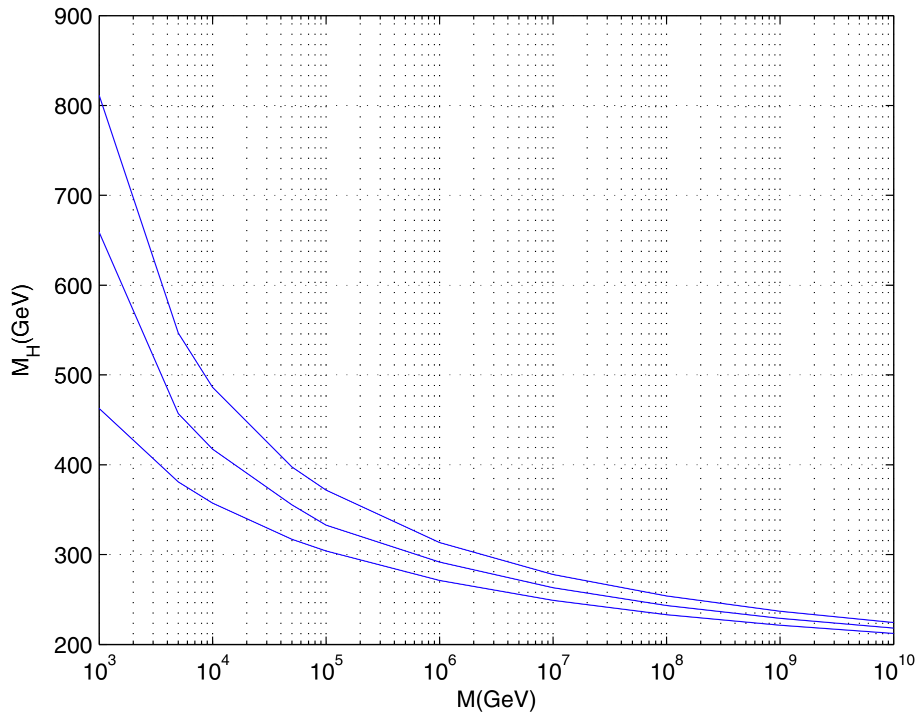

Fig. 1. Top curve, upper bound on unification mass scale M with no higher-than-2-loop input. Middle curve, upper bound with three-five loop contributions to the β-function for λ, but with λ assumed perturbative up to $\lambda_{\rm FP}/2$, as in top curve. Bottom curve, upper bound with three-five loop contributions and concomitant reduction in how large λ can be before it is nonperturbative.

This expression has a bearing both on how $\lambda_{\max}$ is obtained, as well as how rapidly λ itself evolves to $\lambda_{\max}$.

Prior calculations of the upper bound of the unification mass scale assumed $\lambda_{\max}$ was equal (or related) to its "fixed-point" value $\lambda_{\rm FP}$, defined as where the two-loop and one-loop contribution to (8) are equal:

$$Y = 6/13 \quad \text{or} \quad \lambda_{\rm FP} \cong 73\,.$$

In a two-loop world, this would be near a fixed point in the RG equation (1), particularly as λ is so dominant a coupling constant compared to the others in Eq. (1). In fact, people have advocated for various reasons that $\lambda_{\max}$ be[5] $\lambda_{\rm FP}/2$ or even smaller.[6] The top curve of Fig. 1 shows, given a choice of $M_{\rm H}$, the corresponding value of the upper bound M for the unification mass scale, given that $\lambda_{\max} = \lambda_{\rm FP}/2 = 36$. The intermediate curve in Fig. 1 shows for $\lambda_{\max} = \lambda_{\rm FP}/2$ how the upper-bound M on the unification mass scale decreases if the three-, four- and five-loop terms in Eq. (8) are incorporated into the λ β-function (1).

However, the additional β-function terms in (8) make any referencing to $\lambda_{\rm FP}$ irrelevant. The β-function series (8) does not monotonically decrease unless

$Y < 0.084$ ($\lambda < 13.3$). Hence $\lambda_{\max} = 13.3$ is an *upper bound* on the value of λ for which perturbative Higgs sector physics may still be possible, in that four- and five-loop terms in (8) are equal. The evolution of the coupling constant λ should also be inclusive of the three-, four- and five-loop terms of Eq. (8), as in the middle curve, since such terms are comparable when $\lambda_{\max} = 13.3$. When we augment Eq. (1) with these three-five loop terms in Eq. (8), and impose the additional requirement that the upper bound on λ for perturbative physics is 13.3, we obtain the lowest of the three curves in Fig. 1.

Figure 1 shows that a given value for M, the upper bound for the unification mass scale, now corresponds to substantially smaller values of the Higgs mass when Eq. (8) augments the Eq. (1) β-function, and when $\lambda_{\max} = 13.3$. This separation becomes pronounced when $M < 10^5$ GeV. By incorporating Eq. (8), we find that a Higgs mass of 304 GeV can occur in a theory only if unification is prior to 100 TeV; a Higgs mass of 360 GeV can occur only if unification is prior to 10 TeV; and that Higgs mass in excess of 460 GeV would involve nonperturbative physics immediately. In the prior analysis (top curve) this same nonperturbative bound would be in excess of 800 GeV.

We reiterate that the reduction we find in the unification mass-scale upper bound M for a given choice of M_{H} is itself conservatively taken. The choice $\lambda_{\max} = 13.3$ assumes perturbative physics even when three-, four- and five-loop contributions to the β-function(8) are comparable in magnitude. One could argue for $\lambda_{\max} = 6.65$ via whatever reasoning already employed in the past for choosing $\lambda_{\max} = \lambda_{\mathrm{FP}}/2$ instead of λ_{FP}. We also note that the effect of higher-than-2 loop contributions becomes unimportant for Higgs masses in the vicinity of 200 GeV. We find, for example of a 200 GeV Higgs boson mass, that the upper bound on the unification mass scale is 10^{12} GeV; for a 190 GeV Higgs boson mass, the upper bound on the unification mass scale goes up to 10^{15} GeV. In the prior two-loop analysis ($\lambda_{\max} = \lambda_{\mathrm{FP}}/2$) the same values of the unification mass scale are achieved by Higgs masses only 10 GeV or so larger than those quoted above.

Acknowledgments

We are grateful for the support from the Natural Sciences and Engineering Research Council of Canada.

References

1. C. Ford, D. R. T. Jones, P. W. Stephenson and M. B. Einhorn, *Nucl. Phys. B* **395**, 17 (1993) Appendices.
2. V. Elias, *Phys. Rev. D* **20**, 262 (1979).
3. K. Riesselmann, *Acta. Phys. Polon. B* **27**, 3661 (1996).
4. H. Kleinert *et al.*, *Phys. Lett. B* **319**, 545 (1993).
5. T. Hambye and K. Riesselmann, *Phys. Rev. D* **55**, 7255 (1997).
6. K. Riesselmann and S. Willenbrock, *Phys. Rev. D* **55**, 311 (1997).