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RESEARCH ARTICLE

Bernstein, Pick, Poisson and related integral expressions for Lambert W

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The Lambert W function has a number of integral expressions, including integrals of Bernstein, Thorin, Poisson, Stieltjes, Pick and Burniston-Siewert types. We give explicit integral expressions for W for each of these types. We also give integrals for a number of functions containing W.

1. Introduction

The Lambert W function is a multivalued inverse of the mapping $W \mapsto We^W$. Its branches, denoted by W_k $(k \in \mathbb{Z})$, are defined through the equations [10]

$$\forall z \in \mathbb{C}, \qquad W_k(z)\exp(W_k(z)) = z\ , \tag{1}$$

$$W_k(z) \sim \ln_k z \text{ as } \Re z \to \infty\ , \tag{2}$$

where $\ln_k z$ denotes branch k of the natural logarithm [14], and branch cuts for W are placed on the negative real axis. This paper considers mostly the principal branch W_0, which is the branch that maps the positive real axis onto itself, and therefore we abbreviate W_0 as W herein; the $k = -1$ branch is denoted explicitly W_{-1} when discussed. A summary of the properties of W that are relevant to integral representations has been given in [15] and are not repeated.

In [15] we proved, using general arguments, that many functions containing W belong to function classes having integral representations, specifically the Bernstein or Stieltjes function classes. Here we consider a number of additional function classes, namely the class of Pick functions and subclasses of Bernstein functions, including Thorin-Bernstein functions and complete Bernstein functions. A description of the classes can be found in a review paper [4] and a recently published book [23]. For all classes we give explicit integral representations for W and some functions containing W. Finally, we give integrals for W following Poisson [22] and the methods of Burniston–Siewert [6].

2. Explicit Stieltjes representations

We begin with explicit expressions for the Stieltjes transforms for some functions studied in [15]; we follow the definition of Stieltjes function given in [4].

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Definition 2.1 *A function* $f : (0, \infty) \to \mathbb{R}$ *is called a* Stieltjes function *if it admits a representation*

$$f(x) = a + \int_0^\infty \frac{d\sigma(t)}{x+t} \quad (x > 0) \ , \tag{3}$$

where a *is a non-negative constant and* σ *is a positive measure on* $[0, \infty)$ *such that* $\int_0^\infty (1+t)^{-1} d\sigma(t) < \infty$.

The Stieltjes representation for $W(z)/z$ was given in [15] in terms of W itself. We remove the self-reference by giving an integral representation containing only elementary functions.

Theorem 2.2 *The following representation of function* $W(z)/z$ *holds [28].*

$$\frac{W(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{v^2 + (1 - v\cot v)^2}{z + v\csc(v) e^{-v\cot v}} \, dv \ , \quad (|\arg z| < \pi) \ . \tag{4}$$

Proof From [15], we take

$$\frac{W(z)}{z} = \frac{1}{\pi} \int_{1/e}^\infty \frac{1}{z+t} \frac{\Im W(-t)}{t} \, dt \ , \tag{5}$$

and change to the variable $v = \Im W(t)$. From [15, Eq.1.10], this implies

$$t = t(v) = -v\csc(v) e^{-v\cot v} \ . \tag{6}$$

The integral becomes

$$\frac{W(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{v}{t(z-t)} \frac{dv}{v'(t)} \ , \tag{7}$$

Further simplification gives (4). ■

Remark 1 Since the integrand in (4) is an even function in v, the integral admits the symmetric form

$$\frac{W(z)}{z} = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{v^2 + (1 - v\cot v)^2}{z + v\csc(v) e^{-v\cot v}} \, dv \ , \quad (|\arg z| < \pi) \ .$$

This integral has a C^∞ periodic extension and thus the midpoint rule is spectrally convergent for its quadrature (see e.g. [26]).

By Corollary 2.3 in [15] the derivative of W is a Stieltjes function. This guides us to the following theorem [25].

Theorem 2.3 *The derivative of* W *has the Stieltjes integral representation*

$$W'(z) = \frac{W(z)}{z(1+W(z))} = \frac{1}{\pi} \int_0^\pi \frac{dv}{z + v\csc(v) e^{-v\cot v}} \ , \quad (|\arg z| < \pi) \ . \tag{8}$$

Proof Since W' decays at infinity [15], one can write

$$W'(z) = \int_0^\infty \frac{d\mu(t)}{z+t} \ , \tag{9}$$

where the unknown function $\mu(t)$ can be determined using the Stieltjes-Perron inversion formula [12, p. 591]

$$\mu(t) = \frac{1}{\pi} \lim_{s\to 0+} \Im \int_{-\infty}^{-t} W'(\tau + is)\, d\tau$$

for all continuity points on the t-axis. Since $\mu(t)$ is defined up to an arbitrary constant, one can set, after integrating,

$$\mu(t) = \frac{1}{\pi} \lim_{s\to 0+} \Im W(-t + is) = \frac{1}{\pi} \Im W_0(-t) \ , \tag{10}$$

where the limit uses the continuity from above of W on its branch cut. The same result can be obtained using one of Sokhotskyi's formulas [13, p. 138].

To verify that $\mu(t)$ satisfies the conditions in Definition 2.1, we use [15, lemma 1.1] to trim the domain of integration in (9) to $1/e < t < \infty$. In addition, $\mu(t)$ can be regarded as a positive measure such that $d\mu(t)/dt = o(1/t)$ at large t. Therefore $\int_{1/e}^{\infty}(1+t)^{-1} d\mu(t) < \infty$ as required. Thus (9) takes the form

$$W'(z) = \frac{1}{\pi} \int_{1/e}^{\infty} \frac{1}{z+t} \frac{d\Im W_0(-t)}{dt} dt \ . \tag{11}$$

Changing to the variable $v = \Im W_0(-t)$ as before, we obtain (8). ■

Remark 2 Formula (11) can also be found by considerations similar to those used in [15] to prove (5). Moreover, (11) is a result of differentiating (5) with subsequent integration by parts. The representation (11) is also found in [21].

Remark 3 Comparing formulae (5) and (11) shows that the latter can be formally obtained from the former by replacing the ratios $W(z)/z$ and $\mu(t)/t$ respectively with the derivatives $dW(z)/dz$ and $d\mu(t)/dt$, where $\mu(t)$ is defined by (10).

Corollary 2.4

$$\int_0^{\pi} \left\{ \frac{\sin v}{v} e^{v \cot v} \right\}^p dv = \frac{\pi p^p}{p!}, \quad p \in \mathbb{N} \ . \tag{12}$$

Proof The integral (9) can be written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^n}{n!} z^{n-1} = \frac{1}{\pi} \int_0^{\pi} \frac{dv}{z-t} \ , \tag{13}$$

where t is the same as in (6) and the left side is the Taylor series for W' and is convergent for $|z| < 1/e$. Since $|t| > 1/e$ and therefore $|z| < |t|$, we can expand $(z-t)^{-1}$ in non-negative powers of z. Equating the coefficients of like powers of z we obtain the equality

$$(-1)^{n-1} \frac{n^n}{n!} = -\frac{1}{\pi} \int_0^{\pi} \frac{dv}{t^n} \ ,$$

which after substituting for t results in (12). ■

It is obvious that if the integral (12) is known, then going back from it to (13) we find (8). The integral (12) was conjectured by Nuttall for real $p \geq 0$ [20];

Bouwkamp found a more general integral [5], for which Nuttall's conjecture is a special case, using a representation of $\pi p^p/\Gamma(p+1)$ via a Hankel-type integral. Thus the Stieltjes representation of W' allows one to compute the integral (12) and conversely, starting with the integral of Nuttall-Bouwkamp one can obtain formula (8) in a way completely different from that used in the proof of Theorem 2.3. It is interesting to note that the connection between (12) and Lambert W was noted by W.E. Hornor and C.C. Rousseau before W was named (see editorial remarks in [20]).

According to [15, Thm 2.2], the functions $1/(1+W(z))$ and $1/W(z)-1/z$ are Stieltjes functions. The following theorem makes this explicit.

Theorem 2.5 *The following Stieltjes integral representations hold*

$$\frac{1}{1+W(z)} = \frac{1}{\pi}\int_0^\pi \frac{dv}{1+ze^{v\cot v}\sin v/v} \;, \quad (|\arg z|<\pi) \;, \tag{14}$$

$$\frac{1}{W(z)} = \frac{1}{z} + \frac{1}{\pi}\int_0^\pi \frac{v^2+(1-v\cot v)^2}{v\csc(v)\left(v\csc(v)+ze^{v\cot v}\right)}dv \;, \quad (|\arg z|<\pi) \;. \tag{15}$$

Proof The proof follows the methods of proof of Theorem 2.3. ■

Corollary 2.6

$$W(z) = \ln\left[1+\frac{z}{\pi}\int_0^\pi \frac{v^2+(1-v\cot v)^2}{v\csc(v)\left(v\csc(v)+ze^{v\cot v}\right)}dv\right] \;. \tag{16}$$

Proof By substituting (15) in $W(z)=\ln(z/W(z))$. ■

Remark 4 Sokal [25] has pointed out that these Stieltjes representations can be used to obtain those for functions containing $W(1/z)$ by just replacing z with $1/z$. For example, formula (8) yields

$$\frac{W(1/z)}{1+W(1/z)} = \frac{1}{\pi}\int_0^\pi \frac{dv}{1+zv\csc(v)e^{-v\cot v}} \quad (|\arg z|<\pi) \;.$$

3. Bernstein representations

In [15] it was shown that W is a Bernstein function, which means that it admits the Lévy-Khintchine representation

$$W(x) = a+bx+\int_0^\infty \left(1-e^{-x\xi}\right)d\nu(\xi) \;, \tag{17}$$

where ν is a positive measure on $(0,\infty)$ satisfying $\int_0^\infty \xi(1+\xi)^{-1}d\nu(\xi)<\infty$ (the Lévy measure), and $a,b\geq 0$. Since $W(0)=0$ and $\lim_{x\to\infty}W(x)/x=0$, we have $a=0$ and $b=0$. The function $\nu(\xi)$ is identified by the next theorem.

Theorem 3.1 *For the principal branch of W function the following formula holds*

$$W(z) = \int_0^\infty \left(1-e^{-z\xi}\right)\frac{\varphi(\xi)}{\xi}\,d\xi \;, \quad (\Re z\geq 0), \tag{18}$$

where

$$\varphi(\xi) = \frac{1}{\pi}\int_0^{\pi} \exp\left(-\xi v \csc(v) e^{-v\cot v}\right)\, dv\ . \tag{19}$$

Proof We consider the Stieltjes integral form (9) for W' and use the representation $(x+t)^{-1} = \int_0^\infty e^{-(x+t)\xi} d\xi$ to write it in the form

$$W'(x) = \int_0^\infty \left\{\int_0^\infty e^{-\xi t}\, d\mu(t)\right\} e^{-x\xi}\, d\xi\ . \tag{20}$$

Comparing (20) and the result of differentiating (17) we find the relation between measures μ and ν [3]

$$\frac{d\nu}{d\xi} = \frac{1}{\xi}\int_0^\infty e^{-\xi t}\, d\mu(t)\ .$$

Using formula (10) and changing to the variable $v = \Im W(-t)$ as before we obtain

$$d\nu = \frac{\varphi(\xi)}{\xi}\, d\xi\ , \tag{21}$$

where $\varphi(\xi)$ is defined by (19). We collect the intermediate results and take a holomorphic continuation of (17) to the right half-plane $\Re z \geq 0$ where the integral (18) is convergent, in accordance with near-conjugate symmetry (cf. Proposition 3.5 in [23]). ■

In addition to being a Bernstein function, W is a member of the subclass of complete Bernstein functions [15]. Now we show that W also belongs to another subset of Bernstein functions.

Definition 3.2 *[23, Definition 8.1] A Bernstein function f is called a Thorin–Bernstein function if the Lévy measure in* (17) *is such that $t\, d\nu(t)/dt$ is a completely monotonic function.*

Theorem 3.3 *Lambert W is a Thorin–Bernstein function.*

Proof By Theorem 8.2 in [23], it is sufficient to note that $W(x)$ maps $(0,\infty)$ to itself, $W(0) = 0$ and $W'(x)$ is a Stieltjes function. ■

The same theorem asserts the existence of two integral representations for Thorin–Bernstein functions. One of these is precisely (5) and the other is shown in the following theorem.

Theorem 3.4 *The principal branch of the W function can be represented as the integral*

$$W(z) = \frac{1}{\pi}\int_0^{\pi} \ln\left(1 + z\frac{\sin v}{v} e^{v\cot v}\right) dv \quad (|\arg z| < \pi)\ . \tag{22}$$

Proof Integration of (5) by parts gives

$$W(x) = \frac{1}{\pi}\int_{1/e}^{\infty} \ln\left(1 + \frac{x}{t}\right) \frac{d}{dt}\Im W(-t)\, dt\ . \tag{23}$$

By [15, Lemma 1.2], the measure $\Im W(-t)$ satisfies the requirements needed for [23, Theorem 8.2]. Changing to the variable $v = \Im W(-t)$ as before and taking a holomorphic extension of the result to the cut z-plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying near conjugate symmetry, we obtain (22). ■

Remark 1 In the terminology of [23, p. 75], the integral form (23) is the Thorin representation of W function and $\mu(t) = \Im W(-t)/\pi$ is the Thorin measure of W.

Remark 2 Differentiating the representation (22) for $W(z)$ gives formula (8) for $W'(z)$.

Remark 3 The representation (23) (up to changing t to $-t$) was obtained in [7] as a dispersion relation for the principal branch of W function. The representations (18)-(19) and (23) are also found in [21].

Note that by (19) function $\varphi(\xi)$ is completely monotonic, as it should be by Definition 3.2.

4. Pick representations

DEFINITION 4.1 *[4, Definition 4.1] A function $f(z)$ is called a Pick function (or Nevanlinna function) if it is holomorphic in the upper half-plane $\Im z > 0$ and $\Im f \geq 0$ there.*

A Pick function $f(z)$ admits an integral representation [4, Theorem 4.4]

$$f(z) = \alpha_0 + b_0 z + \int_{-\infty}^{\infty} \frac{1 + tz}{(t - z)(1 + t^2)} d\sigma(t) \quad (\Im z > 0) \,, \tag{24}$$

where

$$\alpha_0 = \Re f(i), \quad b_0 = \lim_{y \to \infty} \frac{f(iy)}{iy} \,, \tag{25}$$

and σ is a positive measure which satisfies

$$\lim_{s \to 0+} \frac{1}{\pi} \int_{\mathbb{R}} \Im f(t + is) \varphi(t) dt = \int_{\mathbb{R}} \varphi(t) d\sigma(t) \tag{26}$$

for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support. The formula (24) with the integral written in terms of a measure $d\tilde{\sigma}(t) = \pi(1 + t^2)^{-1} d\sigma(t)$ is called a *Nevanlinna formula* [17, p. 100].

Since $W(z)$ is a holomorphic function in the upper half-plane $\Im z > 0$, where $\Im W(z) > 0$, we have that $W(z)$ is a Pick function. This also follows from the fact [15] that W belongs to the class of complete Bernstein functions because the latter are exactly those Pick functions which are non-negative on the positive real line [23, Theorem 6.7]. Thus W admits a representation (24) and in view of that the following theorem holds.

THEOREM 4.2 *The principal branch of W function can be represented in the form*

$$W(z) = \alpha_0 + \frac{1}{\pi} \int_0^{\pi} K(z, v) t(v) \, dv \quad (|\arg z| < \pi) \,, \tag{27}$$

where $\alpha_0 = \Re W(i) = 0.3746990..,$

$$K(z,v) = \frac{(1+zt(v))\left(v^2+(1-v\cot v)^2\right)}{(z-t(v))\,(1+t^2(v))}\ , \tag{28}$$

and $t(v)$ *is defined by* (6).

Proof We apply formulae (25) and (26) to function $f(z) = W(z)$ to obtain

$$\alpha_0 = \Re W(i)\ ,\quad b_0 = \lim_{y\to\infty} \frac{W(iy)}{iy}\ ,\quad d\sigma(t) = \frac{1}{\pi}\Im W(t)\,dt\ .$$

Thus $b_0 = 0$, and since $\Im W(t) = 0$ for $t \ge -1/e$, we obtain

$$W(z) = \alpha_0 + \frac{1}{\pi}\int_{-\infty}^{-1/e} \frac{1+tz}{(t-z)(1+t^2)}\Im W(t)\,dt \quad (\Im z > 0)\ . \tag{29}$$

By the change of variable $v = \Im W(t)$ in the integral (29) (see (6)) we obtain formula (27), which is also valid in the lower half-plane $\Im z < 0$ in accordance with the near-conjugate symmetry of W. ∎

Corollary 4.3

$$\frac{W(z)}{z} = \gamma_0 \exp\left\{-\frac{1}{\pi}\int_0^\pi K(z,v)t(v)dv\right\} \quad (|\arg z| < \pi)\ , \tag{30}$$

where $\gamma_0 = \exp(-\Re W(i)) = 0.6874961...$

Proof It immediately follows from (27) owing to the identity $W(z)/z = e^{-W(z)}$. ∎

Now we take advantage of the fact that if f is a Stieltjes function then $-f$ and $1/f$ are Pick functions [4]. Therefore, $-W(x)/x$ and $x/W(x)$ are Pick functions that admit a representation (24).

Theorem 4.4 *For the principal branch of the W function the following formulae, with* $K(z,v)$ *defined by* (28)*, hold.*

$$\frac{W(z)}{z} = \beta_0 + \frac{1}{\pi}\int_0^\pi K(z,v)\,dv \quad (|\arg z| < \pi)\ , \tag{31}$$

$$\frac{z}{W(z)} = \eta_0 - \frac{1}{\pi}\int_0^\pi K(z,v)e^{-2v\cot v}\,dv \quad (|\arg z| < \pi)\ , \tag{32}$$

where $\beta_0 = \Re\left[W(i)/i\right] = \Im W(i) = 0.5764127..,$ $\eta_0 = \Re[i/W(i)] = 1.2195314..$.

The constants in (27)—(32) obey $\alpha_0 + i\beta_0 = W(i)$, $\gamma_0 = e^{-\alpha_0} = \beta_0/\cos\beta_0$, and $\eta_0 = \beta_0/(\alpha_0^2+\beta_0^2)$.

We add in one more integral representation associated with the Nevanlinna formula which follows from the result obtained by Cauer [9]. Specifically, based on the Riesz-Herglotz formula [17, p. 99] Cauer proved that if a real symmetric function $f(z)$ with non-negative real part is holomorphic in the right z-half-plane, it can be represented as

$$f(z) = z\left[b + \int_0^\infty \frac{dh(r)}{z^2+r}\right] , \tag{33}$$

where constant $b \geq 0$ and

$$h(r) = \frac{2}{\pi}\lim_{x\to 0}\Re\int_0^{\sqrt{r}} f(x+iy)dy . \tag{34}$$

In fact, the formula (33) follows from the Nevanlinna formula (or (24)) after changing the variable $z \to -iz$, which transforms the upper half-plane onto the right half-plane, and taking into account $f(\bar{z}) = \overline{f(z)}$.

Theorem 4.5 *The following representation of function $W(z)/z$ holds*

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^{\pi/2} \frac{\left[v^2 + (1+v\tan v)^2\right] v\sec(v)e^{v\tan v}}{z^2 + v^2\sec^2(v)e^{2v\tan v}}\tan v\, dv \quad (\Re z > 0) . \tag{35}$$

Proof Since W function meets the above requirements, the formulas (33) and (34) can be applied with the result

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^\infty \frac{\Re W(is)}{z^2+s^2}ds \quad (\Re z > 0) ,$$

where we set $b = 0$, because $\lim_{z\to\infty} W(z)/z = 0$, and $r = s^2$. The integral can be converted to elementary functions by in terms of a parameter v given by

$$\Re W(is) = v\tan v , \quad s = s(v) = v\sec(v)e^{v\tan v} . \tag{36}$$

We obtain

$$\frac{W(z)}{z} = \frac{2}{\pi}\int_0^{\pi/2} \frac{v\tan v}{z^2+s^2(v)}\frac{ds}{dv}\, dv . \tag{37}$$

Completing the simplifications, one obtains (35). ■

5. Poisson integrals

Siméon Poisson was one of many mathematicians who defined and used Lambert W (or a cognate) without naming it. In his 105 page treatise on integration [22], he considered integrals of the following type.

Theorem 5.1 *The following two formulae of Poisson type hold for $x \in (-1/e, e)$*

$$W(x) = \frac{2}{\pi}\int_0^\pi \frac{\cos\frac{3}{2}\theta - xe^{-\cos\theta}\cos\left(\frac{5}{2}\theta + \sin\theta\right)}{1 - 2xe^{-\cos\theta}\cos(\theta+\sin\theta) + x^2e^{-2\cos\theta}}\cos\tfrac{1}{2}\theta\ \mathrm{d}\theta , \tag{38}$$

$$W(x) = -\frac{2}{\pi}\int_0^\pi \frac{\sin\frac{3}{2}\theta + xe^{\cos\theta}\sin\left(\frac{5}{2}\theta - \sin\theta\right)}{1 + 2xe^{\cos\theta}\cos(\theta-\sin\theta) + x^2e^{2\cos\theta}}\sin\tfrac{1}{2}\theta\ \mathrm{d}\theta . \tag{39}$$

Proof We consider the defining equation (1) as an equation $F(W) = 0$ with respect to W, where

$$F(\zeta) = \zeta - xe^{-\zeta} . \tag{40}$$

Let Γ be the positively-oriented circumference of the unit circle $|\zeta| = 1$ in the complex ζ-plane and let G be the interior of Γ. The function $F(\zeta)$ is holomorphic in G and by Rouché's theorem it has a single isolated zero there when $|x| < 1/e$ because in this case $\left|-xe^{-\zeta}\right| < |\zeta|$ on Γ. Therefore, using Cauchy's integral formula with Γ as the integration contour, we can write

$$W = \frac{1}{2\pi i} \int_\Gamma \frac{F'(\zeta)}{F(\zeta)} \zeta \, d\zeta \tag{41}$$

for $|x| < 1/e$. Since $F'(\zeta) = 1 + xe^{-\zeta} = 1 + \zeta$, we obtain

$$W = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}(1 + e^{i\theta})}{1 - xe^{-\cos\theta - i(\theta + \sin\theta)}} d\theta , \tag{42}$$

where we set $\zeta = e^{i\theta}, -\pi \le \theta \le \pi$. Separating the real and imaginary parts of the integrand in (42) we find that the former is an even function of θ whereas the latter is an odd one. Thus, the integral of the imaginary part vanishes, as it should, and the integral of the real part gives double the value of the integral on $[0, \pi]$. As a result, after some re-arrangement, we come to integral (38). The formula (39) can be proved in a similar way (see details in Remark *1* below).

The above proof establishes the domain of validity of the integrals to be at least $-1/e < x < 1/e$. We now show that the upper limit of the domain can be extended from $1/e$ to e. This follows from the fact that W is a single valued function, and therefore $F(\zeta)$, the denominator in (41), has a single zero in G for each x such that $|\zeta| < 1$, i.e. for $-1/e < x < e$. Since Rouché's theorem is a consequence of the argument principle (see e.g. [18]), it is instructive to obtain this result using the latter. To do this, say for integral (38), we apply the argument principle to the function (40) in the case $x > 0$. It is easy to see that $\eta = F(\zeta)$ conformally maps the strip $\{-\infty < \Re\zeta < \infty, -\pi < \Im\zeta < \pi\}$, containing the entire domain G, to the complex η-plane cut along two semi-infinite lines on which $\eta = \xi \pm i\pi, \xi \ge 1 + \ln x$. We also cut the η-plane along the negative real axis to take $|\arg \eta| \le \pi$ in the cut plane and consider an image of Γ which is defined by equations

$$\rho \cos\varphi = \cos\theta - xe^{-\cos\theta} \cos(\sin\theta) , \tag{43a}$$

$$\rho \sin\varphi = \sin\theta + xe^{-\cos\theta} \sin(\sin\theta) , \tag{43b}$$

where $\rho = |\eta|$ and $\varphi = \arg \eta$.

The equations (43) are invariant under transformation $\theta \to -\theta, \varphi \to -\varphi$ and describe a closed curve $\tilde{\Gamma}$ that is symmetric with respect to the real axis in the η-plane. Suppose that while a variable point ζ moves along Γ once in the ζ-plane, the image point $\eta = F(\zeta)$ moves on $\tilde{\Gamma}$ once in the η-plane, making one cycle about the origin. Then the change in argument of η is 2π and therefore, by the argument principle the function $F(\zeta)$ has a single zero in G [18, p. 48]. For this it is necessary that two points on $\tilde{\Gamma}$ corresponding to $\varphi = 0$ and $\varphi = \pi$ are located on the real axis on opposite sides of the origin, i.e. with positive ρ to be measured on the opposite rays. Substituting $\theta = \pi$ in (43) gives $\rho \cos\varphi = -1 - xe$ and $\rho \sin\varphi = 0$. We have $\rho > 0$ only when $\varphi = \pi$; then $\rho = 1 + xe$ is positive for any $x > 0$. When $\theta = 0$, we have $\rho\cos\varphi = 1 - x/e$ and $\rho\sin\varphi = 0$. Now $\varphi = 0$ and $\rho = 1 - x/e > 0$ when $x < e$. Thus for $0 < x < e$ the curve $\tilde{\Gamma}$ encloses the origin. Since for these x the right-hand side of equation (43b) vanishes, i.e. $\Im\eta = 0$ sequentially at $\theta = -\pi, \theta = 0$ and $\theta = \pi$ as θ continuously changes from $-\pi$ to π, the curve $\tilde{\Gamma}$ is traversed once with exactly one cycle about the origin being made. This corresponds to the fact that the inverse of the mapping $\eta = F(\zeta)$ is continuous in the domain bounded by the curve $\tilde{\Gamma}$ and on $\tilde{\Gamma}$ itself and hence $\tilde{\Gamma}$ consists only of simple points [19, Theorem 2.22]. Thus, by the argument principle the function $F(\zeta)$ has a single zero in G. Gathering the results, we conclude that the integral (38) is valid for $x \in (-1/e, e)$. The integral (39) can be considered in a similar manner. ■

Remark 1 The integral (39) is explicitly given by Poisson in [22, sec. 80, p. 501]. He defined, without naming, a function he called y which today we call the tree function $T(x)$, defined by [11, p.127-128] $Te^{-T} = x$, or $T(x) = -W(-x)$. Poisson proved the formula (39) using the Lagrange Inversion Theorem [27, p. 133] and a series expansion of the logarithmic function $-\ln(1-e^{ix}\phi)$ in powers of e^{ix}, where the expansion coefficients ϕ^n/n are exactly the coefficients of the complex exponential Fourier series for the same function. On the other hand, today it is well known [8, p. 143-145] that there is a tight connection between the classical Poisson Formula and the Cauchy Integral Formula. Based on the latter one could give another proof similar to the above one of integral (38).

Remark 2 We can apply the above approach to the equation $W(z) = \ln z - \ln W(z)$. To eliminate a singularity at the origin we compose the integration contour of a small circle of radius, say r, and the unit circle, both centered at the origin and connected through the cut along the negative real axis. Then, making r go to zero, we find for $0 < x < e$

$$W(x) = \psi(x) + \frac{2}{\pi}\int_0^\pi \frac{\cos\dfrac{\theta}{2} + \theta\sin\dfrac{3}{2}\theta - \cos\dfrac{3}{2}\theta\ln x}{1 + 2\theta\sin\theta + \theta^2 - 2\cos\theta\ln x + \ln^2 x}\cos\frac{\theta}{2}d\theta ,$$

where

$$\psi(x) = \int_0^1 \frac{t-1}{\pi^2 + (\ln x + t - \ln t)^2}dt .$$

6. Burniston-Siewert representations

One of the analytic methods for solving transcendental equations is based on a canonical solution of the suitably posed Riemann-Hilbert boundary-value problem [13, p. 183-193]. This method was found and developed by Burniston and Siewert [6]; its versions, variations and applications were also considered by other authors. The method solves a transcendental equation as a closed-form integral formula that can be regarded as an integral representation of the unknown variable. Below we consider such integrals for W function which are based on the results of application of the Burniston-Siewert method to solving equation (1) obtained in paper [1] and the classical work [24].

We start with two formulas derived in [1] and apply them to function (40)

$$W(x) = -F(0)\exp\left\{-\frac{1}{2\pi i}\int_\Gamma \frac{\ln\left(F(\zeta)/\zeta\right)}{\zeta}d\zeta\right\} , \qquad (44)$$

$$W(x) = -\frac{1}{2\pi i}\int_\Gamma \ln\left(\frac{F(\zeta)}{\zeta}\right)d\zeta, \tag{45}$$

where the integration contour Γ is the unit circle $|\zeta| = 1$ and $x \in (-1/e, e)$. Since $F(0) = -x$ and $W(x)/x = e^{-W(x)}$, formula (44) is simplified

$$W(x) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln(F(\zeta)/\zeta)}{\zeta}d\zeta \ . \tag{46}$$

We set $\zeta = e^{i\theta}, -\pi \le \theta \le \pi$. Then, as $F(\zeta)/\zeta = F(e^{i\theta})e^{-i\theta} = R(\theta) + iI(\theta)$, where

$$\begin{aligned} R(\theta) &= 1 - xe^{-\cos\theta}\cos(\theta + \sin\theta), \\ I(\theta) &= \quad xe^{-\cos\theta}\sin(\theta + \sin\theta), \end{aligned}$$

and $d\zeta/\zeta = id\theta$, the integral (46) is reduced to

$$W(x) = \frac{1}{2\pi}\int_0^\pi \ln\left(R^2(\theta) + I^2(\theta)\right)d\theta \ . \tag{47}$$

Similarly, the integral (45) can be represented in the form

$$W(x) = \frac{1}{2\pi}\int_0^\pi \left\{2\arctan(I(\theta)/R(\theta))\sin\theta - \ln\left(R^2(\theta) + I^2(\theta)\right)\cos\theta\right\}d\theta \ , \tag{48}$$

where we have taken into account that $\arg(R(\theta) + iI(\theta)) = \arctan(I(\theta)/R(\theta))$ as $R(\theta) > 0$ for $0 < \theta < \pi$ and $-1/e < x < e$. We note that the integral (47) has a simpler form than (48). Integrals similar to the above with using a function $\tilde{F}(\zeta) = \zeta e^\zeta - x$ in our notations instead of (40) in formulas (44) and (45) (without simplification (46)) are given in [1].

Thus the integrals (47) and (48) representing the principal branch of the Lambert W function are valid in the domain that contains interval $(-1/e, 0)$.

However, there is one more branch that is also a real-valued function on this interval, this is the branch -1 with the range $(-\infty, -1)$ (recall $W_0 > -1$ and $W_0(-1/e) = W_{-1}(-1/e) = -1$) [10]. To obtain a representation of this branch we find a zero of function $\Phi(\zeta) = 1 - xe^{-\zeta}/\zeta$ in $(-\infty, -1)$ for fixed x using an approach [2] (cf. formulae (12) and (8) therein)

$$W_{-1}(x) = -c - \frac{1}{2\pi i}\int_C \ln\frac{\Phi(\zeta)}{\zeta + c}d\zeta \ ,$$

where the circle C is defined by equation $|\zeta + c| = c - 1$ with arbitrary constant $c > 1$ and $-1/e < x < -(2c-1)e^{1-2c}$. The last integral can be written as

$$W_{-1}(x) = 1 - 2c - \frac{1}{2\pi i}\int_C \ln\Phi(\zeta)\,d\zeta \ .$$

To evaluate $\Phi(\zeta)$ on C we set $\Phi(\zeta) = \Phi\left(-c + (c-1)e^{i\theta}\right) = R_c(\theta) + iI_c(\theta)$. Then one can find

$$\begin{aligned} R_c(\theta) =& 1 + xe^\alpha(\alpha\cos\beta + \beta\sin\beta)/(\alpha^2 + \beta^2) \ , \\ I_c(\theta) =& - xe^\alpha(\alpha\sin\beta - \beta\cos\beta)/(\alpha^2 + \beta^2) \ , \end{aligned}$$

where $\alpha = \alpha(\theta) = 1 + 2(c-1)\sin^2(\theta/2)$ and $\beta = \beta(\theta) = (c-1)\sin\theta$. As a result, similarly to (48), we can write

$$W_{-1}(x) = 1-2c+\frac{c-1}{2\pi}\int_0^{\pi}\left\{2\arg(R_c(\theta)+iI_c(\theta))\sin\theta - \ln\left(R_c^2(\theta)+I_c^2(\theta)\right)\cos\theta\right\}d\theta\,.$$

We return to the principal branch and use the result in [24, formula(13)] to write [29]

$$W(z) = 1 + (\ln z - 1)\exp\left(\frac{i}{2\pi}\int_0^{\infty}\ln\left(\frac{\ln z + t - \ln t + i\pi}{\ln z + t - \ln t - i\pi}\right)\frac{dt}{1+t}\right) \qquad (49)$$

or

$$W(z) = 1 + (\ln z - 1)\exp\left\{-\frac{1}{\pi}\int_0^{\infty}\frac{\arg(\ln z + t - \ln t + i\pi)}{1+t}dt\right\}\;, \qquad (50)$$

where $z \notin [-1/e, 0]$. In case of real $z = x > 1/e$, when the expression $\ln z + t - \ln t$ is real and positive (for $t \in (0,\infty)$), the formula (50) is simplified and reduced to

$$W(x) = 1 + (\ln x - 1)\exp\left\{-\frac{1}{\pi}\int_0^{\infty}\arctan\left(\frac{\pi}{\ln x + t - \ln t}\right)\frac{dt}{1+t}\right\} \qquad (51)$$

or, after integrating by parts

$$W(x) = 1 + (\ln x - 1)\exp\left\{-\int_0^{\infty}\frac{t-1}{\pi^2 + (\ln x + t - \ln t)^2}\frac{\ln(1+t)}{t}dt\right\}\;. \qquad (52)$$

We emphasize that the domain $x > 1/e$ of validity of the formulae (51) and (52) is different from that of (47) and (48).

For the case $x \in (-1/e, 0)$, we refer the reader to [24, formulae (32)] where the principal branch W_0 and the branch W_{-1} are represented in the form of a combination of two expressions similar to the right-hand side of (50).

Remark 1 We can regard the integral in the formula (49) as an improper integral depending on parameter $p = \ln z$ and consider it in the limit $p \to \infty$ (when $z \to \infty$). Since the integrand is a continuous function of two variables t and p in the domain under consideration and the integral is uniformly convergent with respect to p, we can take the limit under the integral sign and find that the integral vanishes as the integrand goes to zero. Then the formula (49) reproduces the asymptotic result (2).

Finally we note that by use of elementary complex analysis in [16] there is obtained a common closed form representation for all the branches $W_k(z)$ in the complex z-plane through simple quadratures.

7. Concluding remarks

We have derived various integral representations of the principal branch of the Lambert W function. Equivalently, we can say that we have established by explicit construction that W and some functions of W belong to various function classes. Besides their own importance the derived integral representations have some applications. One of them has been mentioned in connection with finding Nuttall-Bouwkamp integral (12). Other definite integrals appear when taking particular values of z. For example, integrals (4), (14), (15), (35) taken at $z = e$ yield new integrals for π.

$$\pi = \int_0^\pi \frac{v^2 + (1 - v\cot v)^2}{1 + v\csc v e^{-(1+v\cot v)}} dv \ ,$$

$$\pi = \int_{-\pi}^\pi \frac{v\, dv}{v + e^{1+v\cot v}\sin v} \ ,$$

$$\pi = \frac{e}{e-1} \int_0^\pi \frac{v^2 + (1 - v\cot v)^2}{v\csc(v)\,(v\csc(v) + e^{1+v\cot v})} dv \ ,$$

$$\pi = \int_{-\pi/2}^{\pi/2} \frac{\left[v^2 + (1 + v\tan v)^2\right] v\sin v e^{v\tan v - 1}\, dv}{\cos^2 v + v^2 e^{2(v\tan v - 1)}} \ .$$

Another advantage that can be taken of the obtained results is based on a comparison between different representations of the same function. This reveals equivalent forms of the involved integrals. In addition, since some of the integrals are simpler than others, such equations can be regarded as a simplification of the latter. For example, equating integrals (31) and (4) shows that the former can be simplified and reduced to the latter. At last we mention that the Pick representations (27), (30), (31), and (32) can be considered as integrals expressing properties of the kernel $K(z, v)$ defined by (28).

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