Unconstrained Parametric Minimization of a Polynomial: Approximate and Exact

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Abstract. We consider a monic polynomial of even degree with symbolic coefficients. We give a method for obtaining an expression in the coefficients (regarded as parameters) that is a lower bound on the value of the polynomial, or in other words a lower bound on the minimum of the polynomial. The main advantage of accepting a bound on the minimum, in contrast to an expression for the exact minimum, is that the algebraic form of the result can be kept relatively simple. Any exact result for a minimum will necessarily require parametric representations of algebraic numbers, whereas the bounds given here are much simpler. In principle, the method given here could be used to find the exact minimum, but only for low degree polynomials is this feasible; we illustrate this for a quartic polynomial. As an application, we compute rectifying transformations for integrals of trigonometric functions. The transformations require the construction of polynomials that are positive definite.

1 Introduction

Let $n \in \mathbb{Z}$ be even, and let $P_n \in \mathbb{R}[a_0, \ldots, a_{n-1}][x]$ be monic in x, that is,

$$P_n(x) = x^n + \sum_{j=0}^{n-1} a_j x^j \ . \tag{1}$$

A function $L(a_j)$ of the coefficients is required that is a lower bound for $P_n(x)$, i.e., L must satisfy

$$(\forall x) P_n(x) \geq L(a_j) \ . \tag{2}$$

The problem definition does not require that the equality in (2) be realized. If that is also the case, then L is the minimum of P_n:

$$\min_{x \in R} P_n(x) = L_{\min}(a_j) \ . \tag{3}$$

Thus $L_{\min}$ obeys

$$(\forall x) P_n(x) \geq L_{\min}(a_j) \ , \qquad (\exists x) P_n(x) = L_{\min}(a_j) \ , \tag{4}$$

and $L_{\min} \geq L$, where L satisfies (2).

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The problem described has connections to several areas of research, including parametric optimization, quantifier elimination and polynomial positive-definiteness. Much of the work on parametric optimization concerns topics such as the continuity of the optimum as a function of the parameters, or the performance of numerical methods; see, for example, [1, 2, 4]. The following problem was considered in [1].

$$\min\{\lambda^2 x^2 - 2\lambda(1-\lambda)x \mid x \ge 0\} \ .$$

The unique solution for the unconstrained problem is found for $\lambda \neq 0$ to be $-(1-\lambda)^2$, which is realised when x takes the value $\hat{x} = (1-\lambda)/\lambda$. The constrained problem has this solution only for $\lambda \in (0,1)$ and ceases to be smooth at the end points. The unconstrained problem is covered in this paper, although the focus is on higher degree polynomials.

There has been a large amount of work on a related problem in quantifier elimination. For $n = 4$, Lazard [7] and Hong [5] have solved the following problem. Find a condition on the coefficients p, q, r that is equivalent to the statement

$$(\forall x) x^4 + px^2 + qx + r \ge 0 \ . \tag{5}$$

The solution they found is

$$\Big[[256r^3 - 128p^2r^2 + 144pq^2r + 16p^4r - 27q^4 - 4p^3q^2 \ge 0 \wedge 8pr - 9q^2 - 2p^3 \le 0]$$

$$\bigvee [27q^2 + 8p^3 \ge 0 \wedge 8pr - 9q^2 - 2p^3 \ge 0]\Big] \bigwedge r \ge 0 \ . \tag{6}$$

It is clear that a solution of (3) gives a solution of this problem, since (5) is equivalent to the statement $r = -\min(x^4 + px^2 + qx \,|\, x)$. The question of positive definiteness is also related to the current problem. Ulrich and Watson [8] studied this problem for a quartic polynomial, except that they included the constraint $x \in R^+$, the positive real line.

Previous work has all been directed towards calculations of the minimum of a given polynomial. For $n = 2$, the minimum of a quadratic polynomial is a standard result.

$$\min_{x \in \mathbb{R}} P_2(x) = \min(x^2 + a_1 x + a_0) = a_0 - \tfrac{1}{4} a_1^2 \ , \tag{7}$$

and this is attained when $x = -\frac{1}{2} a_1$. For larger n, there is only the standard calculus approach, which uses the roots of the derivative. This, however, is only possible for numerical coefficients, because there is no way of knowing which root corresponds to the minimum. Floating-point approximations to the minimum are easily obtained.

If all of the coefficients of P_{2n} are purely numerical, rather than symbolic, then there are many ways to find the minimum. For example, Maple has the command `minimize` and the command `Optimize:-Minimize`. An example is

```
>minimize( x^4 - 5*x^2 + 4*x ,x);
 RootOf(2 _Z^3 - 5 _Z - 2,index=3 )^4
    - 5*RootOf(2 _Z^3 - 5 _Z - 2,index=3 )^2
    + 4 RootOf(2 _Z^3 - 5 _Z - 2,index=3 )
```

which can be simplified by Maple to

```
-(5/2) RootOf(2 _Z^3-5 _Z-2,index=3)^2
- 3 RootOf(2 _Z^3-5 _Z-2,index=3)
```

The second argument of `RootOf` selects, using an index, the appropriate root of the polynomial.

2 Algorithm for Lower Bound

We now describe a recursive algorithm. In principle, it could be used to find the minimum of a parametric polynomial, and indeed we show this below for a quartic polynomial, but the main intended use is for a simpler lower bound.

Consider a polynomial given by (1). We shall express the lower bound to P_n in terms of that for P_{n-2}. This recursive descent terminates at P_2, for which we have the result (7). The descent is based on the following obvious lemma.

Lemma 1. If $f(x)$ and $g(x)$ are two even-degree monic polynomials, then

$$\inf(f(x) + g(x)) \geq \inf f(x) + \inf g(x).$$

Proof: The equality holds when the minima of f and g are realized at the same critical point x. □

It is convenient at this point to acknowledge the evenness of the degree by changing notation to consider P_{2n}. We apply the lemma by using the standard transformation $x = y - a_{2n-1}/(2na_{2n})$ to remove the term in x^{2n-1} from $P_{2n}(x)$. Thus we have the depressed polynomial

$$P_{2n}(y) = a_{2n}y^{2n} + \sum_{j=0}^{2n-2} b_j y^j. \tag{8}$$

Now, we split P_{2n} into two even-degree polynomials with positive leading coefficients by introducing a parameter k_n satisfying $k_n > 0$ and $k_n > b_{2n-2}$.

$$P_{2n} = [a_{2n}y^{2n} + (b_{2n-2} - k_n)y^{2n-2}] + [k_n y^{2n-2} + \ldots] = P_{2n}^{(1)} + P_{2n}^{(2)} .$$

The minimum of $P_{2n}^{(1)}$ is

$$\inf(P_{2n}^{(1)}) = -\frac{(n-1)^{n-1}(k_n - b_{2n-2})^n}{n^n a_{2n}^{n-1}}$$

which is obtained at the critical points $y^2 = (n-1)(k_n - b_{2n-2})/(na_{2n})$.

Since $\deg P_{2n}^{(2)} = 2n-2 < 2n$, we can recursively compute the minimum and critical point of $P_{2n}^{(2)}$. Let the minimum and the corresponding critical point of $P_{2n}^{(2)}$ be $M(k_{n-1},\ldots,k_2), N(k_{n-1},\ldots,k_2)$ respectively. Then by Lemma 1, we have

$$\inf(P_{2n}) \geq -\frac{(n-1)^{n-1}(k_n - b_{2n-2})^n}{n^n a_{2n}^{n-1}} + M.$$

Therefore, a lower bound for P_{2n} is obtained after recursion in terms of parameters $k_n, k_{n-1}, \ldots, k_2$. If it is possible to choose the k_i such that

$$\frac{(n-1)(k_n - b_{2n-2})}{na_{2n}} = N(k_{i-1},\ldots,k_2)^2 \ , \tag{9}$$

at each recursive step, then an expression for the minimum would be obtained. However, our main aim is to find lower bounds in as simple a form as possible, hence we choose each k_i to satisfy the requirements in a simple way.

Since the k_i will appear in the denominators of expressions, it is not a good idea to allow a value that is too small. Otherwise there will be computational difficulties. A simple choice is $k_i = 1$, but this may not satisfy $1 > b_{2i-2}$. Therefore we have chosen to use

$$k_i = \max(1, 1 + b_{2i-2}) \ .$$

Table 1. A Maple procedure for computing a lower bound on the value of an even-degree monic polynomial

```
BoundPoly:=proc(p,var)
local m,n,a,b,c,redpoly,y,p1,p2,tp,par:
# Input: An even degree (parametric) polynomial p(var).
# Output: a lower bound.
m:=degree(p,var):
if m=0 or modp(m,2)<>0 then error("Bad input") end if;
  a:=coeff(p,var,m):
  b:=coeff(p,var,m-1):
  c:=coeff(p,var,m-2):
  if m=2 then
    (4*a*c-b^2)/(4*a):
  else
    n:=m/2:
    redpoly:=expand(subs(var=y-b/(m*a),p)):
    b:=coeff(redpoly,y,m-2):
    par:=max(1,b+1):
    p1:=a*y^m+(b-par)*y^(m-2):
    p2:=expand(redpoly-p1):
    tp:=(n-1)^(n-1)*(par-b)^n/(n^n*a^(n-1)):
    simplify(-tp+BoundPoly(p2,y)):
  end if
end proc:
```

This has the advantage that the simple value 1 will be selected whenever possible, and otherwise the more complicated value is used. Several other choices were tried, for example, $k_i = 1 + |b_{2i-2}|$. In either case, the results are much simpler if Maple is able to determine the sign of b_{2i-2}, otherwise many unsimplified expressions can appear in the output. The first choice gives the following algorithm, which is presented in Maple syntax in table 1.

3 Examples

Consider the polynomial

$$p = x^6 + x^4 - 2x^3 + x^2 - ax + 2 \ . \tag{10}$$

Applying the algorithm, we obtain

$$30299/17280 - (3/20)a - (1/5)a^2 \ . \tag{11}$$

Using a numerical routine, we can choose varying values of a and compute the numerical minimum and then plot this against the bound just obtained. This is shown in figure 1.

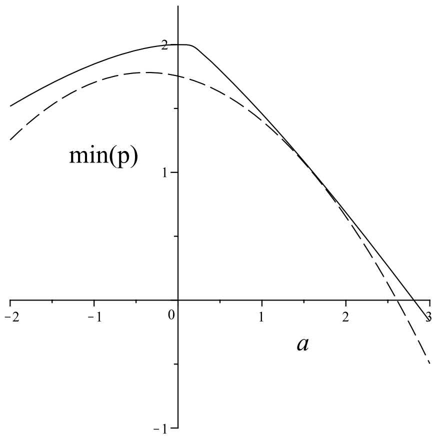

Fig. 1. The minimum of the polynomial $p(x)$ defined in (10) and the lower bound given in (11). The solid line is the exact minimum. Although very close, the two curves never touch.

For different values of a, this example shows both very close bounds and very poor ones. Thus for the case $a = 1.5$, the lower bound on the minimum is 1.078, whereas the true minimum is 1.085. In contrast, for large a, the exact minimum is asymptotically $-5(a/6)^{6/5}$, whereas the bound is $-a^2/5$, so the bound can be arbitrarily bad in that case. However, as shown in the next section, in the intended application, there is no need for a close bound; any bound will be just as good.

A second example shows a different form of output. We assume the condition $a > 0$ and look for a lower bound on

$$p = x^6 + x^4 - 2x^3 + (1+a)x^2 - x + 2 \ . \tag{12}$$

With the Maple assumption `assume(a,positive)`, we obtain the bound

$$\frac{24251 + 24628a}{3456(5+4a)} \ .$$

Notice that since $a > 0$, the denominator is never zero. We can quickly check the accuracy of this bound by trying a numerical comparison. Thus for $a = 10$, the bound takes the value $30059/17280 \approx 1.7395$, while the minimum value is actually 1.9771. For large, positive, a the minimum is asymptotically 2 and the bound is asymptotically $6157/3456 \approx 1.78$, so in this case the asymptotic behaviour is good.

4 The Minimum of a Quartic Polynomial

Although the main implementation aims for a simple lower bound, it has already been stated that the same approach can be used to find an minimum. We show that this is so, but also show the more complicated form of the result, by deriving an exact minimum for a quartic polynomial. As above, we need consider only a depressed quartic.

Theorem 1: If the coefficient $b_1 \neq 0$, the quartic polynomial

$$P_4(x) = x^4 + b_2x^2 + b_1x$$

has the minimum

$$\inf P_4 = b_2k_2 - \tfrac{3}{4}k_2^2 - \tfrac{1}{4}b_2^2 \ , \tag{13}$$

where

$$k_2 = s^{1/3} + \frac{b_2^2}{9s^{1/3}} + \frac{b_2}{3} \ , \tag{14}$$

$$s = \tfrac{1}{4}b_1^2 + \tfrac{1}{27}b_2^3 + \tfrac{1}{36}\sqrt{81b_1^4 + 24b_1^2b_2^3} \ . \tag{15}$$

Moreover, the minimum of P_4 is located at $x = x_{\mathrm{m}} = -\frac{1}{2}b_1/k_2$.

Proof: As above, the polynomial P_4 is split into two by introducing a parameter k_2 satisfying $k_2 > 0$ and $k_2 > b_2$.

$$P_4 = [y^4 + (b_2 - k_2)y^2] + [k_2y^2 + b_1y] = P_4^{(1)} + P_4^{(2)} \ .$$

The minimum of $P_4^{(1)}$ is

$$\inf(y^4 + (b_2 - k_2)y^2) = -\tfrac{1}{4}(k_2 - b_2)^2 \ ,$$

given the restrictions on k_2. The coordinate of this minimum obeys $y^2 = \frac{1}{2}(k_2 - b_2)$. The minimum of $P_4^{(2)}$ is $-b_1^2/2$, by (7), and therefore

$$\inf(P_4) \geq -b_1^2/(4k_2) - \tfrac{1}{4}(k_2 - b_2)^2 \ . \tag{16}$$

Equating the coordinates of the two infima gives an equation for the value of k_2 at which the lower bound equals the minimum of P_4.

$$\frac{k_2 - b_2}{2} = \left(\frac{-b_1}{2k_2}\right)^2 \ .$$

This is equivalent to the cubic

$$k_2^3 - b_2k_2^2 - \tfrac{1}{2}b_1^2 = 0 \ , \tag{17}$$

which equation can also be obtained by maximizing the right side of (16) directly. It is straightforward to show that (17) has a unique positive solution, and furthermore it is always greater than b_2, as was assumed at the start of the derivation. Rewriting (17) in the form

$$\tfrac{1}{2}k_2^2 - \tfrac{1}{2}b_2k_2 = b1^2/(4k_2) \ ,$$

allows the expression (16) to be transformed into the form given in the theorem statement. □

Since (17) has a unique positive solution, its solution takes the form (14) given in the theorem. For some values of the coefficients, the quantity s will be complex, but if $s^{1/3}$ is always evaluated as the principal value, then k_2 given by (14) is the real positive solution.

Theorem 2: For the case $b_1 = 0$, the quartic polynomial

$$P_4(x) = x^4 + b_2x^2 \tag{18}$$

has the minimum

$$\inf P_4 = -\max(0, -b_2/2)^2 \ ,$$

at the points $x_{\mathrm{m}}^2 = \max(0, -b_2/2)$.

Proof: By differentiation. □

Implementation. The discussion here is in the same spirit as the discussion in [3]. The following issues must be addressed by the implementer, taking into account the facilities available in the particular CAS.

For a polynomial with numerical coefficients in the rational-number field Q, the infima can be algebraic numbers of degrees 1, 2 or 3. If the formulae (13) and (14) are used for substitution, the answer will always appear to be an algebraic number of degree 3, and the simplification of such numbers into lower degree forms cannot be relied on in some systems. Therefore, if it is accepted that the system should return the simplest expressions possible, then the best strategy in this case is not to use (14), but instead to solve the cubic equation (17) directly. Even if simplicity is not an issue, roundoff error in the Cardano formula often results in a small nonzero imaginary part in k_2.

For symbolic coefficients, the main problem is the specialization problem [3]. Since Theorem 1 excludes $b_1 = 0$, it is important to see what would happen if the formulae (13) and (14) were returned to a user and later the user substituted coefficients giving $b_1 = 0$. Substituting $b_1 = 0$ into (14) gives

$$k_2 = \tfrac{1}{3}(b_2^3)^{1/3} + b_2^2/3(b_2^3)^{1/3} + b_2/3$$

For $b_2 > 0$ this gives $k_2 = b_2$, while for $b_2 < 0$ it simplifies to $k_2 = 0$. For $b_2 = 0$, the system should report a divide by zero error. Thus for $b_2 \neq 0$, (13) and (14) work even for $b_1 = 0$, although it should be noted that the position of the minimum, $-b_1/2k_2$, will give a divide by zero error for all $b_2 < 0$. It is important to remember in this discussion that the mathematical properties of (13) and (14). Thus, the fact that it is possible to obtain the correct result for $b_1 = b_2 = 0$ by taking limits is not relevant; what is relevant is how a CAS will manipulate the expressions.

An alternative implementation can use the fact that some CAS have functions for representing one root of an equation directly. In particular, Maple has the `RootOf` construction, but in order to specify the root uniquely, an interval must be supplied that contains it. The left side of (17) is $-\frac{1}{2}b_1^2$ for $k = 0, b_2$ and hence the interval can start at $\max(0, b_2)$. By direct calculation, the left side is positive at $|b_2| + b_1^2/6 + 1$. An advantage of this approach is the fact that $b_1 = b_2 = 0$ is no longer an exceptional case, at least for the value of the minimum: the position still requires separate treatment.

5 Application to Integration

Let $\psi, \phi \in \mathbb{R}[x, y]$ be polynomials over $\mathbb{R}$, the field of real numbers. A rational trigonometric function over $\mathbb{R}$ is a function of the form

$$T(\sin z, \cos z) = \frac{\psi(\sin z, \cos z)}{\phi(\sin z, \cos z)} \ . \tag{19}$$

The problem considered here is the integration of such a function with respect to a real variable, in other words, to evaluate $\int T(\sin x, \cos x)\ \mathrm{d}x$ with $x \in \mathbb{R}$. The particular point of interest lies in the continuity properties of the expression obtained for the integral. General discussions of the existence of discontinuities in expressions for integrals have been given by [6] and [10].

A simple example shows the difficulty to be faced. The integral below was evaluated as shown by all the common computer algebra programs (Maple, Mathematica and others); notice that the integral depends on a symbolic parameter a.

$$U(x) = \frac{(a\cos^4 x + 3\sin^2 x \cos^2 x)}{\cos^6 x + (a \sin x \cos^2 x + \sin^3 x)^2} \ , \tag{20}$$

$$\int U(x)\, \mathrm{d}x = \arctan(a \tan x + \tan^3 x) \ . \tag{21}$$

It is a simple calculation to see that the integrand $U(x)$ is continuous at $x = \pi/2$, with $U(0) = 0$, but the expression for the integral is discontinuous at the same point, having a jump of π. We have

$$\lim_{x\uparrow\pi/2} \arctan(a\tan x + \tan^3 x) - \lim_{x\downarrow\pi/2} \arctan(a\tan x + \tan^3 x) = \pi \ .$$

The notion of a rectifying transformation was introduced in [6], and can be applied to this situation.

The general problem is to rectify expressions of the form $\arctan[P(u)]$, where $P \in \mathbb{R}[u]$, and without loss of generality is monic. Moreover, $u = \tan x$, where x is chosen according to the properties of the integrand. We note first the identity

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy} + \begin{cases} \operatorname{sgn}(y)\pi \ , & \text{for } 1+xy<0 \ , \\ 0 \ , & \text{otherwise.} \end{cases} \tag{22}$$

We shall use this in a formal sense, dropping the piecewise constant. The two cases of P of even degree and P of odd degree are treated separately. For P of even degree, we transform as follows.

$$\arctan P(u) \to \arctan P(u) - \arctan(1/k) \to \arctan \frac{P-1/k}{1+P/k} = \arctan \frac{kP-1}{k+P} \ .$$

The first step simply adds a constant to the result of the integration. The second step uses formula (22), dropping the piecewise constant. The final expression will now be continuous provided

$$\forall u \in \mathbb{R}, \quad P(u) + k > 0 \ .$$

The problem, therefore, is to choose k so that this condition is satisfied. Notice that since $P(u)$ is even degree and monic, it will always be possible to satisfy the condition, and the problem is to find an expression for k. Also note that in the example, P contains a parameter, so a simple calculus exercise will not be sufficient to determine k.

For P of odd degree, we transform as follows.

$$\begin{aligned} \arctan(P(u)) &\to \arctan(P(u)) - \arctan u/k + \arctan u/k - \arctan u + x \\ &\to \arctan \frac{P(u) - u/k}{1 + P(u)u/k} + \arctan \frac{u/k - u}{1+u^2/k} + x \ , \\ &= \arctan \frac{kP-u}{k+uP} + \arctan \frac{u-ku}{k+u^2} \ . \end{aligned} \tag{23}$$

The first step in the transformation uses the formal identity

$$\arctan u = \arctan(\tan x) \to x \ .$$

The second step combines the inverse tangents in pairs, again dropping the piecewise constants. This will be a continuous expression provided

$$\forall u \in \mathbb{R}, \quad k + uP(u) > 0 \ .$$

Since P has odd degree, uP has even degree, so again k exists. Our aim is therefore to obtain an expression for k in each case.

For the specific integral example given in (21), we have that $uP = u^4 + au^2$, and the above routine gives the lower bound $k = -1/4\,(\max(1, a+1) - a)^2$. This value can now be used in (23).

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