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Indefinite integration as term rewriting: integrals containing tangent

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We describe the development of a term-rewriting system for indefinite integration; it is also called a rule-based evaluation system. The development is separated into modules, and we describe the module for a wide class of integrands containing the tangent function.

1. Introduction

Programming styles in computer-algebra systems are frequently described as either term-rewriting based, or computationally based. For example, Mathematica is widely recognized as a rewrite language [1], whereas Maple is rarely described this way. The distinction is mostly one of emphasis, since all available systems include elements of both styles of programming. The dichotomy can be seen more specifically in programming to evaluate indefinite integrals, also called primitives or anti-derivatives. For symbolic integration, some of the best-known approaches are computationally based. Thus the Risch algorithm [8] and the Rothstein-Trager-Lazard-Rioboo algorithm [9, 10, 5] are both computational algorithms. These algorithms and others like them are not universally applicable, however, and for many integrals rule-based rewriting is needed and has advantages, some of which we discuss below.

Doubts have been expressed about the viability of large-scale term rewriting [2]; the present scheme is more algorithmic and deterministic than earlier term rewriting schemes, and we prefer the description rule-based scheme for the integration scheme presented here. A particular success of rule-based schemes is the popularity of software that can display the steps of a calculation. This is variously called 'display step' or 'single step'. Examples of software offering this include WolframAlpha and Derive, and many calculus tutorial programs.

The rule-based integration scheme that is considered here [7, 6] consists of a public-domain repository of transformation rules for indefinite integration, together with utility files that allow it to be utilized by various computer-algebra systems. The repository is *not* a table of integrals; it is a compact set of rules, much smaller than a table covering the same domain. Also simple correctness is not the only aim of the development. The *quality* of integral expressions is judged by a number of criteria, which are used to decide whether an integration rule should be accepted. Assume that an integrand $f(x)$ has a proposed primitive $F(x)$. Selection is based on the following criteria, which are discussed further in the next section.

- Correctness: we require $F'(x) = f(x)$.
- Simplicity: we seek the simplest form for an integral. We adopt a pragmatic approach and aim for the shortest expression length.
- Continuity: we aim to ensure that all of the expressions for integrals are continuous on domains of maximum extent [4].
- Æsthetics: we employ a number of principles to select for mathematical beauty where possible.
- Utility: the rules should facilitate the aforementioned 'show-step' application. See Sec. 2.5.
- Efficiency: the path to a result should be as direct as possible, and the set of rules should be compact.

A more detailed description of the repository is given below.

2. Discussion of selection criteria

The following example allows us to discuss several aspects of our overall aim. We compare 5 possible expressions for an integral.

$$\begin{aligned}\int \sqrt{-2\tan x}\,dx = &-\tfrac{1}{2}\ln\left(2-2\tan x-2\sqrt{-2\tan x}\right)\\ &-\arctan\left(\sqrt{-2\,\tan x}-1\right)\\ &+\tfrac{1}{2}\ln\left(2-2\tan x+2\sqrt{-2\tan x}\right)\\ &-\arctan\left(\sqrt{-2\tan x}+1\right)\ ,\end{aligned} \tag{1}$$

$$\int \sqrt{-2\tan x}\,dx = \tag{2}$$

$$\frac{\sqrt{-\mathrm{Tan}[x]}}{2\sqrt{\mathrm{Tan}[x]}}\Big(-2\mathrm{ArcTan}\left[1-\sqrt{2}\sqrt{\mathrm{Tan}[x]}\right]$$
$$+2\mathrm{ArcTan}\left[1+\sqrt{2}\sqrt{\mathrm{Tan}[x]}\right]$$
$$+\mathrm{Log}\left[1-\sqrt{2}\sqrt{\mathrm{Tan}[x]}+\mathrm{Tan}[x]\right]$$
$$-\mathrm{Log}\left[1+\sqrt{2}\sqrt{\mathrm{Tan}[x]}+\mathrm{Tan}[x]\right]\Big)\ , \tag{3}$$

$$= \ln\cos x + \ln(1+\sqrt{-2\tan x}-\tan x) + \arctan\frac{1+\tan x}{\sqrt{-2\tan x}}\ , \tag{4}$$

$$= -\arctan\frac{\sqrt{-2\tan x}}{1+\tan x} + \operatorname{arctanh}\frac{\sqrt{-2\tan x}}{1-\tan x}\ , \tag{5}$$

$$= \arctan\frac{1+\tan x}{\sqrt{-2\tan x}} + \operatorname{arctanh}\frac{1-\tan x}{\sqrt{-2\tan x}}\ . \tag{6}$$

In the above equations, expression (1) comes from Maple 16; (3) comes from Mathematica 8; (4) to (6) come from the present project. We now consider these expressions under the headings listed above.

2.1. *Correctness*

With f and F denoting an integrand and primitive as above, we note that the test $F' = f$ requires a definition of differentiation. Here, we choose it to be complex differentiation. This is reflected in the absence of absolute values around the arguments of the logarithm functions in the example. The debate between computer-algebra users over

$$\int \frac{dx}{x} = \begin{cases} \ln x\ , & \text{Complex} \\ \ln|x|\ , & \text{Real} \end{cases} \tag{7}$$

is of long standing. In the current repository, all transformation rules are valid for complex quantities. In the example the integrand itself becomes imaginary for intervals $(n\pi, n\pi+\pi/2)$ with $n \in \mathbb{Z}$.

Note that the verification of the test $F' = f$ by a computer-algebra system is not itself a trivial step. Maple cannot complete a verification of (4) in any straightforward way, for example.

2.2. *Simplicity*

The last 3 expressions are clearly shorter and simpler than the first two. A constant problem in computer-algebra systems is expression swell, and every part of such systems should be striving to keep results succinct. This increases the utility of a system.

Integral expressions cannot necessarily be simplified automatically into their shortest forms. There are two reasons for this. First, the shortest form may differ by a constant from the given expression, and therefore algebraic manipulation alone cannot succeed in finding it. Secondly, branch cuts could make the given expression differ algebraically from the shortest form; another way to say this is that the two expressions differ by a piecewise constant. Therefore an integration system should aim to obtain the simplest form directly, since algebraic simplification is not likely to be successful. This can be seen particularly in the results of Risch integration, which are obtained without regard for branch cuts.

2.3. *Continuity*

The integrand $\sqrt{-2\tan x}$ is singular at $x = n\pi + \pi/2$, and is otherwise continuous. At $x = n\pi$ the function changes from real to imaginary, but is continuous at those points. Therefore expressions for its integral should also be continuous except possibly at $x = (n+1/2)\pi$. We see that expression (5) has discontinuities at $\tan x = \pm 1$. Therefore the other expressions are preferred. Combining continuity with simplicity, we can therefore reduce our choice of preferred expressions to (6) and (4). An additional point to note is that the integrand is integrable at its singular points, and one could ask for the integral expression to be continuous there. None of the expressions has a defined value at $x = n\pi + \pi/2$ and therefore the evaluation of definite integrals with these endpoints must be evaluated as a one-sided limit.

It should be further noted that requirements for continuity and simplicity might conflict. Consider a different example: the integral

$$\int \frac{x^2+2}{x^4-3x^2+4}\,dx = \arctan\frac{x}{2-x^2}\ , \tag{8}$$

$$= \arctan\left(2x+\sqrt{7}\right) + \arctan\left(2x-\sqrt{7}\right)\ . \tag{9}$$

The discontinuous expression is shorter, and would be judged simpler by most users. This is a simple example using the Lazard-Rioboo algorithm [5]; as the degrees of the polynomials grow, the difference in length between the two expressions increases.

2.4. *Æsthetics*

> Beauty is the first test. There is no permanent place in the world for ugly mathematics.
>
> – G. H. Hardy [3]

If we compare (4) and (6), we see that (6) uses two related functions, namely arctan and arctanh. Although we know that arctanh can be expressed in logarithms, the look of (6) with its many symmetries has greater mathematical beauty than the other expressions.

A similar principle concerns the relation between the form of the integrand and the form of its integral.

Consider the following example, which comes from MAPLE 16.

$$\int \sqrt{2\tan x}\, dx = \frac{\sqrt{\tan x}\cos x \arccos(\cos x - \sin x)}{\sqrt{\cos x \sin x}} - \ln\left(\cos x + \sqrt{2}\sqrt{\tan x}\cos x + \sin x\right) \ . \quad (10)$$

In contrast to (1), also from Maple, this form introduces a jumble of functions not seen in the integrand. Also the obvious symmetry of $x \to -x$ which relates problems (1) and (10) is not preserved in the answers. We prefer, then, to echo as much as possible the functional forms seen in the integrand.

2.5. *Utility*

The way in which rules are encoded will be reflected when a user single-steps through a derivation. Some systems are happy to use many substitutions on the way to the final expression. Although this reflects the way a human might solve the problem, it makes for difficult reading, since the user is probably not taking notes along the way. A simple example is an integral of the form $\int f(ax+b)\, dx$. Although a human might immediately write $y = ax + b$ and then work in y, we prefer to live with the longer form, and save the user from having to keep track of the substitutions.

3. Format for rules

Each entry in the repository has three functional parts, which together define a rule. The repository entries may contain other information, such as rule derivations and literature citations, which do not have a functional role.

1. The transformation. This maps an integral to an expression which contains terms that are free of integrals and terms containing new (simpler) integrals.
2. Validity conditions. Since the integrals in part 1 usually contain parameters, these conditions ensure the correctness of the transformation.
3. Simplification conditions. These conditions ensure that the transformation is desirable, meaning that any new integral or integrals will lead, after further transformations, to a solution of the original problem.

An important design objective is that the conditions defining the rules are mutually exclusive, meaning that once parameters are specified for any integrand, only one set of conditions will evaluate to true, and therefore only one rule can be applied to any particular case.

As an example, consider transformation (21) in the appendix. The factor $(m + n + 1)$ on the left side must clearly be non-zero for the equation to act as a transformation from left to right. Therefore the condition $m + n + 1 \neq 0$ must be listed in the validity field of the rule. In addition, the transformation reduces the exponent of $T_1 = \tan(c + dx)$ from m to $m - 1$, and clearly this requires $m \geq 1$ in order to qualify for a simplification. Thus this becomes a condition in the simplification field. We can note in passing that the case $m + n + 1 = 0$ does not invalidate the equation, which now reduces to a simple exact integral, but simply prevents the equation being used as a transformation.

A similar effect can be seen in transformation (18). Here, the case $C = 0$ reduces the transformation to a trivial identity.

4. Functions containing tangent

The new module for integration addresses functions of the form

$$\tan^m(c+dx)(a+b\tan(c+dx))^n * (A + B\tan(c+dx) + C\tan^2(c+dx)) \ ,$$

where $a, b, c, d, A, B, C \in \mathbb{C}$ and $m, n \in \mathbb{R}$ are arbitrary. This expression will not always be integrable, and the aim is to evaluate all cases in which it is integrable. This can even include cases in which m and n remain symbolic.

It is worth commenting on this choice for the integrand, particularly the presence of the last factor. The appendix contains a set of recurrence relations which allow us to simplify the integrand to a point where it can be evaluated explicitly. It is found that the final factor is necessary to the recurrence relations. Even if we set $B = C = 0$ in the above expression, one step of the reduction of the integral will introduce the additional terms. Remarkably, the case $a^2 + b^2 = 0$ allows a simpler form of integrand to be reduced, and the appropriate relations are also included in the appendix.

One standard approach to such integrals, used by other systems, is the substitution $u = \tan(c + dx)$. This removes all trigonometric functions from the integral, and converts the problem to a quasi-rational function in u. Another similar substitution is the Weierstrass substitution $u = \tan((c + dx)/2)$. We do not follow this option. From a mathematical point of view, it brings with it problems of inverting the transformation, since arctan is a branched function. From a programming point of view, the tactic reduces the independence of the module. The integral problem is moved to other systems, or other modules in the same system, that handle non-trigonometric functions. Also, we stated earlier the aesthetic principle that we try to stay with the functions appearing in the integrand; this applies both to the derivations and the final expressions.

The rule-based approach for the above integrands uses 12 transformations listed in the appendix to iteratively reduce integrands to forms for which the evaluation is known. The final rule is often called a termination rule. We have not listed all the rules here, because with the application conditions and termination rules, there are too many to print. We now give an example of the rules in action.

5. Example reduction

We consider a specific problem. In order to focus on the principal idea, we have chosen the numerical constants so that some terms in the recurrence will have zero coefficients and keep the expressions short.

$$\int \tan(1+i+x)* \frac{\left(4-12\tan(1+i+x)+9\tan(1+i+x)^2\right)}{(2-3\tan(1+i+x))^{3/2}}\,dx\ . \tag{11}$$

To select one of the recurrences below, the program compare the integrand with the standard form above and identifies $m=1$ and $n=-3/2$. Since $n\leq -1$, we select recurrence (20). Because in this case $Ab^2-abB+a^2C=0$, the recurrence simplifies even further. Thus we obtain

$$-\frac{1}{13}\int \frac{\tan(1+i+x)(-26+39\tan(1+i+x))}{\sqrt{2-3\tan(1+i+x)}}\,dx\ . \tag{12}$$

Our choice of constants now allows us to simplify the integrand algebraically. In practice this step is not performed by appealing to the simplification routines of the host system. Rather, it is coded within the rule-based system as a separate rule. This is necessary because the general host simplification function will ignore our aesthetic principle of working with the original functions, in this case tangent. We obtain

$$\int \tan(1+i+x)\sqrt{2-3\tan(1+i+x)}\,dx\ . \tag{13}$$

This is next transformed using

$$dn\int T_1T_2^n\,dx \longrightarrow T_2^n - dn\int T_2^{n-1}T_3(b,-a,0)\,dx$$

to obtain

$$\int \frac{3+2\tan(1+i+x)}{\sqrt{2-3\tan(1+i+x)}}\,dx + 2\sqrt{2-3\tan(1+i+x)}\ . \tag{14}$$

Next an algebraic manipulation rule is used, valid provided $A^2+B^2\neq 0$, $a^2+b^2\neq 0$

$$\int \frac{A+B\tan(c+dx)}{\sqrt{a+b\tan(c+dx)}}\,dx \longrightarrow \frac{1}{2}(A-Bi)\int \frac{1+i\tan(c+dx)}{\sqrt{a+b\tan(c+dx)}}\,dx + \frac{1}{2}(A+Bi)\int \frac{1-i\tan(c+dx)}{\sqrt{a+b\tan(c+dx)}}\,dx \tag{15}$$

followed by if $A^2+B^2=0$ and $bA+aB\neq 0$ then

$$\int \frac{A+B\tan(c+dx)}{\sqrt{a+b\tan(c+dx)}}\,dx \longrightarrow -\frac{2B\,\mathrm{arctanh}\left[\frac{\sqrt{a+b\tan(c+dx)}}{\sqrt{a+\frac{bA}{B}}}\right]}{d\sqrt{a+\frac{bA}{B}}} \tag{16}$$

This rule is used twice to obtain the final expression

$$+2\sqrt{2-3\tan(1+i+x)} - \sqrt{2-3i}\,\mathrm{arctanh}\left[\frac{\sqrt{2-3\tan(1+i+x)}}{\sqrt{2-3i}}\right] - \sqrt{2+3i}\,\mathrm{arctanh}\left[\frac{\sqrt{2-3\tan(1+i+x)}}{\sqrt{2+3i}}\right]$$

6. Concluding remarks

Although one aim of this project is to develop a system-independent repository of integration rules, at present the host computer algebra system influences the form of some rules. Since the first step in identifying which rule to apply is to identify the pattern of the integrand, the underlying pattern recognition functions of the host system will influence the selection of rules. For example, in Mathematica, $(\sin(c+dx))^{-1}$ is represented internally as $\csc(c+dx)$, and $\tan(c+dx)^{-1}$ as $\cot(c+dx)$. Hence, the current module must contain entries for both tangent and cotangent (other modules must contain entries for sine and cosecant). A system which stored reciprocals of tangents differently might need a modification of the database.

A second way in which the host system influences the repository is through simplification. We have already noted that some algebraic simplifications are coded as rule-based transformations, because otherwise the system simplifier would destroy the patterns we prefer. Our inability to specify our requirements to system simplifiers forces us to include algebraic simplification within the repository, which is a significant increase in size.

7. Appendix

We list the recurrence relations used in the integration scheme [11]. To save space and to show the structure more clearly, we use the abbreviations

$$T_1 = \tan(c+dx)\ , \qquad T_2 = a + b\tan(c+dx)\ ,$$
$$T_3(A,B,C) = A + B\tan(c+dx) + C\tan^2(c+dx)\ .$$

Then the recurrences, valid for all $A, B, C \in \mathbb{C}$, are

$$d(m+1)\int T_1^m T_2^n T_3(A,B,C)\,dx = \tag{17}$$
$$AT_1^{m+1}T_2^n + d\int T_1^{m+1}T_2^{n-1}T_3(\hat{A},\hat{B},\hat{C})\,dx\ ,$$
$$\hat{A} = aB(m+1) - Abn\ ,$$
$$\hat{B} = (bB - aA + aC)(m+1)\ ,$$
$$\hat{C} = bC(m+1) - Ab(m+n+1)\ .$$

$$d(m+n+1)\int T_1^m T_2^n T_3(A,B,C)\,dx = \tag{18}$$
$$CT_1^{m+1}T_2^n + d\int T_1^m T_2^{n-1}T_3(\hat{A},\hat{B},\hat{C})\,dx\ ,$$
$$\hat{A} = Aa(m+n+1) - C(m+1)a\ ,$$
$$\hat{B} = (aB + bA - bC)(m+n+1)\ ,$$
$$\hat{C} = aCn + bB(m+n+1)\ .$$

$$bd(n+1)\left(a^2+b^2\right)\int T_1^m T_2^n T_3\,dx = \tag{19}$$
$$\left(Ab^2 - abB + a^2C\right)T_1^m T_2^{n+1} + d\int T_1^{m-1}T_2^{n+1}\hat{T}_3\,dx\ ,$$
$$\hat{A} = -\left(Ab^2 - abB + a^2C\right)m\ ,$$
$$\hat{B} = b(bB + aA - aC)(n+1)\ ,$$
$$\hat{C} = (m+n+1)(aB - Ab)b - ma^2C + (n+1)b^2C\ .$$

$$ad(n+1)\left(a^2+b^2\right)\int T_1^m T_2^n T_3\,dx = \tag{20}$$
$$-\left(Ab^2 - abB + a^2C\right)T_1^{m+1}T_2^{n+1} + d\int T_1^m T_2^{n+1}\hat{T}_3\,dx$$
$$\hat{A} = A\left(a^2(n+1) + b^2(m+n+2)\right)$$
$$-a(bB - aC)(m+1)\ , \hat{B} = a(aB - bA + bC)(n+1),$$
$$\hat{C} = \left(Ab^2 - abB + a^2C\right)(m+n+2)\ .$$

$$bd(m+n+1)\int T_1^m T_2^n T_3(A,B,C)\,dx = \tag{21}$$
$$CT_1^m T_2^{n+1} - d\int T_1^{m-1}T_2^n\hat{T}_3\,dx\ ,$$
$$\hat{A} = aCm\ , \quad \hat{B} = b(C-A)(m+n+1)\ ,$$
$$\hat{C} = aCm - bB(m+n+1)\ .$$

$$ad(m+1)\int T_1^m T_2^n T_3(A,B,C)\,dx = \tag{22}$$
$$AT_1^{m+1}T_2^{n+1} + d\int T_1^{m+1}T_2^n\hat{T}_3\,dx\ ,$$
$$\hat{A} = aB(m+1) - Ab(m+n+2)\ ,$$
$$\hat{B} = -a(A-C)(m+1)\ , \quad \hat{C} = -Ab(m+n+2)\ .$$

The following 6 transformations, from (23) to (28), require the condition $a^2 + b^2 = 0$. Note that the third argument of T_3 is now always 0.

$$d(m+1)\int T_1^m T_2^n T_3(A,B,0)\,dx = \tag{23}$$
$$AaT_1^{m+1}T_2^{n-1} - d\int T_1^{m+1}T_2^{n-1}T_3(\hat{A},\hat{B},0)\,dx\ ,$$
$$\hat{A} = Ab(n-1) - (Ab + Ba)(m+1)\ ,$$
$$\hat{B} = Aa(m+n) - Bb(m+1)\ .$$

$$d(m+n)\int T_1^m T_2^n T_3(A,B,0)\,dx = \tag{24}$$
$$BbT_1^{m+1}T_2^{n-1} + d\int T_1^m T_2^{n-1}T_3(\hat{A},\hat{B},0)\,dx\ ,$$
$$\hat{A} = Aa(n+m) - Bb(m+1)\ ,$$
$$\hat{B} = Ba(n-1) + (Ab + Ba)(m+n)\ .$$

$$2a^2nd\int T_1^m T_2^n T_3(A,B,0)\,dx = \tag{25}$$
$$BbT_1^m T_2^n + d\int T_1^{m-1}T_2^{n+1}T_3(\hat{A},\hat{B},0)\,dx\ ,$$
$$\hat{A} = (Ab - Ba)m\ , \quad \hat{B} = Bb(m-n) + Aa(m+n)\ .$$

$$2a^2nd\int T_1^m T_2^n T_3(A,B,0)\,dx = \tag{26}$$
$$-a(aA + bB)T_1^{m+1}T_2^n + d\int T_1^m T_2^{n+1}\hat{T}_3(\hat{A},\hat{B},0)\,dx\ ,$$
$$\hat{A} = bB(m+1) + aA(m+2n+1)\ ,$$
$$\hat{B} = (aB - Ab)(m+n+1)\ .$$

$$ad(m+n)\int T_1^m T_2^n T_3(A,B,0)\,dx = \qquad (27)$$

$$aBT_1^m T_2^n + d\int T_1^{m-1} T_2^n \hat{T}_3(\hat{A},\hat{B},0)\,dx\ ,$$

$$\hat{A} = -aBm\ , \quad \hat{B} = Aam + (Aa - Bb)n\ .$$

$$ad(m+1)\int T_1^m T_2^n T_3(A,B,0)\,dx = \qquad (28)$$

$$aAT_1^{m+1} T_2^n + d\int T_1^{m+1} T_2^n \hat{T}_3(\hat{A},\hat{B},0)\,dx\ ,$$

$$\hat{A} = Abn - Ba(m+1)\ , \quad \hat{B} = Aa(m+n+1)\ .$$

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