# Solutions for Problem Set 1 MATH 4122/9022 

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January 29, 2018

1 (i) Reflexivity: obvious Symmetry: obvious (because all conditions are iff) Transitivity: Suppose $x \sim y$ and $y \sim z$ then we have $x \in A \Longleftrightarrow y \in$ $A \Longleftrightarrow z \in A, \forall A \in \mathcal{A}$. This gives us transitivity.

1 (ii) Given $x \in X$, we have $y \in[x]_{\sim}$ only if $y \in A$ for all $A \in \mathcal{A}$ such that $x \in A$. Hence, $[x]_{\sim} \subset \bigcap\{A \in \mathcal{A}: x \in A\}$. Conversely, if $y \notin[x]_{\sim}$ then $y \notin A_{0}$ for some $A_{0} \in \mathcal{A}$ with $x \in A_{0}$. Then, $y \notin \bigcap\{A \in \mathcal{A}: x \in A\}$. The containment $[x]_{\sim} \in \mathcal{A}$ follows from the fact that the above intersection is finite.

1 (iii) By part (ii) and the finiteness of $X$, it suffices to show that if $A \in \mathcal{A}$ and $x \in A$ then $[x]_{\sim} \subset A$. The latter though is just a consequence of the definition of $\sim$.

The problems assigned from the text
2.1 Consider $X=\{1,2,3\}$ and $\mathcal{M}=\{\emptyset,\{1\},\{2\},\{3\}, X\}$. It can easily be seen that $\mathcal{M}$ is a monotone class. However, $\mathcal{M}$ is not a $\sigma$-algebra because $\{1\} \cup\{2\} \notin \mathcal{M}$.
2.3 We may construct a counter example. Consider $X=\mathbb{Z}_{+}$. For $i \geq 1$, let $\mathcal{A}_{i}$ be the $\sigma$-algebra consisting of all the subsets of $\{1,2, \ldots, i\}$ and their complements. Consider $A_{i}=\{2 i\}$. Now $A_{i} \in \bigcup_{i=1}^{\infty} \mathcal{A}_{i}$, but $\bigcup_{i=1}^{\infty} A_{i} \notin$ $\bigcup_{i=1}^{\infty} \mathcal{A}_{i}$. The latter is because $\bigcup_{i=1}^{\infty} \mathcal{A}_{i}$ contains only sets that are either finite or co-finite.
2.4 We may use a similar counter example as above, except for $A_{i}=\{2,4, \ldots, 2 i\}$.
$2.5 X \in \mathcal{B}$ because $X=f^{-1}(Y) . \emptyset \in \mathcal{B}$ because $\emptyset=f^{-1}(\emptyset)$. If $B \in \mathcal{B}$ then $B=f^{-1}(A)$ for some $A \in \mathcal{A}$. Then, $B^{c}=\left(f^{-1}(A)\right)^{c}=f^{-1}\left(A^{c}\right)$ is in $\mathcal{B}$. The equality $f^{-1}\left(\cup_{i=1}^{\infty} A_{i}\right)=\cup_{i=1}^{\infty} f^{-1}\left(A_{i}\right)$ gives us the closure under countable unions.
2.6 We first prove that $\mathcal{A}$ is infinite. This implies that it is uncountable by the solution of 2.8 . We choose $A_{0} \in \mathcal{A}, A_{0} \neq \emptyset$. Then by the property given we have $A_{0}=A_{1} \cup A_{1}^{\prime}$, with $A_{1} \cap A_{1}^{\prime}=\emptyset$, and $A_{1}, A_{1}^{\prime} \neq \emptyset$. Inductively, from $A_{n} \in \mathcal{A}$, we may again find $A_{n+1}, A_{n+1}^{\prime}$. These sets form a sequence
$A_{0} \supsetneq A_{1} \supsetneq \cdots \supseteq A_{n} \supsetneq A_{n+1} \supsetneq \ldots$. Further we have $A_{n} \neq \emptyset, \forall n \in \mathbb{N}$, and $A_{n} \in \mathcal{A}, \forall n \in \mathbb{N}$. Thus $|\mathcal{A}| \geq \aleph_{0}$, and hence by $2.8|\mathcal{A}|>\aleph_{0}$.
2.7 We have $\emptyset \in \mathcal{A}$ because (the constant function) $0 \in \mathcal{F} . X \in \mathcal{A}$ because $1 \in \mathcal{F}$. Further, for any $A \subset X, A \in \mathcal{A} \Longrightarrow \chi_{A} \in \mathcal{F} \Longrightarrow 1-\chi_{A} \in$ $\mathcal{F} \Longrightarrow A^{c} \in \mathcal{A}$. Now we prove closure under infinite intersections because it is simpler. Note that if $A_{i} \in \mathcal{A}$ then $\chi_{\cap_{i=1}^{n} A_{i}}=\prod_{i=1}^{n} \chi_{A_{i}} \in \mathcal{F}$. Now $\chi_{\cap_{i=1}^{n} A_{i}} \rightarrow \chi_{\cap_{i=1}^{\infty} A_{i}}$ as $n \rightarrow \infty$, and hence $\chi_{\cap_{i=1}^{\infty} A_{i}} \in \mathcal{F} \Longrightarrow \cap_{i=1}^{\infty} A_{i} \in \mathcal{A}$.
2.8 We use the first problem of the assignment as a hint. Consider the same equivalence relation. Suppose this time that $\mathcal{A}$, a $\sigma$-algebra, has a countable and infinite number of members. If there are a finite number of equivalence classes of $\sim$, then $\mathcal{A}$ is finite. However if there are an infinite number of equivalence classes then all their countable unions are in $\mathcal{A}$ which implies that $|\mathcal{A}|>\aleph_{0}$, which is a contradiction.
3.1 Consider $B_{i}$ pairwise disjoint. Define $A_{n}=\cup_{i=1}^{n} B_{i}$. Then we see that $\cup_{i=1}^{\infty} A_{i} \rightarrow \cup_{i=1}^{\infty} B_{i}$ as $n \rightarrow \infty$. Thus we have, $\mu\left(\cup_{i=1}^{\infty} A_{i}\right)=\lim _{i \rightarrow \infty} \mu\left(A_{i}\right)=$ $\lim _{i \rightarrow \infty} \sum_{j=1}^{i} \mu\left(B_{i}\right)=\mu\left(\cup_{i=1}^{\infty} A_{i}\right)$.
3.2 Consider $A_{n}=\cup_{i=1}^{\infty} B_{i} \backslash \cup_{i=1}^{n} B_{i}$. Now $\mu\left(A_{i}\right)+\sum_{i=1}^{n} \mu\left(B_{i}\right)=\mu\left(\cup_{i=1}^{\infty} B_{i}\right)$. (Note that $\mu\left(B_{i}\right)<\infty$ because of the finiteness condition) Taking the limits of both sides we see that $\mu$ is a measure.
3.4 The result follows immediately, when all the quantities involved are finite, from adding the relations $\mu(A \backslash B)+\mu(B \cap A)=\mu(A)$ and $\mu(A \cup B)=$ $\mu(A \backslash B)+\mu(B)$. It is easy to check that the relation holds in the case when any of the measures is infinite.
3.5 The required property reduces to the consideration of the interchange of the order of the double sum $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_{n} \mu_{n}\left(A_{i}\right)$, where the $A_{i}$ are pairwise disjoint members of $\mathcal{A}$. In the case where all the summands are finite, this can be done because if any rearrangement yielded a finite sum, then all rearrangements would yield a finite sum. The case with even a single infinite summand is trivial.
3.6 This follows from the fact that, if $\left\{A_{i}\right\}_{i}$ is a pairwise disjoint family in $\mathcal{A}$, then $\left\{A_{i} \cap B\right\}_{i}$ is also a pairwise disjoint family. One then easily checks the countable additivity property from the definition of $\nu$ in terms of $\mu$.
3.7 This is true. Suppose that $\left\{A_{i}\right\}_{i}$ is a pairwise disjoint family of sets of $\mathcal{A}$. Then we have $\lim _{n \rightarrow \infty} \lim _{k \rightarrow \infty} \sum_{i=1}^{k} \mu_{n}\left(A_{i}\right)=\sup _{n \in \mathbb{N}} \sup _{k \in \mathbb{N}} \sum_{i=1}^{k} \mu_{n}\left(A_{i}\right)=$ $\sup _{k \in \mathbb{N}} \sup _{n \in \mathbb{N}} \sum_{i=1}^{k} \mu_{n}\left(A_{i}\right)$, which proves that $\mu$ is a measure. Note that monotonocity is essential in the previous argument. In the monotonically decreasing case the finiteness is essential as it allows us to interchange two limits in the corresponding double sequence.

