# Solutions for Problem Set 8 MATH 4122/9022 

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7.14 The posted solution to 7.14 (Problem Set 7) was incorrect. Here is an elementary solution. Denote the integral in the question by $I_{n}$ and make the change of variable $y=n x$ so, integrating by parts, we get

$$
\begin{aligned}
I_{n} & =\frac{1}{n} \int_{0}^{\infty} e^{-y} \sin (n / y) d y=\frac{1}{n^{2}} \int_{0}^{\infty} y^{2} e^{-y}(\cos (n / y))^{\prime} d y= \\
& =\frac{1}{n^{2}}\left[\left.y^{2} e^{-y} \cos (n / y)\right|_{0} ^{\infty}-\int_{0}^{\infty}\left(y^{2} e^{-y}\right)^{\prime} \cos (n / y) d y\right] \\
& =-\frac{1}{n^{2}} \int_{0}^{\infty}\left(2 y-y^{2}\right) e^{-y} \cos (n / y) d y .
\end{aligned}
$$

Since $|\cos (n / y)| \leq 1$ for all $y \geq 0$, it follows that $\left|I_{n}\right| \leq \frac{1}{n^{2}} \int_{0}^{\infty}\left|2 y-y^{2}\right| e^{-y} d y$. The integral on the right is finite, hence $\lim _{n \rightarrow \infty} I_{n}=0$.
11.6 Since $m_{2}(A)=\int_{[0,1]} m\left(s_{x}(A)\right) m(d x)=1$ it follows that $m\left(s_{x}(A)\right) \leq 1$ on $[0,1]$ (of course, we also have $m\left(s_{x}(A)\right) \geq 0$ in $[0,1]$ ). Indeed, if we assume $m\left(s_{x}(A)\right)>1$ then $\int_{[0,1]} m\left(s_{x}(A)\right) m(d x)>\int_{[0,1]} m(d x)=1$, which contradicts the fact that $m_{2}(A)=1$. So, the function $x \mapsto 1-m\left(s_{x}(A)\right)$ is nonnegative on $[0,1]$ and, by hypothesis $m_{2}(A)=\int_{[0,1]}\left[m\left(s_{x}(A)\right)-1\right] m(d x)=0$. By Proposition 8.1 the result follows.
11.8 Let $\mathcal{F}$ be the family of all Lebesgue measurable subsets $E \subset[0,1]^{2}$ for which $\int_{E} f=0$. From the given condition it is straightforward to see that $\mathcal{F}$ contains all the rectangles $I \times J$ where $I=\left[a_{1}, a_{2}\right] \subset[0,1]$ and $J=\left[b_{1}, b_{2}\right] \subset[0,1]$ : first note that the condition implies $\int_{\left[a_{1}, a_{2}\right] \times\left[0, b_{1}\right]} f=0$ and $\int_{\left[b_{1}, b_{2}\right] \times\left[0, a_{1}\right]} f=0$; then we have $\left[0, a_{2}\right] \times\left[0, b_{2}\right]=(I \times J) \cup\left(\left[0, a_{1}\right] \times\left[0, b_{1}\right]\right) \cup\left(\left[a_{1}, a_{2}\right] \times\left[0, b_{1}\right]\right) \cup$ $\left(\left[b_{1}, b_{2}\right] \times\left[0, a_{1}\right]\right)$, where the union is disjoint. Applying the given condition and elementary properties of the integral, this proves the statement we made above. Again, applying the elementary properties of the Lebesgue integral, it follows that all elementary sets are contained in $\mathcal{F}$. Hence, all Borel measurable sets in $[0,1]^{2}$ are contained in $\mathcal{F}$, because $\mathcal{F}$ is a $\sigma$-algebra. Now, let $A \subset[0,1]^{2}$ be the set over which $f=f^{+} \geq 0$. Since $A$ is Lebesgue measurable, there exists a

Borel measurable subset $B \subset A$ such that $m(A \backslash B)=0$ (Proposition 4.14(4)). By the above, $0=\int_{B} f=\int_{B} f^{+}$which implies that $f=f^{+}=0$ a.e. on $B$, hence $f=0$ a.e. on $A$. Similarly, there is a Borel set $C \subset A^{c}=\{(x, y) \in$ $\left.[0,1]^{2}: f(x, y)=f^{-}(x, y)\right\}$ so, we deduce in the same way as above that $f=0$ a.e. on $A^{c}$, which proves the result.
11.9 We have:

$$
\begin{equation*}
\left|\frac{x^{2}-y^{2}}{\left(x^{2}+y^{2}\right)^{3 / 4}} \log (4+\sin x)\right| \leq \frac{x^{2}+y^{2}}{\left(x^{2}+y^{2}\right)^{3 / 4}} \log 5=\log 5 \cdot\left(x^{2}+y^{2}\right)^{1 / 4} \tag{0.1}
\end{equation*}
$$

The function $(x, y) \mapsto \log 5 \cdot\left(x^{2}+y^{2}\right)^{1 / 4}$ is continuous on $[0,1]^{2}$ hence Lebesgue (in fact, Riemann) integrable. By (0.1) we obtain
$\int_{[0,1]^{2}}\left|\frac{x^{2}-y^{2}}{\left(x^{2}+y^{2}\right)^{3 / 4}} \log (4+\sin x)\right| d m_{2}(x, y) \leq \int_{[0,1]^{2}} \log 5 \cdot\left(x^{2}+y^{2}\right)^{1 / 4} d m_{2}(x, y)<\infty$,
where $m_{2}$ is the two-dimensional Lebesgue measure. By Fubini's theorem the result follows.
11.10 (1) Define $R_{n}:=\bigcup_{k=1}^{n}\left[\frac{k-1}{n}, \frac{k}{n}\right]^{2}$, for all integers $n \geq 1$. Clearly, $R_{n}$ is Borel measurable and it is not difficult to see that $D=\cap_{n=1}^{\infty} R_{n}$, which proves that $D$ is Borel measurable.
(2) On one hand,

$$
\begin{aligned}
\int_{X} \int_{Y} \chi_{D}(x, y) \mu(d y) m(d x) & =\int_{X} \mu\left(s_{x}(D)\right) m(d x) \\
& =\int_{X} 1 m(d x)=1
\end{aligned}
$$

since, for every fixed $x \in X, s_{x}(D)$ contains exactly one point. On the other hand,

$$
\begin{aligned}
\int_{Y} \int_{X} \chi_{D}(x, y) m(d x) \mu(d y) & =\int_{Y} m\left(s_{y}(D)\right) \mu(d y) \\
& =\int_{Y} 0 \mu(d y)=0
\end{aligned}
$$

since, for every fixed $y \in Y, s_{y}(D)$ contains exactly one point and its Lebesgue measure is 0 .
This does not contradict Fubini's theorem because the counting measure $\mu$ is not $\sigma$-finite on $[0,1]$ (otherwise, we would get that $[0,1]$ is a countable union of finite sets, which is absurd).
11.11 Let us compute the first integral:

$$
\begin{aligned}
\iint f(x, y) d y d x & =\int_{0}^{\infty} \int_{0}^{\infty} f(x, y) d y d x \\
& =\int_{0}^{\infty}\left(\int_{0}^{x} f(x, y) d y+\int_{x}^{x+1} f(x, y) d y+\int_{x+1}^{x+2} f(x, y) d y+\int_{x+2}^{\infty} f(x, y) d y\right) d x \\
& =\int_{0}^{\infty}\left(\int_{x}^{x+1} 1 d y+\int_{x+1}^{x+2}(-1) d y\right) d x=\int_{0}^{\infty}(1-1) d x=0
\end{aligned}
$$

where we used that, by the definition of $f, \int_{0}^{x} f(x, y) d y=\int_{x+2}^{\infty} f(x, y) d y=0$. To compute the second integral, note that $\int_{2}^{\infty} \int_{0}^{\infty} f(x, y) d x d y=\int_{2}^{\infty}\left(\int_{y-2}^{y-1}(-1) d x+\right.$ $\left.\int_{y-1}^{y} 1 d x\right) d y=0$. So

$$
\begin{aligned}
\iint f(x, y) d x d y & =\int_{0}^{2} \int_{0}^{\infty} f(x, y) d x d y \\
& =\int_{0}^{1}\left(\int_{0}^{y} 1 d x\right) d y+\int_{1}^{2}\left(\int_{0}^{y-1}(-1) d x+\int_{y-1}^{y} 1 d x\right) d y \\
& =\int_{0}^{1} y d y+\int_{1}^{2}(-y+1+1) d y=1 / 2+(-4 / 2+1 / 2+2)=1
\end{aligned}
$$

Let $A \subset \mathbb{R}^{2}$ be the set on which $f \neq 0$. Then $|f|=1$, so, if we let $m_{2}$ denote the Lebesgue measure on $\mathbb{R}^{2}, \iint|f| d m_{2}=\int_{A} 1 d m_{2}=m_{2}(A)=\infty$, hence $f$ is not integrable on $\mathbb{R}^{2}$. This is the reason why the above does not contradict Fubini's theorem.
3. For every $n \geq 2$ and $a \geq 0$ denote by $S_{n}^{a}$ the set $S$ defined in the question. We prove by induction that $\int_{\mathbb{R}^{n}} \chi_{S_{n}^{a}}=a^{n} / n!$. Let $m_{n}$ denote the Lebesgue measure on $\mathbb{R}^{n}$. Note that $S_{n}^{a}$ is a closed, bounded region of $\mathbb{R}^{n}$ hence $m_{n}\left(S_{n}^{a}\right)<\infty$, which implies that $\chi_{S_{n}^{a}}$ is integrable, so we can apply Fubini. For $n=2$ we have

$$
\int_{\mathbb{R}^{2}} \chi_{S_{2}^{a}} d m_{2}=\int_{S_{n}^{a}} d m_{2}=\int_{0}^{a} \int_{0}^{a-x_{1}} d x_{2} d x_{1}=a^{2}-\left.\frac{x_{1}^{2}}{2}\right|_{0} ^{a}=\frac{a^{2}}{2}
$$

Now, suppose that for all $a>0, \int_{\mathbb{R}^{n}} \chi_{S_{n}^{a}}=\frac{a^{n}}{n!}$ for some $n \geq 2$. Then,

$$
\int_{\mathbb{R}^{n+1}} \chi_{S_{n+1}^{a}} d m_{n+1}=\int_{0}^{a}\left(\int_{0}^{a-x_{1}} \cdots \int_{0}^{a-x_{1}-\cdots-x_{n}} d x_{n+1} \ldots d x_{2}\right) d x_{1}
$$

Note that $a-x_{1} \geq 0$ and the integral between brackets is equal to $\int_{\mathbb{R}^{n}} \chi_{s_{n}^{a-x_{1}}}=$ $\left(a-x_{1}\right)^{n} / n$ !, by the induction hypothesis. So,

$$
\int_{\mathbb{R}^{n+1}} \chi_{S_{n+1}} d m_{n+1}=\int_{0}^{a}\left(a-x_{1}\right)^{n} / n!d x_{1}=a^{n+1} /(n+1)!
$$

11.1 Recall that if $(X, \mathcal{A})$ is measurable, a complex-valued function $f: X \rightarrow \mathbb{C}$, $f=u+i v$ is said to be measurable if $u$ and $v$ are measurable (as real-valued functions). Also, if $(X, \mathcal{A}, \mu)$ is a measure space, $f$ is integrable if $\int_{X}|u|+|v| d \mu<$ $\infty$, which is the same as saying that both $u$ and $v$ are integrable.
A version of Fubini's theorem for a complex-valued function $f: X \times Y \rightarrow \mathbb{C}$, $f=u+i v$ can be stated as in the real case (Theorem 11.3) by keeping all the conditions and statements the same, except for conditions $(a),(b)$ which we replace with: either $u, v$ are both integrable, or both nonnegative, or at least one of $u, v$ is nonnegative and the other one is integrable. The proof of the statement follows by applying Fubini's theorem for real-valued functions to $u$ and $v$, as required. Note that the new conditions introduced above allow us to do so. In these notes we prove only point (5).

$$
\begin{aligned}
\int f(x, y) d(\mu \times \nu)(x, y) & =\int u(x, y) d(\mu \times \nu)(x, y)+i \int v(x, y) d(\mu \times \nu)(x, y) \\
& =\iint u(x, y) d \mu(x) d \nu(y)+i \iint v(x, y) d \mu(x) d \nu(y) \\
& =\int\left(\int(u+i v) d \mu(x)\right) d \nu(y)=\iint f(x, y) d \mu(x) d \nu(y) .
\end{aligned}
$$

We used Theorem 11.3 in the second equality, then the linearity of the integral together with Proposition 7.5 regarding multiplying an integral with a complex constant. The other equality follows similarly.
5. We compute $\int_{0}^{\infty} \frac{\sin x}{x} d x$ (implicitly showing that the integral converges) which would also give us the required limit. Using the two hints provided in the question we get

$$
\begin{aligned}
\int_{0}^{\infty} \frac{1}{x} \sin x d x & =\frac{1}{2 i} \int_{0}^{\infty}\left(\int_{0}^{\infty} e^{-t x} d t\right)\left(e^{i x}-e^{-i x}\right) d x \\
& =\frac{1}{2 i} \int_{0}^{\infty} \int_{0}^{\infty}\left(e^{(-t+i) x}-e^{(-t-i) x}\right) d t d x \\
& =\frac{1}{2 i} \int_{0}^{\infty} \int_{0}^{\infty}\left[e^{-t x}(\cos x+i \sin x)-e^{-t x}(\cos x-i \sin x)\right] d t d x \\
& =\int_{0}^{\infty} \int_{0}^{\infty} e^{-t x} \sin x d t d x=\int_{0}^{\infty} \int_{0}^{\infty} e^{-t x} \sin x d x d t
\end{aligned}
$$

Let $I:=\int_{0}^{\infty} e^{-t x} \sin x d x$. Using integration by parts, we obtain

$$
I=1-t\left(\left.e^{-t x} \sin x\right|_{0} ^{\infty}+t I\right)
$$

which gives $I=\frac{1}{1+t^{2}}$. Then, our integral becomes $\int_{0}^{\infty} \frac{1}{1+t^{2}} d t=\left.\arctan \right|_{0} ^{\infty}=$ $\frac{\pi}{2}$.
6. If $t_{1}<t_{2}$, then $\left\{x:|f(x)| \geq t_{2}\right\} \subset\left\{x:|f(x)| \geq t_{1}\right\}$ and both sets are measurable, so $\mu_{f}\left(t_{2}\right) \leq \mu_{f}\left(t_{1}\right)$, which proves monotonicity. As every monotone function, $\mu_{f}$ is thus Borel measurable, since for any $s \in \mathbb{R}, \mu_{f}^{-1}((s, \infty))$ is either the empty set or an interval.
7. Using the hint,

$$
\begin{aligned}
\int_{X}|f(x)|^{p} d \mu(x) & =\int_{X}\left(\int_{0}^{|f(x)|} p t^{p-1} d t\right) d \mu(x) \\
& =\int_{X}\left(\int_{0}^{\infty} p t^{p-1} \chi_{[0,|f(x)|]}(t) d t\right) d \mu(x) \\
& =\int_{0}^{\infty} p t^{p-1}\left(\int_{X} \chi_{[0,|f(x)|]}(t) d \mu(x)\right) d t \\
& =\int_{0}^{\infty} p t^{p-1}\left(\int_{X} \chi_{A_{t}}(x) d \mu(x)\right) d t \\
& =\int_{0}^{\infty} p t^{p-1} \mu\left(A_{t}\right) d t \\
& =\int_{0}^{\infty} \mu_{f}(t) p t^{p-1} d t .
\end{aligned}
$$

In the third equality we were able to use Fubini because the function $(x, t) \mapsto$ $p t^{p-1} \chi_{[0,|f(x)|]}(t)$ is nonnegative (and of course, because both $\mu$ and the Lebesgue measure on $\mathbb{R}$ are $\sigma$-finite).

